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A NOTE ON THE CONVEXITY OF THE INDEFINITE JOINT NUMERICAL RANGE*

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Abstract. This note investigates the convexity of the indefinite joint numerical range of a tuple of Hermitian matrices in the setting of Krein spaces. Its main result is a necessary and sufficient condition for convexity of this set. A new notion of "quasi-convexity" is introduced as a refinement of pseudo-convexity.

Key words. Krein space, Joint numerical range, Convexity.

AMS subject classifications. 46C20, 47A12.

1. Introduction. The concept of numerical range of linear operators on a Hilbert space was introduced by Toeplitz ([16]) and has been generalized in several directions.The theory of numerical ranges of operators on a Krein space has also been considered by some authors (e.g., see [2, 3, 5, 6, 12, 14] and the references therein).

An n-dimensional Krein space is a complex vector space $\mathcal{K}$ with an indefinite inner product $[\cdot,\cdot]$. In this space, we can choose a basis $\{\xi_1, \xi_2, \dots, \xi_r, \xi_{r+1}, \dots, \xi_n\}$ satisfying

$$[\xi_i, \xi_j] = 0$$

for $1 \le i < j \le n$, $[\xi_i, \xi_i] > 0$ for $1 \le i \le r$ and $[\xi_i, \xi_i] < 0$ for $r+1 \le i \le n$. If $r = n$, then the space $\mathcal{K}$ is a Hilbert space. If $r = 0$, then the metric $[\cdot,\cdot]$ introduces a Hilbert space structure in $\mathcal{K}$. So we assume that $0 < r < n$. We identify the space $\mathcal{K}$ with the vector space $\mathbb{C}^n$ via the isomorphism $x_1\xi_1 + \cdots + x_n\xi_n \mapsto (x_1, \dots, x_n)^T$. Assuming that $\langle \xi_i, \xi_j \rangle = \delta_{ij}$, the standard inner product $\langle \cdot, \cdot \rangle$ is given by

$$\langle (u_1, \dots, u_n)^T, (v_1, \dots, v_n)^T \rangle = u_1\overline{v_1} + \cdots + u_n\overline{v_n}.$$

We set the real invertible matrix $J = \mathrm{diag}(a_1, \dots, a_n)$ with signature $(r, n-r)$. Then

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the metric $[\cdot,\cdot]$ is associated with the inner product $\langle\cdot,\cdot\rangle$ by

$$[\xi,\eta]=\langle J\xi,\eta\rangle$$

for ξ,η in $\mathcal{K}=\mathbb{C}^n$.

Let $\mathcal{H}_n$ denote the real vector space of $n\times n$ Hermitian matrices. Suppose $JH=(JH_1,\ldots,JH_k)\in\mathcal{H}_n^k$. The *J-joint numerical range* $W_J(H)$ of $(H_1,\ldots,H_k)$, was introduced in [12] and is defined as

$$W_J(H_1,\ldots,H_k)=\{([H_1x,x],\ldots,[H_kx,x])/[x,x]:\ x\in\mathbb{C}^n,\ [x,x]\neq 0\}\subseteq\mathbb{R}^{k\times 1}.$$

Given a linear operator T acting on $\mathcal{K}$, the set

$$W_J(T)=\{[Tx,x]/[x,x]:\ x\in\mathcal{K},\ [x,x]\neq 0\},$$

is called the *J-numerical range* of T. Its boundary lies on an algebraic curve. The method to compute the equation of the boundary is given in [15]. The J-numerical range can be viewed as one of generalized numerical ranges (cf. [8, 11]). In view of the representation $T=T_1+iT_2$, where $JT_1=\dfrac{JT+T^*J}{2}$ and $JT_2=\dfrac{JT-T^*J}{2i}$ are Hermitian matrices, $W_J(T)$ is the J-joint numerical range of (T_1,T_2).

In the sequel, we also consider the sets

$$\begin{aligned}W_J^{\pm}(H)&=W_J^{\pm}(H_1,\ldots,H_n)\\&=\{\pm([H_1x,x],\ldots,[H_kx,x]):\ x\in\mathbb{C}^n,\ [x,x]=\pm 1\}\subseteq\mathbb{R}^{k\times 1}.\end{aligned}$$

When $J=I$, $W_J(H)=W_J^+(H)$ reduces to the usual *joint numerical range*

$$W(H)=\{x^*H_1x,\ldots,x^*H_kx:\ x\in\mathbb{C}^n,\ x^*x=1\}\subseteq\mathbb{R}^{k\times 1},$$

which has attracted the attention of several researchers (e.g., see [4, 7] and references therein). This concept is useful in theoretical and applied subjects, in particular, control systems (e.g. see [9] and their references).

For $JH\in\mathcal{H}_n^k$, the following properties can be easily checked.

(P1) $W_J(H)=W_J^+(H)\cup W_J^-(H)$.

(P2) $W_{(-J)}^+(-H)=W_J^-(H)$.

Having in mind property (P2), we can focus our study on $W_J^+(H)$ and translate the results on this set to $W_J^-(H)$ and $W_J(H)$. We observe that $W_J^{\pm}(H)$ need not be closed or bounded, however it is always connected. In fact, since $\{x\in\mathbb{C}^n:\ x^*Jx=1\}$ is connected, the same happens with the set

$$V_J=\{xx^*:\ x\in\mathbb{C}^n,\ x^*Jx=1\}.$$

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Thus, $W_J^+(H)$ is the image of V_J under the linear map $\phi_H : \mathcal{H}_n \to \mathbb{R}^{k\times 1}$ defined by

$$\phi_H(X) = (\mathrm{tr}\,(JH_1X), \ldots, \mathrm{tr}\,(JH_kX)), \quad X \in \mathcal{H}_n,$$

where $\mathrm{tr}\, Y$ denotes the trace of the matrix Y.

We recall that a subset S in $\mathbb{R}^{k\times 1}$ has affine dimension m if there exists $v \in S$ such that the linear span of $S - v$ has dimension m. Having in mind that $\mathrm{rank}(\phi_H)$ equals the dimension of $\mathrm{span}\{H_1, \ldots, H_k\}$ and $V_H \subseteq \mathcal{H}_2$ has real affine dimension 4, we conclude that $W_J^+(H)$ has affine dimension at most 4.

In studies of the numerical range and its generalizations, a useful technique is performing a convenient reduction to the 2×2 case. For instance, in results of convexity [1], this approach is useful, as well as when devising an effective procedure for generating the numerical range of an arbitrary n by n complex matrix reducing the general case to the bi-dimensional one. The following theorem was obtained in [13] and treats this case.

Theorem 1.1. *Suppose* $J, JH_1, \ldots, JH_k \in \mathcal{H}_2$, $H = (H_1, \ldots, H_k,)$ *and* $\mathrm{span}\{J, JH_1, \ldots, JH_k\}$ *has dimension* $m \le 4$. *Then* $W_J^+(H)$ *has affine dimension* $m - 1$ *and* $W_J^+(H)$ *is: a singleton if* $m = 1$*; a open or closed half line or a straight line if* $m = 2$*; a closed one-component hyperbola with interior, an open two-dimensional half plane, or a two-dimensional plane if* $m = 3$*; a closed one-component hyperboloid without interior if* $m = 4$.

As a simple consequence of this result, we can conclude that for $J, JH_1, \ldots, JH_k \in \mathcal{H}_2$ and $H = (H_1, \ldots, H_k)$, the set $W_J^+(H)$ is convex if and only if $\mathrm{span}\{J, JH_1, \ldots, JH_k\}$ has dimension less than 4.

Since the works of Toeplitz [16], it is known that the joint numerical range is not, in general, convex. There exists a vast literature on the (non)convexity of $W(H)$. The joint numerical range of a triple of 2×2 Hermitian matrices is typically not convex. The situation is analogous if $k \ge 4$ with the non convexity of $W(H_1, \ldots, H_k)$ (cf. [10]). As a consequence, there has been great emphasis on the study of sufficient conditions for the convexity of $W(H_1, \ldots, H_k)$ with $k > 3$. Interestingly, if $H_1, H_2, H_3 \in \mathcal{H}_n$ where $n > 2$, then $W(H_1, H_2, H_3)$ is convex (cf. [1, 4]) In view of this fact, it seems natural to ask whether $W_J(H_1, H_2, H_3)$ is convex.

Besides this introductory section, this note consists of two more sections. In Section 2, an example answers negatively the above question. In Section 3, a necessary and sufficient condition for convexity of the J-joint numerical range is obtained.

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2. Preliminary examples of non-convexity. Let us consider

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, H_1 = JH_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$H_2 = JH_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, H_3 = JH_3 = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We recall that

$$W_J^+(H_1, H_2, H_3) = \left\{ \left(\frac{\langle JH_1\xi, \xi\rangle}{\langle J\xi, \xi\rangle}, \frac{\langle JH_2\xi, \xi\rangle}{\langle J\xi, \xi\rangle}, \frac{\langle JH_3\xi, \xi\rangle}{\langle J\xi, \xi\rangle} \right) : \xi \in \mathbb{C}^3, \langle \xi, \xi\rangle = 1, \langle J\xi, \xi\rangle > 0 \right\},$$

where $\langle \xi, \eta\rangle = \eta^*\xi$ for $\xi, \eta \in \mathbb{C}^3$. Since the equation

$$\langle A(c\xi), (c\xi)\rangle = \langle A\xi, \xi\rangle$$

holds for any linear operator A, any unit vector ξ and any complex number c with modulus 1, we conclude that the set $W(J, H_1, H_2, H_3)$ coincides with the following set

$$\{(\langle J\xi, \xi\rangle, \langle H_1\xi, \xi\rangle, \langle H_2\xi, \xi\rangle, \langle H_3\xi, \xi\rangle) : \xi \in \Omega\}$$

for

$$\Omega = \{(\cos\phi\cos\psi\exp(i\theta), \cos\phi\sin\psi\exp(i\eta), \sin\phi)^T : \\ 0 \le \phi < \pi/4, 0 \le \psi \le 2\pi, 0 \le \theta \le 2\pi, 0 \le \eta \le 2\pi\}.$$

For the vectors $\xi \in \Omega$, the following relations hold

$$\begin{aligned} \langle J\xi, \xi\rangle &= \cos(2\phi), \\ \langle H_1\xi, \xi\rangle &= \frac{\cos(2\phi) + 1}{2}\cos(2\psi), \\ \langle H_2\xi, \xi\rangle &= \frac{\cos(2\phi) + 1}{2}\sin(2\psi)\cos(\theta - \eta), \\ \langle H_3\xi, \xi\rangle &= \frac{\cos(2\phi) + 1}{2}\sin(2\psi)\sin(\theta - \eta). \end{aligned}$$

Thus, we may assume that $0 \le \theta \le 2\pi$ and $\eta = 0$ for the determination of the set $W_J^+(H_1, H_2, H_3)$. Since

$$\left\{ \frac{1}{2}\frac{\cos(2\phi) + 1}{\cos(2\phi)} : 0 \le \phi < \pi/4 \right\} = [1, +\infty),$$

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we obtain

$$\begin{aligned} &W_J^+(H_1,H_2,H_3)\\ &=\{(t\cos(2\psi),t\sin(2\psi)\cos\theta,t\sin(2\psi)\sin\theta):1\le t,0\le\psi\le 2\pi,0\le\theta\le 2\pi\}\\ &=\{(x,y,z)\in\mathbb{R}^3:x^2+y^2+z^2\ge 1\}.\end{aligned}$$

Hence, the connected set $W_J^+(H_1,H_2,H_3)$ is not convex, since the two points $(0,0,1)$, $(0,0,-1)$ belong to it but their midpoint $(0,0,0)$ does not belong to $W_J^+(H_1,H_2,H_3)$.

By a similar process, we can prove that $W_J^-(H_1,H_2,H_3)=\mathbb{R}^3$, and so $W_J(H_1,H_2,H_3)=\mathbb{R}^3$. We can also construct a 3-tuple of Hermitian matrices (H_1,H_2,H_3) on a 4-dimensional Krein space of type $(2,2)$ for which $W_J^+(H_1,H_2,H_3)$ is not convex. Indeed, let us now consider $\widetilde{J}=I_2\oplus -I_2$ and the direct sums of the Pauli matrices with 2×2 zero blocks,

$$\widetilde{H}_1=\widetilde{J}\widetilde{H}_1=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\oplus\begin{bmatrix}0&0\\0&0\end{bmatrix},$$

$$\widetilde{H}_2=\widetilde{J}\widetilde{H}_2=\begin{bmatrix}0&1\\1&0\end{bmatrix}\oplus\begin{bmatrix}0&0\\0&0\end{bmatrix},\quad \widetilde{H}_3=\widetilde{J}\widetilde{H}_3=\begin{bmatrix}0&i\\-i&0\end{bmatrix}\oplus\begin{bmatrix}0&0\\0&0\end{bmatrix}.$$

Let

$$\widetilde{\xi}=\begin{bmatrix}\cos\theta\cos\phi\cos\psi\mathrm{e}^{i\alpha}\\ \cos\theta\cos\phi\sin\psi\mathrm{e}^{i\beta}\\ \cos\theta\sin\phi\mathrm{e}^{i\gamma}\\ \sin\theta\end{bmatrix},$$

for $\theta,\phi,\psi,\alpha,\beta,\gamma\in[0,2\pi]$. We find

$$\widetilde{\xi}^*\widetilde{J}\widetilde{\xi}=\frac{1}{4}(-2+2\cos(2\phi)+2cos(2\theta)+4\cos(2\phi)cos(2\theta)),$$

$$\widetilde{\xi}^*\widetilde{H}_1\widetilde{\xi}=\frac{1}{4}\cos(2\psi)(1+\cos(2\phi)+\cos(2\theta)+\cos(2\phi)cos(2\theta)),$$

$$\widetilde{\xi}^*\widetilde{H}_2\widetilde{\xi}=\frac{1}{4}\cos(\alpha-\beta)\sin(2\psi)(1+\cos(2\phi)+\cos(2\theta)+\cos(2\phi)\cos(2\theta)),$$

$$\widetilde{\xi}^*\widetilde{H}_3\widetilde{\xi}=\frac{1}{4}\sin(\beta-\alpha)\sin(2\psi)(1+\cos(2\phi)+\cos(2\theta)+\cos(2\phi)cos(2\theta)).$$

For

$$R(\theta,\phi)=\left|\frac{1+\cos(2\phi)+\cos(2\theta)+\cos(2\phi)cos(2\theta)}{-2+2\cos(2\phi)+2cos(2\theta)+4\cos(2\phi)\cos(2\theta)}\right|,$$

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and

$$D(\theta,\phi) = (1+\cos(2\phi)+\cos(2\theta)+\cos(2\phi)\cos(2\theta)),$$

we have

$$\{R(\theta,\phi) : \theta,\phi \in [0,2\pi], D(\theta,\phi) < 0\} = [0,+\infty[$$

and

$$\{R(\theta,\phi) : \theta,\phi \in [0,2\pi], D(\theta,\phi) > 0\} = \left[\frac{2}{3},+\infty\right[.$$

Thus, we get

$$W^+_{\widetilde{J}}(\widetilde{H}_1,\widetilde{H}_2,\widetilde{H}_3) = \left\{(x,y,z) : x^2+y^2+z^2 \geq \frac{2}{3}\right\}, \quad W^-_{\widetilde{J}}(\widetilde{H}_1,\widetilde{H}_2,\widetilde{H}_3) = \mathbb{R}^3.$$

3. Convexity. A linear operator H on $\mathcal{K}$ is said to be *J-Hermitian* if it satisfies

$$[H\xi,\eta] = [\xi,H\eta]$$

for $\xi,\eta \in \mathcal{K}$.

THEOREM 3.1. *Assume that $\mathcal{K}$ is a Krein space with dimension $n \geq 3$ and (H_1,H_2,H_3) is an ordered triple of J-Hermitian operators on $\mathcal{K}$. If the joint numerical range $W(J,JH_1,JH_2,JH_3)$ is convex, then:*

(1) *The J-joint numerical ranges*

$$W^+_J(H_1,H_2,H_3) = \left\{\left(\frac{[H_1\xi,\xi]}{[\xi,\xi]},\frac{[H_2\xi,\xi]}{[\xi,\xi]},\frac{[H_3\xi,\xi]}{[\xi,\xi]}\right) : \xi \in \mathcal{K}, [\xi,\xi] > 0\right\},$$

$$W^-_J(H_1,H_2,H_3) = \left\{\left(\frac{[H_1\xi,\xi]}{[\xi,\xi]},\frac{[H_2\xi,\xi]}{[\xi,\xi]},\frac{[H_3\xi,\xi]}{[\xi,\xi]}\right) : \xi \in \mathcal{K}, [\xi,\xi] < 0\right\},$$

are convex.

(2) *For $(x_1,y_1,z_1) \in W^+_J(H_1,H_2,H_3)$ and $(x_2,y_2,z_2) \in W^-_J(H_1,H_2,H_3)$, we have*

$$\{\lambda(x_1,y_1,z_1)+(1-\lambda)(x_2,y_2,z_2) : 1 \leq \lambda\} \subset W^+_J(H_1,H_2,H_3),$$

$$\{\lambda(x_1,y_1,z_1)+(1-\lambda)(x_2,y_2,z_2) : \lambda \leq 0\} \subset W^-_J(H_1,H_2,H_3).$$

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Proof. (1) We consider the Hermitian matrices K_j defined by

$$K_j = JH_j$$

for $j = 1, 2, 3$. We assume that $W(J, K_1, K_2, K_3)$ is convex. Let $(x_1, y_1, z_1), (x_2, y_2, z_2) \in W_J^+(H_1, H_2, H_3)$ and $0 < \lambda < 1$. There exist $\xi, \eta \in \mathbb{C}^n$ such that $\xi^*\xi = 1,\ \eta^*\eta = 1$ and

$$(t, X_1, Y_1, Z_1) = (\xi^* J\xi, \xi^* K_1\xi,\ \xi^* K_2\xi,\ \xi^* K_3\xi),$$

$$(s, X_2, Y_2, Z_2) = (\eta^* J\eta, \eta^* K_1\eta,\ \eta^* K_2\eta,\ \eta^* K_3\eta),$$

with

$$(x_1, y_1, z_1) = \left(\frac{X_1}{t}, \frac{Y_1}{t}, \frac{Z_1}{t}\right), \quad (x_2, y_2, z_2) = \left(\frac{X_2}{s}, \frac{Y_2}{s}, \frac{Z_2}{s}\right).$$

We claim that $(x_3, y_3, z_3) = \lambda(x_1, y_1, z_1) + (1 - \lambda)(x_2, y_2, z_2) \in W_J^+(H_1, H_2, H_3)$. For

$$\mu = \frac{\lambda s}{(1 - \lambda)t + \lambda s},$$

which satisfies $0 < \mu < 1$, let consider $(u, X_3, Y_3, Z_3) = \mu(t, X_1, Y_1, Z_1) + (1 - \mu)(s, X_2, Y_2, Z_2)$, which is obviously in $W(J, K_1, K_2, K_3)$. It follows that

$$\left(1, \frac{X_3}{u}, \frac{Y_3}{u}, \frac{Z_3}{u}\right) = \frac{\mu t}{\mu t + (1 - \mu)s}(1, x_1, y_1, z_1) + \frac{(1 - \mu)s}{\mu t + (1 - \mu)s}(1, x_2, y_2, z_2),$$

or, since $\lambda = \dfrac{\mu t}{\mu t + (1 - \mu)s}$,

$$\left(1, \frac{X_3}{u}, \frac{Y_3}{u}, \frac{Z_3}{u}\right) = \lambda(1, x_1, y_1, z_1) + (1 - \lambda)(1, x_2, y_2, z_2).$$

Thus,

$$(x_3, y_3, z_3) = \left(\frac{X_3}{u}, \frac{Y_3}{u}, \frac{Z_3}{u}\right).$$

Clearly, there exists $\zeta \in \mathbb{C}^n$ such that $\zeta^*\zeta = 1$ and

$$(u, X_3, Y_3, Z_3) = (\zeta^* J\zeta, \zeta^* K_1\zeta,\ \zeta^* K_2\zeta,\ \zeta^* K_3\zeta),$$

which proves the claim. Thus, the convexity of $W_J^+(H_1, H_2, H_3)$ is proved under the assumption that $W(J, K_1, K_2, K_3)$ is convex. Similarly, if $W(J, K_1, K_2, K_3)$ is convex, then $W_J^-(H_1, H_2, H_3)$ is also convex.

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(2) Let $(x_1, y_1, z_1) \in W_J^+(H_1, H_2, H_3)$, $(x_2, y_2, z_2) \in W_J^-(H_1, H_2, H_3)$ and $1 \leq \lambda < +\infty$. There exist $\xi, \eta \in \mathbb{C}^n$ such that $\xi^*\xi = 1$, $\eta^*\eta = 1$ and

$$(t, X_1, Y_1, Z_1) = (\xi^* J\xi, \xi^* K_1\xi,\ \xi^* K_2\xi,\ \xi^* K_3\xi),$$

$$(s, X_2, Y_2, Z_2) = (\eta^* J\eta, \eta^* K_1\eta,\ \eta^* K_2\eta,\ \eta^* K_3\eta),$$

with

$$(x_1, y_1, z_1) = \left(\frac{X_1}{t}, \frac{Y_1}{t}, \frac{Z_1}{t}\right), \quad (x_2, y_2, z_2) = \left(\frac{X_2}{s}, \frac{Y_2}{s}, \frac{Z_2}{s}\right).$$

Notice that $t > 0,\ s < 0$. We claim that $(x_3, y_3, z_3) = \lambda(x_1, y_1, z_1) + (1-\lambda)(x_2, y_2, z_2) \in W_J^+(H_1, H_2, H_3)$. For

$$\mu = \frac{\lambda s}{(1-\lambda)t + \lambda s},$$

which satisfies $0 < \mu < 1$, let consider $(u, X_3, Y_3, Z_3) = \mu(t, X_1, Y_1, Z_1) + (1-\mu)$ (s, X_2, Y_2, Z_2), which is in $W(J, K_1, K_2, K_3)$ by the assumption of the convexity of this set. It follows that

$$\left(1, \frac{X_3}{u}, \frac{Y_3}{u}, \frac{Z_3}{u}\right) = \frac{\mu t}{\mu t + (1-\mu)s}(1, x_1, y_1, z_1) + \frac{(1-\mu)s}{\mu t + (1-\mu)s}(1, x_2, y_2, z_2),$$

or, since $\lambda = \dfrac{\mu t}{\mu t + (1-\mu)s}$,

$$\left(1, \frac{X_3}{u}, \frac{Y_3}{u}, \frac{Z_3}{u}\right) = \lambda(1, x_1, y_1, z_1) + (1-\lambda)(1, x_2, y_2, z_2).$$

Thus,

$$(x_3, y_3, z_3) = \left(\frac{X_3}{u}, \frac{Y_3}{u}, \frac{Z_3}{u}\right).$$

Clearly, there exists $\zeta \in \mathbb{C}^n$ such that $\zeta^*\zeta = 1$ and

$$(u, X_3, Y_3, Z_3) = (\zeta^* J\zeta, \zeta^* K_1\zeta,\ \zeta^* K_2\zeta,\ \zeta^* K_3\zeta),$$

which proves the claim. We have shown that $\{\lambda(x_1, y_1, z_1) + (1-\lambda)(x_2, y_2, z_2) : 1 \leq \lambda\} \subset W_J^+(H_1, H_2, H_3)$. Similarly we can show that $\{\lambda(x_1, y_1, z_1) + (1-\lambda)(x_2, y_2, z_2) : \lambda \leq 0\} \subset W_J^-(H_1, H_2, H_3)$. $\square$

Definition 3.2. The set $W_J(H_1, H_2, H_3)$ is said to be quasi-convex if the following conditions hold together:

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(1) If $(x_1, y_1, z_1), (x_2, y_2, z_2) \in W_J^+(H_1, H_2, H_3)$, then

$$\{\lambda(x_1, y_1, z_1) + (1-\lambda)(x_2, y_2, z_2) : 0 \le \lambda \le 1\} \subset W_J^+(H_1, H_2, H_3);$$

(2) If $(x_1, y_1, z_1), (x_2, y_2, z_2) \in W_J^-(H_1, H_2, H_3)$, then

$$\{\lambda(x_1, y_1, z_1) + (1-\lambda)(x_2, y_2, z_2) : 0 \le \lambda \le 1\} \subset W_J^-(H_1, H_2, H_3);$$

(3) If $(x_1, y_1, z_1) \in W_J^+(H_1, H_2, H_3)$ and $(x_2, y_2, z_2) \in W_J^-(H_1, H_2, H_3)$, then

$$\{\lambda(x_1, y_1, z_1) + (1-\lambda)(x_2, y_2, z_2) : 1 \le \lambda\} \subset W_J^+(H_1, H_2, H_3),$$

and

$$\{\lambda(x_1, y_1, z_1) + (1-\lambda)(x_2, y_2, z_2) : \lambda \le 0\} \subset W_J^-(H_1, H_2, H_3).$$

Next we show that the converse of Theorem 3.1 holds.

THEOREM 3.3. *Assume that $\mathcal{K}$ is a Krein space with dimension $n \ge 3$ and (H_1, H_2, H_3) is an ordered triple of J-Hermitian operators on $\mathcal{K}$. If the J-joint numerical range $W_J(H_1, H_2, H_3)$ is quasi-convex, then the joint numerical range $W(J, JH_1, JH_2, JH_3)$ is convex.*

Proof. Suppose that $W_J(H_1, H_2, H_3)$ is quasi-convex. Assume that $s(1, x_1, y_1, z_1)$, $t(1, x_2, y_2, z_2) \in W(J, JH_1, JH_2, JH_3)$, $s, t \neq 0$. First consider the case $s, t > 0$, so that consider (x_1, y_1, z_1), $(x_2, y_2, z_2) \in W_J^+(H_1, H_2, H_3)$. Then there exist $\xi, \eta \in \mathbb{C}^n$ such that $\xi^*\xi = 1$, $\eta^*\eta = 1$ satisfying

$$s(1, x_1, y_1, z_1) = (\xi^* J\xi, \xi^* K_1\xi,\ \xi^* K_2\xi,\ \xi^* K_3\xi),$$

$$t(1, x_2, y_2, z_2) = (\eta^* J\eta, \eta^* K_1\eta,\ \eta^* K_2\eta,\ \eta^* K_3\eta).$$

For $(x_3, y_3, z_3) = \lambda(x_1, y_1, z_1) + (1-\lambda)(x_2, y_2, z_2)$, let $\zeta \in \mathbb{C}^n$ be such that $\zeta^*\zeta = 1$ and satisfy the following relation

$$u(1, x_3, y_3, z_3) = (\zeta^* J\zeta, \zeta^* K_1\zeta,\ \zeta^* K_2\zeta,\ \zeta^* K_3\zeta).$$

Clearly,

$$u(1, x_3, y_3, z_3) = \mu s(1, x_1, y_1, z_1) + (1-\mu)t(1, x_2, y_2, z_2),$$

for $\mu = \lambda t/(\lambda t + (1-\lambda)s)$.

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The cases $s, t < 0$ and $s < 0, t > 0$ are treated similarly except for the case $\lambda s + (1-\lambda)t = 0$. Since $W(J, JH_1, JH_2, JH_3)$ is compact, the result follows by using a limiting process. $\square$

Let (K_0, K_1, K_2, K_3) be a 4-tuple of 3×3 Hermitian matrices. The following question arises in a natural way: Is the joint numerical range $W(K_0, K_1, K_2, K_3)$ convex? Following [10], we can provide an example of non convexity. Let

$$K_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $W(K_0, K_1, K_2, K_3)$ is expressed as

$$W(K_0, K_1, K_2, K_3) = \{(t, (t+1)/2\, x, (t+1)/2\, y, (1+t)/2\, z) : -1 \leq t \leq 1, \;\; x^2+y^2+z^2 = 1\},$$

and so $W(K_0, K_1, K_2, K_3)$ lies on the hyper surface

$$x^2 + y^2 + z^2 - \frac{(t+1)^2}{4} = 0.$$

Gutkin, Jonckheere and Karow's [10] Theorems 5.1 and 5.5 provide a sufficient condition for the joint range $W(K_0, K_1, K_2, K_3)$ to be convex. We consider the characteristic equation

$$\det(\lambda I_3 - (tK_0 + xK_1 + yK_2 + zK_3)) = 0$$

for every point $(t, x, y, z) \in \mathbb{R}^4$ on the unit sphere $t^2 + x^2 + y^2 + z^2 = 1$. If the eigenvalues $\lambda_1(t, x, y, z), \lambda_2(t, x, y, z), \lambda_3(t, x, y, z)$ are simple for every point $(t, x, y, z) \in \mathbb{S}^3$, then the additional condition concerning the eigenspace of the first eigenvalue of $tK_0 + xK_1 + yK_2 + zK_3$ in Theorem 5.1 in [10] is automatically satisfied, and so $W(K_0, K_1, K_2, K_3)$ is convex.

We shall provide an example of a 4-tuple of Hermitian matrices satisfying this condition. Let

$$K_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix},$$

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$$K_2 = \begin{bmatrix} 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 0 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0 & i/\sqrt{2} & 0 \\ -i/\sqrt{2} & 0 & i/\sqrt{2} \\ 0 & -i/\sqrt{2} & 0 \end{bmatrix}.$$

Then we have

$$\det(\lambda I_3 - (tK_0 + xK_1 + yK_2 + zK_3)) = \lambda\,(\lambda^2 - (t^2 + x^2 + y^2 + z^2)).$$

Thus, the eigenvalues of $tK_0+xK_1+yK_2+zK_3$ are $\{1, 0, -1\}$ for every point (t, x, y, z) of the unit sphere $\mathbb{S}^3$. We observe that the matrices in this example belong to the basis of the real vector space of 3×3 Hermitian traceless matrices known as Gell-Mann matrices, which play an important role in quark physics. We have many examples of convex numerical ranges $W(JH_0, JH_1, JH_2, JH_3)$ by a small perturbation of Gell-Mann matrices.

REFERENCES

[1] Y.H. Au-Yeung and N.K. Tsing. An extension of the Hausdorff -Toeplitz theorem on the numerical range. *Proc. Amer. Math. Soc.*, 89:215–218, 1983.

[2] N. Bebiano, R. Lemos, J. da Providência, and G. Soares. On the geometry of numerical ranges in spaces with indefinite inner product. *Linear Algebra Appl.*, 399:17–34, 2005.

[3] N. Bebiano, H. Nakazato, A. Nata, and J. da Providência. The boundary of the Krein space tracial numerical range, an algebraic approach and a numerical algorithm. *Ann. Mat. Pura Appl.*, 189:539–551, 2010.

[4] P. Binding and C.K. Li. Joint numerical ranges of Hermitian matrices and simultaneous diagonalization. *Linear Algebra Appl.*, 151:157–168, 1991.

[5] N. Bebiano and J. da Providência. Numerical ramnge of quadratic operators in spaces with an indefinte metric. *Electron. J. Linear Algebra*, 26:141–157, 2013.

[6] N. Bebiano and J. da Providência. Indefinite higher-rank numerical ranges. *Linear Multilinear Algebra*, 60:1009–1026, 2012.

[7] M.-T. Chien and H. Nakazato. Strict convexity of the joint c-numerical range. *Linear Algebra Apl.*, 438:1305–1321, 2013.

[8] W.-S. Cheung and N.-K. Tsing. The C-numerical range of matrices is star-shaped. *Linear Multilinear Algebra*, 41:245–250, 1996.

[9] M.K.H. Fan and A.L. Tits. m-form numerical range and the computation of the structured singular value. *IEEE Trans. Automat. Control*, 33:284–289, 1988.

[10] E. Gutkin, E.A. Jonckheere, and M. Karow. Convexity of the joint numerical range: topological and differential geometrical viewpoints. *Linear Algebra Appl.*, 376:143–171, 2004.

[11] C.K. Li. Some convexity theorems for the generalized numerical ranges. *Linear Multilinear Algebra*, 40:235–240, 1996.

[12] C.K. Li and L. Rodman. Numerical ranges of operators in spaces with an indefinite inner metric. *Proc. Amer. Math. Soc.*, 126:973–982, 1998.

[13] C.K. Li and L. Rodman. H-Joint numerical ranges. *Bull. Aust. Math. Soc.*, 66:105–117, 2002.

[14] C.K. Li, N.K. Tsing, and F. Uhlig. Numerical ranges of an operator on an indefinite inner product space. *Electron. J. Linear Algebra*, 1:1–17, 1996.

[15] P.J. Psarrakos. Numerical range of linear pencils. *Linear Algebra Appl.*, 317:127–141, 2000.

[16] O. Toeplitz. Das algebraische Analogon zu einer Satze von Fejée. *Math. Z.*, 2:187–197, 1918.