

EXACT RELAXATION FOR THE SEMIDEFINITE MATRIX
RANK MINIMIZATION PROBLEM WITH EXTENDED
LYAPUNOV EQUATION CONSTRAINT*

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Abstract. The semidefinite matrix rank minimization, which has a broad range of applications in system control, statistics, network localization, econometrics and so on, is computationally NP-hard in general due to the noncontinuous and non-convex rank function. A natural way to handle this type of problems is to substitute the rank function into some tractable surrogates, most popular ones of which include the convex trace norm and the non-convex Schatten p -norm relaxations with $p \in (0, 1)$. The corresponding exactness of these relaxations have absorbed great attention and interest from researchers both in mathematics and engineering fields. In this paper, a special semidefinite matrix rank minimization problem with the extended Lyapunov equation constraint arising from low-order optimal control is considered and shown to possess the desired exact relaxation properties by exploiting the special structures of the involved linear transformation and by developing some essential properties and features on rank function and the semidefinite matrix cone.

Key words. Semidefinite matrix minimization, Exact relaxation, Schatten p -norm, Extended Lyapunov equation, Multidimensional scaling.

AMS subject classifications. 15B48, 90C22, 90C59, 93B60.

1. Introduction. Semidefinite matrix rank minimization is to minimize the rank of a matrix variable under some affine constraints over the semidefinite matrix cone. The mathematical model is of the form:

$$(P) \quad \min\{\text{rank}(X) : \mathcal{A}(X) = b, X \in \mathcal{S}_+^n\},$$

where $X \in \mathcal{S}^n$ is the matrix variable, $\mathcal{A} : \mathcal{S}^n \rightarrow \mathbb{R}^m$ is a linear transformation and $b \in \mathbb{R}^m$. Here and in the sequel, \mathcal{S}^n denotes the space of all $n \times n$ real symmetric matrices, \mathcal{S}_+^n is the semidefinite matrix cone consisting of all positive semidefinite matrices in \mathcal{S}^n and $\text{rank}(X)$ is the rank of X which is the number of all nonzero eigenvalues of X . Problem (P) has gained plenty of recent attention in both mathematical

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and engineering fields, owing to its wide applications in system control [3, 15, 16, 18], statistics [4, 13, 20], network localization [10], econometrics, signal processing, quantum information, and many others [2].

Mathematically, problem (P) is difficult to solve due to the discontinuity and non-convexity of the objective function. From the viewpoint of computation complexity, it is generally NP-hard since it includes the cardinality minimization as a special case when the matrix variable is restricted to be diagonal [1, 11]. A natural way to make it tractable is to employ some appropriate surrogates for the rank function. Most popular ones among them include the convex nuclear norm (indeed the trace norm for positive semidefinite matrices) heuristic [16], and the non-convex Schatten p -norm relaxation with $p \in (0, 1)$ [10]. Since for any $X \in \mathcal{S}_+^n$, its Schatten p -norm, termed as $\|X\|_p$, can be reduced to $\|X\|_p = \left[\sum_{i=1}^n \lambda_i^p(X) \right]^{1/p}$ with $\lambda_i(X)$ the i -th eigenvalue of X , and its trace norm, denoted by $tr(X)$, is exactly $tr(X) = \sum_{i=1}^n \lambda_i(X)$, we can merge these two relaxation counterparts into the following unified form:

$$(\bar{P}) \quad \min\{\|X\|_p^p : \mathcal{A}(X) = b, X \in \mathcal{S}_+^n\}$$

with $p \in (0, 1]$.

As the extensive and fruitful study on various algorithms solving low rank solutions based on the tractable problem (\bar{P}), a fundamental question arises: under what conditions the solutions of problem (\bar{P}) coincide with the desired minimal rank solutions of problem (P). Stimulated by the exact recovery theory on the compressed sensing (which is indeed the diagonal case in our problem), several strong assumptions are imposed on the involved linear transformation \mathcal{A} , such as the semi-RIP condition [19], the s -semigoodness [12], and even some null space property to ensure the uniqueness of feasible solutions [21]. However, most these conditions are not easy to verify for deterministic linear transformations. This to some extent might hinder the extensive applicability of the relaxation approach, especially for those practical problems in the areas of system control, and positioning and localization. Fortunately, based on some specific structures of the involved linear transformation, together with some inherent properties of the constant in the affine constraints, Mesbahi and Papavassilopoulos [16], and Parrilo [17] showed that the trace norm relaxation succeeded to produce a minimal rank solution if the feasible set takes the form $\left\{ X \in \mathcal{S}^n : X - \sum_{i=1}^k M_i X M_i^\top - Q \in \mathcal{S}_+^n, X \in \mathcal{S}_+^n \right\}$ with $Q \in \mathcal{S}_+^n$. Even though the equivalence of this semidefinite matrix rank minimization and its trace norm relaxation does not hold resulting from the non-uniqueness of minimal rank solutions, it still inspires us to consider some other special minimization problems which can possess the desired exact relaxation with any $p \in (0, 1]$.

In this paper, we concentrate on a special semidefinite matrix rank minimization problem, which is to pursue the minimal rank solutions of the extended Lyapunov equation. Related applications can be found in the literature of linear-quadratic optimal control problems [7, 8, 9, 14]. The corresponding mathematical model can be formulated as

$$(P_0) \quad \begin{array}{ll} \min & \text{rank}(X) \\ \text{s.t.} & AX + XA^\top + WXW^\top + BB^\top = 0, \\ & X \in \mathcal{S}_+^n, \end{array}$$

where $A, W \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are given matrices. The relaxation counterpart is of the following form:

$$(P_p) \quad \begin{array}{ll} \min & \|X\|_p^p \\ \text{s.t.} & AX + XA^\top + WXW^\top + BB^\top = 0, \\ & X \in \mathcal{S}_+^n, \end{array}$$

with $p \in (0, 1]$.

Our aim is to establish the exact relaxation theory for this special semidefinite matrix rank minimization problem. By employing the matrix analysis, together with properties of the rank function and features of the semidefinite matrix cone, we show that problems (P_0) and (P_p) are equivalent and have a common unique solution. These results can be regarded as an important part of the refinement to the exact relaxation theory for general matrix rank minimization, which makes it significant both in theory and in practice.

The organization of this paper is as follows. Some fundamental properties and features of the rank function and semidefinite matrix cone are recalled and developed in Section 2. The main results on exact relaxation are stated in Section 3. Conclusions are drawn in Section 4.

2. Preliminaries. This section is devoted to recalling and developing some fundamental properties and features of the matrix rank function and the semidefinite matrix cone.

LEMMA 2.1 (Spectral Decomposition, [6]). *For any $X \in \mathcal{S}^n$, there exist some orthogonal matrix $P = (v_1 \cdots v_n) \in \mathbb{R}^{n \times n}$ and real vector $\lambda(X) = (\lambda_1(X), \dots, \lambda_n(X))^\top \in \mathbb{R}^n$ such that*

$$X = P \text{Diag}(\lambda(X)) P^\top = \sum_{i=1}^n \lambda_i(X) v_i v_i^\top.$$

This is called the spectral decomposition of X and $\lambda_1(X), \dots, \lambda_n(X)$ are its eigenvalues with the corresponding eigenvectors $v_i \in \mathbb{R}^n$, $i = 1, \dots, n$.

LEMMA 2.2 ([5]). Let $X = [x_{ij}] \in \mathcal{S}_+^n$ with its spectral decomposition $X = \sum_{i=1}^n \lambda_i(X) v_i v_i^\top$, where $(v_1 \cdots v_n)$ is some orthogonal matrix in $\mathbb{R}^{n \times n}$. We have

- (i) $\lambda_i(X) \geq 0$, for any $i = 1, \dots, n$;
- (ii) $x_{ii} \geq 0$, and $x_{ii} x_{jj} \geq x_{ij}^2$, for any $i, j = 1, \dots, n$;
- (iii) $\text{tr}(X) = \sum_{i=1}^n \lambda_i(X) = \sum_{i=1}^n x_{ii}$.

LEMMA 2.3. For any given $X \in \mathbb{R}^{n \times m}$ and $A \in \mathcal{S}_{++}^n$, if $X^\top A X = 0$, then $X = 0$.

Proof. Note that

$$0 = X^\top A X = X^\top A^{1/2} A^{1/2} X = (A^{1/2} X)^\top (A^{1/2} X).$$

Thus, every eigenvalue of $(A^{1/2} X)^\top (A^{1/2} X)$ is zero, which means that all singular value of $A^{1/2} X$ is zero as well. This implies that $A^{1/2} X = 0$. Using the invertibility of $A^{1/2}$, the desired result follows. \square

COROLLARY 2.4. For any $X, Y \in \mathcal{S}_+^n$ with $X \succeq Y$, we have $\text{rank}(X) \geq \text{rank}(Y)$ and $\|X\|_p^p \geq \|Y\|_p^p$ for any $0 < p \leq 1$. Moreover, if $X \neq Y$, then $\|X\|_p^p > \|Y\|_p^p$ for any $0 < p \leq 1$.

Proof. The first part follows directly from Lemma 2.2 (i) and the nondecreasing of the function $f(t) := t^p$ for any $t \geq 0$ and $p \in (0, 1]$. For the moreover part, assume on the contrary that $\|X\|_p^p = \|Y\|_p^p$, that is $\sum_{i=1}^n (\lambda_i(X)^p - \lambda_i(Y)^p) = 0$. It is known from Corollary 7.7.4 in [5] that $\lambda_i(X) \geq \lambda_i(Y)$ for any $i = 1, \dots, n$ when λ_i 's are arranged in the nondecreasing order. Together with the nondecreasing of $f(t)$, we immediately get that $\lambda_i(X) = \lambda_i(Y)$ for any $i = 1, \dots, n$. Thus, $\text{tr}(X) = \text{tr}(Y)$. On the other hand, since $X - Y \succeq 0$ and $X \neq Y$, it follows that $\text{tr}(X) - \text{tr}(Y) = \text{tr}(X - Y) = \langle X - Y, I \rangle > 0$. This comes to a contradiction. Thus, the desired strict inequality holds. \square

LEMMA 2.5. For any given $A \in \mathcal{S}_+^n$ and any of its principal submatrix $A_r \in \mathcal{S}_+^r$ with $r \leq n$, we have $\|A\|_p^p \geq \|A_r\|_p^p$ for any $0 < p \leq 1$.

Proof. Let $\lambda_1(A), \lambda_2(A), \dots, \lambda_n(A)$ be the eigenvalues of A , and $\lambda_1(A_r), \lambda_2(A_r), \dots, \lambda_r(A_r)$ be the ones of A_r , both in a non-increasing order. By definition of the matrix Schatten- p norm, together with the positive semidefiniteness of A and A_r , we know that $\|A\|_p^p = \sum_{i=1}^n \lambda_i(A)^p$, $\|A_r\|_p^p = \sum_{i=1}^r \lambda_i(A_r)^p$. It is known from Corollary 3.1.3 in [6] that for any $i = 1, \dots, r$, $\lambda_i(A_r) \leq \lambda_i(A)$. Combining with the nonde-

creasing of the function $f(t) = t^p$ for any $t \geq 0$ and $0 < p \leq 1$, the desired result follows. \square

It is worth mentioning that all assertions in Corollary 2.4 and Lemma 2.5 can be extended to the case that $p > 1$ via a similar proof.

PROPOSITION 2.6. *For any given $A, W \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ and any $\beta > 0$ satisfying that $A - \beta I$ is invertible, set $M_1 := (A - \beta I)^{-1}(A + \beta I)$, $M_2 := \sqrt{2\beta}(A - \beta I)^{-1}W$ and $H := 2\beta(A - \beta I)^{-1}BB^\top(A - \beta I)^{-\top}$. Then the following three systems are equivalent:*

- (a) $AX + XA^\top + WXW^\top + BB^\top = 0, X \in \mathcal{S}_+^n$;
- (b) $X - \sum_{i=1}^2 M_i X M_i^\top = H, X \in \mathcal{S}_+^n$.
- (c) $X - \sum_{i=1}^2 M_i X M_i^\top - H \in \mathcal{S}_+^n, X \in \mathcal{S}_+^n, \langle X - \sum_{i=1}^2 M_i X M_i^\top - H, X \rangle = 0$.

Proof. The equivalence between (a) and (b) can be obtained by direct calculation, and it is trivial to have (c) if (b) holds. Thus, it remains to showing that if the semidefinite linear complementarity system is consistent at X in (c), then the equality $X - \sum_{i=1}^2 M_i X M_i^\top = H$ is valid. For simplicity, denote $F(X) := X - \sum_{i=1}^2 M_i X M_i^\top$. Note that

$$(2.1) \quad \langle F(X), F(X) - H \rangle = \|F(X) - H\|_F^2 + \langle H, F(X) - H \rangle \geq 0,$$

where the last inequality follows from the semidefiniteness of H and $F(X) - H$ from (c), and the self-duality of \mathcal{S}_+^n . On the other hand,

$$(2.2) \quad \begin{aligned} \langle F(X), F(X) - H \rangle &= \langle F(X), F(X) - H \rangle - \langle X, F(X) - H \rangle \\ &= - \left\langle \sum_{i=1}^2 M_i X M_i^\top, F(X) - H \right\rangle \\ &\leq 0, \end{aligned}$$

where the first equality follows from the complementarity in (c) and the last inequality from the semidefiniteness of $\sum_{i=1}^2 M_i X M_i^\top$. Together with (2.1), it yields that $\langle F(X), F(X) - H \rangle = 0$, which further implies that $\|F(X) - H\|_F^2 = 0$, and hence, $X - \sum_{i=1}^2 M_i X M_i^\top - H = 0$. This completes the proof. \square

3. Main result. This section is dedicated to the exact relaxation theory for the semidefinite matrix rank minimization problem with extended Lyapunov equation constraint. Before we give the main exact relaxation theorem, an important proposition is stated which will serve as an essential preparation for the sequel analysis.

PROPOSITION 3.1. *For any given $A, W \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ satisfying $\mathcal{F} := \{X \in \mathcal{S}_+^n : AX + XA^\top + WXW^\top + BB^\top = 0\} \neq \emptyset$, there exists a unique matrix $X^* \in \mathcal{F}$ such that $X \succeq X^*$ for any $X \in \mathcal{F}$.*

Proof. For simplicity, we use $F(X)$ to denote $X - \sum_{i=1}^2 M_i X M_i^\top$, and $\bar{\mathcal{F}}$ to denote the set $\{X \in \mathcal{S}_+^n : F(X) - H \in \mathcal{S}_+^n\}$, where M_i and H are defined as in Proposition 2.6. It follows from the equivalence in Proposition 2.6 that

$$(3.1) \quad \mathcal{F} = \{X \in \mathcal{S}_+^n : F(X) - H \in \mathcal{S}_+^n, \langle F(X) - H, X \rangle = 0\} \subset \bar{\mathcal{F}}.$$

By employing Lemma II.1 in [16], we know that there exists a unique \bar{X} such that $\bar{X} \in \bar{\mathcal{F}}$ such that

$$(3.2) \quad X \succeq \bar{X}, \quad \forall X \in \bar{\mathcal{F}}.$$

Together with (3.1), we obtain that

$$(3.3) \quad X \succeq \bar{X}, \quad \forall X \in \mathcal{F}.$$

Now we claim that $\bar{X} \in \mathcal{F}$, which is sufficient to show that $\langle F(\bar{X}) - H, \bar{X} \rangle = 0$. Assume on the contrary that $\langle F(\bar{X}) - H, \bar{X} \rangle \neq 0$. By the semidefiniteness of both $F(\bar{X}) - H$ and \bar{X} , we have $\langle F(\bar{X}) - H, \bar{X} \rangle > 0$. Let $\bar{X} = [Q_1 \ Q_2] \begin{bmatrix} \text{Diag}(\lambda(\bar{X})) & 0 \\ 0 & 0 \end{bmatrix} [Q_1 \ Q_2]^\top$

be its spectral decomposition with $\lambda(\bar{X}) = (\lambda_1(\bar{X}), \dots, \lambda_r(\bar{X}))^\top$ and $r = \text{rank}(\bar{X}) > 0$. From (i) in Lemma 2.2, we have $\lambda_i(\bar{X}) > 0$ for any $i = 1, \dots, r$. Set $V := \sum_{i=1}^2 M_i \bar{X} M_i^\top + H$. It follows from the semidefiniteness of \bar{X} and H that $V \in \mathcal{S}_+^n$.

Applying the semidefiniteness of $F(\bar{X}) - H$ and V , we have

$$0 \preceq [Q_1 \ Q_2]^\top (F(\bar{X}) - H) [Q_1 \ Q_2] = \begin{bmatrix} \text{Diag}(\lambda(\bar{X})) - Q_1^\top V Q_1 & -Q_1^\top V Q_2 \\ -Q_2^\top V Q_1 & -Q_2^\top V Q_2 \end{bmatrix},$$

and $Q_2^\top V Q_2 \succeq 0$. Thus, $\text{Diag}(\lambda(\bar{X})) - Q_1^\top V Q_1 \in \mathcal{S}_+^r \setminus \{0\}$, $Q_2^\top V Q_2 \in \mathcal{S}_+^{n-r} \cap (-\mathcal{S}_+^{n-r}) = \{0\}$. Relying on (ii) in Lemma 2.2, we further get $Q_1^\top V Q_2 = 0$. Henceforth,

$$F(\bar{X}) - H = Q_1 (\text{Diag}(\lambda(\bar{X})) - Q_1^\top V Q_1) Q_1^\top.$$

For any $\epsilon > 0$ and any $Z = Q_1 Z_1 Q_1^\top \in \mathcal{S}_+^n$ with $Z_1 \in \mathcal{S}_+^r$, it follows that

$$\begin{aligned} F(\bar{X} - \epsilon Z) - H &= (\bar{X} - \epsilon Z) - \sum_{i=1}^2 M_i (\bar{X} - \epsilon Z) M_i^\top - H \\ &= F(\bar{X}) - H - \epsilon Z + \epsilon \sum_{i=1}^2 M_i Z M_i^\top \\ &\succeq F(\bar{X}) - H - \epsilon Z \\ &= Q_1 (\text{Diag}(\lambda(\bar{X})) - Q_1^\top V Q_1 - \epsilon Z_1) Q_1^\top. \end{aligned}$$

By choosing $Z_1 := \text{Diag}(\lambda(\bar{X})) - Q_1^\top V Q_1 \in \mathcal{S}_+^r \setminus \{0\}$, and $\epsilon \in (0, 1)$, we can verify that

$$F(\bar{X} - \epsilon Z) - H \succeq F(\bar{X}) - H - \epsilon Z = (1 - \epsilon)(F(\bar{X}) - H) \in \mathcal{S}_+^n,$$

$$\bar{X} - \epsilon Z = Q_1 \left((1 - \epsilon)\text{Diag}(\lambda(\bar{X})) + \epsilon Q_1^\top V Q_1 \right) Q_1^\top \in \mathcal{S}_+^n \setminus \{0\}.$$

Thus, $\bar{X} - \epsilon Z \in \bar{F}$, and $\bar{X} - \epsilon Z \neq \bar{X}$. Combining with the fact $\bar{X} \succeq \bar{X} - \epsilon Z$, this comes to a contradiction to (3.2). Therefore, we have proven our claim that $\bar{X} \in \mathcal{F}$. By setting $X^* := \bar{X}$, the desired result follows from (3.3). \square

THEOREM 3.2. *Let $A, M \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. If \mathcal{F} (defined as in Proposition 3.1) is nonempty, then problems (P_0) and (P_p) are equivalent and share a common unique solution X^* with $\text{rank}(X^*) \geq \text{rank}(B)$.*

Proof. It is known from Proposition 3.1 that there exists some unique $X^* \in \mathcal{F}$ such that $X \succeq X^*$ for any $X \in \mathcal{F}$. Therefore, for any $X \in \mathcal{F}$ with $X \neq X^*$, it follows from Lemma 2.4 that $\|X\|_p^p > \|X^*\|_p^p$ for any $p \in (0, 1]$ and $\text{rank}(X) \geq \text{rank}(X^*)$. This indicates that X^* is the unique solution to problem (P_p) and also a solution to problem (P_0) . Now we proceed to show the solution uniqueness of problem (P_0) . Assume on the contrary that there exists some $Y \in \mathcal{F}$ with $Y \neq X^*$ and $\text{rank}(Y) = \text{rank}(X^*)$. Let $X^* = [Q_1 \ Q_2] \begin{bmatrix} \text{Diag}(\lambda(X^*)) & 0 \\ 0 & 0 \end{bmatrix} [Q_1 \ Q_2]^\top$ be its spectral decomposition with $\lambda(X^*) = (\lambda_1(X^*), \dots, \lambda_r(X^*))^\top$ and $r = \text{rank}(X^*)$. Note that $\text{Diag}(\lambda(X^*)) \in \mathcal{S}_{++}^r$ and

$$[Q_1 \ Q_2]^\top (Y - X^*) [Q_1 \ Q_2] = \begin{bmatrix} -\text{Diag}(\lambda(X^*)) + Q_1^\top Y Q_1 & Q_1^\top Y Q_2 \\ Q_2^\top Y Q_1 & Q_2^\top Y Q_2 \end{bmatrix}.$$

This comes to $Q_1^\top Y Q_1 \in \mathcal{S}_{++}^r$. Utilizing the Schur complement theorem [6, p. 100, Exercise 8], we know that

$$\begin{aligned} r &= \text{rank}(Y) = \text{rank}(Q_1^\top Y Q_1) + \text{rank}(M^*) \\ &= r + \text{rank}(M^*), \end{aligned}$$

where $M^* := Q_2^\top Y Q_2 - Q_2^\top Y Q_1 (Q_1^\top Y Q_1)^{-1} Q_1^\top Y Q_2$. It further derives that

$$(3.4) \quad Q_2^\top Y Q_2 - Q_2^\top Y Q_1 (Q_1^\top Y Q_1)^{-1} Q_1^\top Y Q_2 = 0.$$

Choose some sufficiently small $\epsilon > 0$ such that $X^* - \epsilon Q_1 Q_1^\top \in \mathcal{S}_+^n$. By the semidefiniteness of $[Q_1 \ Q_2]^\top (Y - X^* + \epsilon Q_1 Q_1^\top) [Q_1 \ Q_2]$, it follows from the Schur complement theorem that $Q_2^\top Y Q_2 - Q_2^\top Y Q_1 (Q_1^\top Y Q_1 - (\text{Diag}(\lambda(X^*)) - \epsilon I_r))^{-1} Q_1^\top Y Q_2 \succeq 0$. Combining with (3.4), we have

$$(3.5) \quad Q_2^\top Y Q_1 [(Q_1^\top Y Q_1)^{-1} - L] Q_1^\top Y Q_2 \succeq 0.$$

where $L := (Q_1^\top Y Q_1 - (\text{Diag}(\lambda(X^*) - \epsilon I_r))^{-1})$. Noting that $Q_1^\top Y Q_1 \succ L^{-1} \succ 0$, it yields that

$$(3.6) \quad (Q_1^\top Y Q_1)^{-1} - L \in (-S_{++}^r).$$

Thus,

$$-Q_2^\top Y Q_1 [(Q_1^\top Y Q_1)^{-1} - L] Q_1^\top Y Q_2 \succeq 0.$$

Combining with (3.5), we have $Q_2^\top Y Q_1 [(Q_1^\top Y Q_1)^{-1} - L] Q_1^\top Y Q_2 = 0$. Invoking (3.6), it follows from Lemma 2.3 that $Q_1^\top Y Q_2 = 0$. Henceforth, $Y = Q_1 Q_1^\top Y Q_1 Q_1^\top$. By choosing sufficiently small $\delta > 0$, we can get $X_0 := X^* - \delta(Y - X^*) = Q_1((1 - \delta)\text{Diag}(\lambda(X^*)) - \delta(Q_1^\top Y Q_1))Q_1^\top \succeq 0$. Evidently, $X_0 \in \mathcal{F}$ and $X^* \succeq X_0$ with $X^* \neq X_0$. This contradicts to the fact that $X \succeq X^*$ for any $X \in \mathcal{F}$. Here the solution uniqueness is concluded. Observe that $X^* = V + H \succeq H$. It follows from Corollary 2.4 that $\text{rank}(X^*) \geq \text{rank}(H) = \text{rank}(B)$. This completes the proof. \square

It is worth mentioning that the extended Lyapunov equation turns out to be the continuous-time or discrete-time Lyapunov equation if $W = 0$ or $A = -\frac{1}{2}I$. Therefore, the exact relaxation theorem, as discussed in Theorem 3.2, holds for both of these two specific cases. Meanwhile, compared with the existing exact relaxation results on semidefinite matrix rank minimization proposed in [16, 18], we focus on the equality constraints other than inequalities, and the surrogate of the rank function used as the objective function, is more general which includes the convex (when $p = 1$) and non-convex (when $0 < p < 1$) heuristics.

4. Conclusions. Semidefinite matrix rank minimization problems are generally hard to solve due to the discontinuity and nonconvexity of the rank function. In this paper, we have dealt with a special case, the extended Lyapunov equation case from system control. Based on the structures of the involved linear transformation, the properties of matrix rank function and the features of the semidefinite matrix cone, we have established the exact relaxation theory for this special semidefinite matrix rank minimization problems and its convex and non-convex relaxation problems. The proposed results are of importance both in theory and in practice since they can be served as an important refinement for the exact relaxation theory for general matrix rank minimization, and are useful in low-order optimal control problems.

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