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NEW RESULTS ON NONSINGULAR POWER LCM MATRICES*

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Abstract. Let e and n be positive integers and $S = \{x_1, \ldots, x_n\}$ be a set of n distinct positive integers. The $n \times n$ matrix having eth power $[x_i, x_j]^e$ of the least common multiple of x_i and x_j as its (i, j)-entry is called the eth power least common multiple (LCM) matrix on S, denoted by $([S]^e)$. The set S is said to be gcd closed (respectively, lcm closed) if $(x_i, x_j) \in S$ (respectively, $[x_i, x_j] \in S$) for all $1 \le i,\ j \le n$. In 2004, Shaofang Hong showed that the power LCM matrix $([S]^e)$ is nonsingular if S is a gcd-closed set such that each element of S holds no more than two distinct two prime factors. In this paper, this result is improved by showing that if S is a gcd-closed set such that every element of S contains at most two distinct prime factors or is of the form $p^l qr$ with p, q and r being distinct primes and $1 \le l \le 4$ being an integer, then except for the case that $e = 1$ and 270, 520, 810, 1040 $\in S$, the power LCM matrix $([S]^e)$ on S is nonsingular. This gives an evidence to a conjecture of Hong raised in 2002. For the lcm-closed case, similar results are established.

Key words. Gcd-closed set, Lcm-closed set, Greatest-type divisor, Power LCM matrix, Nonsingularity.

AMS subject classifications. 11C20, 11A05, 15B36.

1. Introduction.

Let $S = \{x_1, \ldots, x_n\}$ be a set of n distinct positive integers. For any integer x and y, we use (x, y) and $[x, y]$ to denote their greatest common divisor and least common multiple, respectively. The matrix having (x_i, x_j) (resp., $[x_i, x_j]$) as its (i, j)-entry is called the greatest common divisor (GCD) matrix (resp., least common multiple (LCM) matrix) defined on S, denoted by $((x_i, x_j))$ (resp., $([x_i, x_j])$). A set S is called *factor closed* if all divisors of $x \in S$ are also in S. In 1876, Smith [24] obtained that the determinant of GCD matrix $((x_i, x_j))$ on a factor-closed set S is the product $\prod_{i=1}^{n} \varphi(x_i)$, where φ is Euler's totient function and the determinant

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of LCM matrix $([x_i, x_j])$ on a factor-closed set S is the product $\prod_{i=1}^{n} \varphi(x_i)\pi(x_i)$ with π being the multiplicative function and being defined for the prime power p^r by $\pi(p^r) = -p$. Since then this topic has received a lot of attention from many authors and particularly became extremely active in the past decades (see, for example, [1]–[23] and [25]–[30]).

In [2], Beslin and Ligh generalized Smith's result on a gcd-closed set S (i.e., $(x_i, x_j) \in S$ for all integers i and j with $1 \le i, j \le n$) by showing that $\det((x_i, x_j)) = \prod_{k=1}^{n} \alpha_k$, where $\alpha_k = \sum_{\substack{d|x_k, d\nmid x_t \\ x_t<x_k}} \varphi(d)$. In [3], Bourque and Ligh proved that the LCM matrix $([x_i, x_j])$ on a gcd-closed set S is $\prod_{k=1}^{n} x_k^2\beta_k$ with $\beta_k = \sum_{\substack{d|x_k, d\nmid x_t \\ x_t<x_k}} g(d)$, where the arithmetical function g is defined by $g(m) = \frac{1}{m} \sum_{d|m} d\mu(d)$ and μ is the Möbius function. In [11], Hong extended Beslin and Ligh's result by showing that the determinant of power LCM matrix $([x_i, x_j]^e)$ defined on a gcd-closed set S is given as

$$\det([x_i, x_j]^e) = \prod_{k=1}^{n} x_k^{2e} \alpha_{e,k}, \tag{1.1}$$

where

$$\alpha_{e,k} = \alpha_{e,k}(x_1, \dots, x_k) = \sum_{\substack{d|x_k, d\nmid x_t \\ x_t<x_k}} (\frac{1}{\zeta_e} * \mu)(d) \tag{1.2}$$

and ζ_e is the arithmetical function defined by $\zeta_e(x) = x^e$. Clearly, $\alpha_{1,k} = \alpha_k$.

Nonsingularity is an important topic in the field of power GCD matrices and power LCM matrices. From Smith's result [24], one can easily deduce that the GCD matrix $((x_i, x_j))$ and LCM matrix $([x_i, x_j])$ defined on a factor-closed set S are always nonsingularity. It is known that the GCD matrix $((x_i, x_j))$ on any gcd-closed set S is nonsingular (see, for instance, Theorem 3 of [1]). In 1992, Bourque and Ligh [3] conjectured that the LCM matrix $([x_i, x_j])$ on any gcd-closed set is nonsingular. Haukkanen et al. [8] gave a counterexample to this conjecture when $n = 9$. By introducing the concept of greatest-type divisor, Hong [9] obtained a great reduced formula for the determinant of the LCM matrix $([x_i, x_j])$ on any gcd-closed set S and then he proved that the Bourque-Ligh conjecture is true when $n \le 7$. That is, the LCM matrix $([x_i, x_j])$ on any gcd-closed set $S = \{x_1, \dots, x_n\}$ is nonsingular if $n \le 7$. However, for $n \ge 8$, Hong [9] showed that there exist gcd-closed sets $S = \{x_1, \dots, x_n\}$ such that $\alpha_n = 0$. Therefore the Bourque-Ligh conjecture is not true when $n \ge 8$. Let e be a given positive integer. By [4], we know that the eth power GCD matrix $((x_i, x_j)^e)$ on any gcd-closed set S is nonsingular. By [9], we see that the LCM matrix

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$([x_i, x_j])$ on any gcd-closed set S is not always nonsingular. However, it is still unclear whether the eth power LCM matrix $([x_i, x_j]^e)$ on any gcd-closed set S is nonsingular or not with $e \geq 2$ being a positive integer. Hong [11] proposed the following conjecture which answered this problem.

Conjecture. [11] Let e be a given positive integer. Then there exists a positive integer $k(e)$, depending only on e, such that if $n \leq k(e)$, then the power LCM matrix $([x_i, x_j]^e)$ on any gcd-closed set $S = \{x_1, \ldots, x_n\}$ is nonsingular. But for $n \geq k(e)+1$, there exists a gcd-closed set $S = \{x_1, \ldots, x_n\}$ such that the power LCM matrix $([x_i, x_j]^e)$ on S is singular.

For any integer $x > 1$, $\omega(x)$ denotes the number of distinct prime factors of x and $\omega(1) = 0$. Regarding the above conjecture, Hong showed the following interesting result.

Theorem 1.1. [14] *Let $e \geq 1$ be an integer and S be a gcd-closed set with $\max_{x \in S}\{\omega(x)\} \leq 2$. Then the power LCM matrix $([x_i, x_j]^e)$ on S is nonsingular.*

Furthermore, Hong, Shum and Sun [20] considered the case $\max_{x \in S}\{\omega(x)\} = 3$, i.e., $pqr, p^2qr, p^3qr \in S$. They proved that the following result is true.

Theorem 1.2. [20] *Let $S = \{x_1, \ldots, x_n\}$ be a gcd-closed set satisfying every element is of the form pqr, or p^2qr, or p^3qr, where p, q, r are distinct primes. If either $e = 1$ and $270, 520 \notin S$ or $e \geq 2$, then the power LCM matrix $([x_i, x_j]^e)$ on S is nonsingular.*

In this paper, our main goal is to continue to study the nonsingularity of the power LCM matrices. We consider the next case $p^4qr \in S$. Incorporated with the above Theorems 1.1 and 1.2, we have the following improved result.

Theorem 1.3. *Let $e \geq 1$ be an integer and S be a gcd-closed set such that each element x of S satisfies that $\omega(x) \leq 2$ or $x = p^l qr$ with $l \leq 4$ being a positive integer and p, q and r being distinct prime numbers. Then except for the case that $e = 1$ and 270, 520, 810, 1040 $\in S$, the power LCM matrix $([x_i, x_j]^e)$ on S is nonsingular.*

The paper is organized as follows. In Section 2, we present several basic lemmas which are needed for the proof of Theorem 1.3. Then in Section 3, we give the proof of Theorem 1.3. In Section 4, as an application of Theorem 1.3, we establish similar results when S is *lcm closed* (i.e., $[x_i, x_j] \in S$ for all $1 \leq i, j \leq n$).

The present paper depends heavily on Hong's methods developed in his previous papers [9], [14], [15] and [20]. Throughout this paper, we let $S = \{x_1, \ldots, x_n\}$ be a gcd-closed set and let $|A|$ denote the cardinality of any finite set A.

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2. Preliminaries. In this section, we state some known definitions and lemmas which are needed in the proof of Theorem 1.3.

DEFINITION 2.1. [9] Let T be a given positive integers set. For any $a, b \in T$, we say that a is a greatest-type divisor of b in T, if $a \mid b, a < b$ and it can be deduced that $c = a$ from $a \mid c, c \mid b, c < b$ and $c \in T$.

DEFINITION 2.2. [20] Let e and k be positive integers and $Z = \{z_1, \dots, z_k\}$ be a set of k distinct positive integers. Then the function $\beta_{e,k}$ on Z is called a *gcd power function* on Z if $\beta_{e,k}$ is defined by

$$\beta_{e,k}(z_1, \dots, z_k) = \begin{cases} \dfrac{1}{z_1^e}, & \text{if } k = 1, \\ \dfrac{1}{z_k^e} + \displaystyle\sum_{r=1}^{k-1}(-1)^r \sum_{1 \le i_1 < \cdots < i_r \le k-1} \frac{1}{(z_{i_1}, \dots, z_{i_r}, z_k)^e}, & \text{if } k \ge 2, \end{cases}$$

where $(z_{i_1}, \dots, z_{i_r}, z_k)$ denotes the greatest common divisor of $z_{i_1}, \dots, z_{i_r}$ and z_k, respectively.

LEMMA 2.3. [15] *Let $R_k = \{y_{k,1}, y_{k,2}, \dots, y_{k,l_k}\}$, where $y_{k,1} < y_{k,2} < \cdots < y_{k,l_k}$, $l_1 = 0$, $l_2 = 1$, $l_3 = 1$ and $1 \le l_k \le k-2$ for $k \ge 4$, be the set of all the greatest-type divisors of x_k in S. If $\alpha_{e,k}$ is defined as in (1.2), then*

$$\alpha_{e,k} = \beta_{e,l_k+1}(y_{k,1}, y_{k,2}, \dots, y_{k,l_k}, x_k).$$

In the following lemmas, we let $R_k = \{y_1, y_2, \dots, y_m\}$ be the set of all greatest-type divisors of x_k $(1 \le k \le n)$ in S, where $y_1 < y_2 < \cdots < y_m$. Then $R_k \ne \emptyset$ when $k \ge 2$ and $R_1 = \emptyset$. If $m \ge 2$, we define $M^{(m)} = \bigcup_{r=2}^m M_r^{(m)}$, where $M_r^{(m)} = \{(y_{i_1}, \dots, y_{i_r}) | 1 \le i_1 < \cdots < i_r \le m\}$ $(2 \le r \le m)$, and the $(y_{i_1}, \dots, y_{i_r})$ denotes the greatest common divisor of $y_{i_1}, \dots, y_{i_r}$. So, $(y_{i_1}, \dots, y_{i_r}) \in M^{(m)}$ and $|M^{(m)}| \ge 1$.

LEMMA 2.4. [20] *Let $m \le 2$ be a positive integer. Then we have*

$$\beta_{e,m+1}(y_1, \dots, y_m, x_k) \ne 0.$$

LEMMA 2.5. [20] *Let $m \ge 3$ be a positive integer. If $|M^{(m)}| = 1$, then*

$$\beta_{e,m+1}(y_1, \dots, y_m, x_k) \ne 0.$$

LEMMA 2.6. [20] *If $|M^{(3)}| \le 3$, then $\beta_{e,4}(y_1, y_2, y_3, x_k) \ne 0$.*

LEMMA 2.7. [14] *If $x_k = p^l q^f$ $(1 \le k \le n)$, where p and q are distinct primes, l and f are positive integers, then $\alpha_{e,k} \ne 0$.*

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3. Lemmas and proof of Theorem 1.3. Throughout let $S = \{x_1, \ldots, x_n\}$ be a gcd-closed set. Without loss of generality, we may let $1 \le x_1 < \cdots < x_n$. In the following lemmas, we always let $x_k = p^4qr$ $(1 \le k \le n)$ be a given positive integer, where p, q, r are distinct prime numbers. By the remark at the end of [11], we may let $k \ge 8$. Clearly, $S = \{x_1, \ldots, x_k\}$ is a gcd-closed set. Let $R_k = \{y_1, y_2, \ldots, y_m\}$ be the set of all greatest-type divisors of x_k in S, where $y_1 < y_2 < \cdots < y_m$. Then by using Lemma 2.3, one has

$$\alpha_{e,k} = \beta_{e,m+1}(y_1, y_2, \ldots, y_m, x_k). \tag{3.1}$$

Note that $y_i \not| y_j$ for all $1 \le i \ne j \le m$. We observe that

$$R_k \subseteq \{p, p^2, p^3, p^4, q, pq, p^2q, p^3q, p^4q, r, pr, p^2r, p^3r, p^4r, qr, pqr, p^2qr, p^3qr\}.$$

If $m \le 2$, from Lemma 2.4 and (3.1) we obtain that $\alpha_{e,k} \ne 0$. So we need only to treat with the case $m \ge 3$. We assume that $m \ge 3$ in Lemmas 3.1 to 3.5 below.

Lemma 3.1. *If at least one of p, q, r is in R_k, then $\alpha_{e,k} \ne 0$.*

Proof. If $p \in R_k$, then $R_k \subseteq \{q, r, qr, p\}$. Note that since $m \ge 3$, we have $R_k = \{p, q, r\}$. Thus, $|M^{(3)}| = 1$, and consequently, the result follows from Lemma 2.5 and (3.1). Now we suppose that $p \notin R_k$.

If both $q \in R_k$ and $r \in R_k$, then we have $R_k \subseteq \{p^2, p^3, p^4, q, r\}$. Since $m \ge 3$, $R_k = \{p^2, q, r\}$, or $\{p^3, q, r\}$, or $\{p^4, q, r\}$. This leads to $|M^{(3)}| = 1$. Thus, the result follows from Lemma 2.5 and (3.1). If $q \in R_k$ and $r \notin R_k$, then $R_k \subseteq \{p^2, p^3, p^4, q, pr, p^2r, p^3r, p^4r\}$. By $m \ge 3$, we have $R_k = \{p^2, q, pr\}$, or $\{p^3, q, pr\}$, or $\{p^4, q, pr\}$, or $\{p^3, q, p^2r\}$, or $\{p^4, q, p^2r\}$, or $\{p^4, q, p^3r\}$. This leads to $|M^{(3)}| = 2$, and hence, from Lemma 2.6 and (3.1), the result follows immediately. For $q \notin R_k$ and $r \in R_k$, we can show $\alpha_{e,k} \ne 0$ by using the same arguments as in the case $q \in R_k$ and $r \notin R_k$. So Lemma 3.1 is proved. ☐

Lemma 3.2. *If p, q, $r \notin R_k$, and either $p^2 \in R_k$ or $p^3 \in R_k$, then $\alpha_{e,k} \ne 0$.*

Proof. Since $p, q, r \notin R_k$, one has

$$R_k \subseteq \{p^2, p^3, p^4, pq, p^2q, p^3q, p^4q, pr, p^2r, p^3r, p^4r, qr, pqr, p^2qr, p^3qr\}.$$

If $p^2 \in R_k$, then we consider the following two cases:

Case 1. $qr \notin R_k$. In this case, $R_k \subseteq \{p^2, pq, pr, pqr\}$. By $m \ge 3$, we have $R_k = \{p^2, pq, pr\}$, then we can observe $M^{(3)} = \{p\}$. Thus, $|M^{(3)}| = 1$, and consequently, we can obtain $\alpha_{e,k} \ne 0$ by Lemma 2.5 and (3.1).

Case 2. $qr \in R_k$. Then $R_k \subseteq \{p^2, pq, pr, qr\}$. For $m \ge 3$, we have $R_k = \{pq, qr, p^2\}$, or $R_k = \{pr, qr, p^2\}$, or $R_k = \{pq, pr, qr, p^2\}$. For the former two cases,

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we deduce that $|M^{(3)}| = 3$. Then the result follows from Lemma 2.6 and (3.1). For the latter case, $R_k = \{pq, pr, qr, p^2\}$. From (3.1) and [20], we have

$$\alpha_{e,k} = \beta_{e,5}(pq, pr, qr, p^2, p^4qr) < \beta_{e,5}(pq, pr, qr, p^2, p^2qr) < 0.$$

If $p^3 \in R_k$, then we also consider the following two cases:

Case 2.1. $qr \notin R_k$. In this case, $R_k \subseteq \{p^3, pq, p^2q, pr, p^2r, pqr, p^2qr\}$. We can deduce $p^2qr \notin R_k$. Assume $p^2qr \in R_k$, then it is easy to show that $R_k \subseteq \{p^3, p^2qr\}$ and $m \le 2 < 3$. This is clearly a contradiction. If $pqr \notin R_k$, then in this case, $R_k \subseteq \{p^3, pq, p^2q, pr, p^2r\}$. Then $R_k = \{p^3, pq, pr\}$, or $R_k = \{p^3, p^2q, p^2r\}$, or $R_k = \{p^3, pq, p^2r\}$, or $R_k = \{p^3, p^2q, pr\}$. This leads to $|M^{(3)}| \le 2$, and hence, the desired result follows by Lemma 2.6 and (3.1). If $pqr \in R_k$, In this case, $R_k \subseteq \{p^3, p^2q, p^2r, pqr\}$. Then $R_k = \{p^3, p^2q, pqr\}$, or $R_k = \{p^3, p^2r, pqr\}$, or $R_k = \{p^3, p^2q, p^2r, pqr\}$. For the former two cases, we have $|M^{(3)}| = 3$, and hence, by using Lemma 2.6 and (3.1) the desired result follows immediately. For the latter case, $R_k = \{p^3, p^2q, p^2r, pqr\}$, we have

$$\alpha_{e,k} = \beta_{e,5}(p^3, p^2q, p^2r, pqr, p^4qr) < \beta_{e,5}(p^3, p^2q, p^2r, pqr, p^3qr).$$

By using the Hong-Shum-Sun result in [20], we get that $\alpha_{e,k} < 0$.

Case 2.2. $qr \in R_k$. Then $R_k \subseteq \{p^3, pq, p^2q, pr, p^2r, qr\}$. Since $m \ge 3$, we see that $R_k = \{p^iq, qr, p^3\}$, $i \in \{1, 2\}$, or $R_k = \{p^jr, qr, p^3\}$, $j \in \{1, 2\}$, or $R_k = \{p^iq, p^jr, qr, p^3\}$, $i, j \in \{1, 2\}$. For the former two cases, we have $|M^{(3)}| = 3$. So the result follows from Lemma 2.6 and (3.1). For the case $R_k = \{p^iq, p^jr, qr, p^3\}$, with $i, j \in \{1, 2\}$, by Hong et al. [20], we know that $\alpha_{e,k} = \beta_{e,5}(p^iq, p^jr, qr, p^3, p^3qr) \neq 0$. This ends the proof of Lemma 3.2. $\square$

LEMMA 3.3. *Let p, q, r, $qr \notin R_k$ and $p^4 \in R_k$. Then each of the following is true:*

(i) *If either $e \ge 2$ or $x_k \notin \{810, 1040\}$, then $\alpha_{e,k} \neq 0$.*

(ii) *For $x_k = 810$ or 1040, there exists a gcd-closed set $S = \{x_1, \ldots, x_k\}$, where $1 \le x_1 < \cdots < x_{k-1} < x_k$, such that $\alpha_{1,k} = 0$.*

Proof. Since $p^4 \in R_k, qr \notin R_k$, we have

$$R_k \subseteq \{p^4, pq, p^2q, p^3q, pr, p^2r, p^3r, pqr, p^2qr, p^3qr\}.$$

We can deduce $p^3qr \notin R_k$. If $p^3qr \in R_k$, then it is easy to show that $R_k \subseteq \{p^4, p^3qr\}$ and $m \le 2 < 3$. This is a contradiction. If $p^2qr \in R_k$, then $R_k \subseteq \{p^4, p^3q, p^3r, p^2qr\}$. We can easily deduce $R_k = \{p^4, p^3q, p^3r\}$, or $R_k = \{p^4, p^3q, p^2qr\}$, or $R_k = \{p^4, p^3r, p^2qr\}$, or $R_k = \{p^4, p^3q, p^3r, p^2qr\}$. For the former three cases, we

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have $|M^{(3)}| \leq 3$. The desired result follows immediately from Lemma 2.6 and (3.1). For the latter case, $R_k = \{p^4, p^3q, p^3r, p^2qr\}$, we have

$$\begin{aligned}\alpha_{e,k} &= \beta_{e,5}(p^4, p^3q, p^3r, p^2qr, p^4qr)\\ &= \frac{1}{(p^4qr)^e} - \frac{1}{(p^4)^e} - \frac{1}{(p^3q)^e} - \frac{1}{(p^3r)^e} - \frac{1}{(p^2qr)^e} + \frac{2}{(p^3)^e} + \frac{1}{(p^2q)^e} + \frac{1}{(p^2r)^e} - \frac{1}{(p^2)^e}\\ &= \frac{1}{(p^4qr)^e}[1 - (p^e + r^e - p^er^e)(p^e + q^e - p^eq^e)] < 0.\end{aligned}$$

If $pqr \in R_k$, then $R_k \subseteq \{p^4, p^2q, p^3q, p^2r, p^3r, pqr\}$. Hence, one can easily deduce that $R_k = \{p^4, p^iq, pqr\}$, $i \in \{2, 3\}$, or $R_k = \{p^4, p^jr, pqr\}$, $j \in \{2, 3\}$, or $R_k = \{p^4, p^iq, p^jr, pqr\}$, $i, j \in \{2, 3\}$. For the former two cases, we have $|M^{(3)}| = 3$, and hence, the result follows by Lemma 2.6 and (3.1). For the latter case, $R_k = \{p^4, p^iq, p^jr, pqr\}$, $i, j \in \{2, 3\}$.

For $i = j = 3$, we have

$$\alpha_{e,k} = \beta_{e,5}(p^4, p^3q, p^3r, pqr, p^4qr) = \frac{\Delta}{(p^4qr)^e},$$

where

$$\Delta = 1 - q^er^e - p^er^e - p^eq^e - p^{3e} + (2p^eq^er^e + p^{3e}r^e + p^{3e}q^e - p^{3e}q^er^e).$$

If $e \geq 2$, then

$$2p^eq^er^e + p^{3e}r^e + p^{3e}q^e - p^{3e}q^er^e = p^eq^er^e(2 - \frac{p^{2e}}{3}) + p^{3e}r^e(1 - \frac{q^e}{3}) + p^{3e}q^e(1 - \frac{r^e}{3}) < 0.$$

Hence, $\alpha_{e,k} < 0$.

Now let $e = 1$. In this case, $\Delta = 1 - qr - pr - pq - p^3 + (2pqr + p^3r + p^3q - p^3qr)$. Consider the following cases:

(a) $p = 2$. In this case, we have $\Delta = -(5qr - 6q - 6r + 7)$. Since $q \geq 3$, we have $5qr - 6q - 6r + 7 = q(5r - 6) - 6r + 7 \geq 3(5r - 6) - 6r + 7 = 9r - 11 > 0$. Thus, $\Delta < 0$ and $\alpha_k < 0$.

(b) $p \geq 3$. If both $q \neq 2$ and $r \neq 2$, we have $2pqr + p^3r + p^3q - p^3qr = pqr(2 - \frac{p^2}{3}) + p^3r(1 - \frac{q}{3}) + p^3q(1 - \frac{r}{3}) < 0$. We conclude that $\Delta < 0$, and therefore, $\alpha_k < 0$. If $q = 2$, then

$$\Delta = 1 - 2r - 2p + 3pr + p^3 - p^3r = -p[p^2(r - 1) - 3r] + 1 - 2p - 2r.$$

Since $p \geq 3$, $p^2(r - 1) - 3r \geq 9(r - 1) - 3r = 6r - 9 > 0$. Thus, $\Delta < 0$. This shows that $\alpha_k < 0$. For the case $r = 2$, we can also prove that $\alpha_k < 0$.

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For $i = j = 2$, we have

$$\alpha_{e,k} = \beta_{e,5}(p^4, p^2q, p^2r, pqr, p^4qr) = \frac{\Delta}{(p^4qr)^e}$$

with

$$\Delta = 1 - q^e r^e - p^{2e} r^e - p^{2e} q^e - p^{3e} + (2p^{2e} q^e r^e + p^{3e} r^e + p^{3e} q^e - p^{3e} q^e r^e).$$

Assume that $e \geq 2$. Then

$$2p^{2e}q^e r^e + p^{3e} r^e + p^{3e} q^e - p^{3e} q^e r^e = p^{2e} q^e r^e (2 - \frac{p^e}{2}) + p^{3e}[\frac{q^e}{4}(4 - r^e) + \frac{r^e}{4}(4 - q^e)] < 0.$$

Hence, $\alpha_{e,k} < 0$.

Now we return to the case $e = 1$. In this case,

$$\Delta = 1 - qr - p^2r - p^2q - p^3 + (2p^2qr + p^3r + p^3q - p^3qr).$$

Consider the following cases:

(a) $p = 2$. Then $\Delta = 1 - [(q-4)(r-4) - 8]$. Suppose that $(q-4)(r-4) - 8 \neq 1$. Thus, $\alpha_k \neq 0$. If $(q-4)(r-4) - 8 = 1$, then we can get $q - 4 = 1$ and $r - 4 = 9$, or $q - 4 = 9$ and $r - 4 = 1$. Thus, $q = 5$ and $r = 13$, or $q = 13$ and $r = 5$. Let

$$\begin{aligned} S &= \{2, 2^2, 2 \times 5, 2 \times 13, 2^4, 2^2 \times 5, 2^2 \times 13, 2 \times 5 \times 13, 2^4 \times 5 \times 13\} \\ &= \{2, 4, 10, 16, 20, 26, 52, 130, 1040\}. \end{aligned}$$

Clearly, S is gcd closed and $\alpha_9(2, 4, 10, 16, 20, 26, 52, 130, 1040) = 0$.

(b) $p = 3$. We consider $\Delta = -10qr + 18q + 18r - 26$. Thus, $\Delta = 0$ if and only if $q = 2, r = 5$, or $q = 5, r = 2$. If we let $p = 3, q = 2, r = 5$ and $S = \{3, 2 \times 3, 3^2, 3 \times 5, 3^4, 3^2 \times 5, 3^2 \times 2, 3 \times 2 \times 5, 3^4 \times 2 \times 5\} = \{3, 6, 9, 15, 18, 30, 45, 81, 810\}$, then S is a gcd-closed set and $\alpha_k = \alpha_9(3, 6, 9, 15, 18, 30, 45, 81, 810) = 0$. For $p = 3, q = 5, r = 2$, we can reduce the same result. For $q > 5$, $5qr - 9q - 9r + 13 = q(5r - 9) - 9r + 13 > 5(5r - 9) - 9r + 13 = 16r - 32 \geq 0$. It means that $\Delta < 0$ and $\alpha_k < 0$.

(c) $p = 5$. $\Delta = -4(19qr - 25q - 25r + 31)$. We consider $19qr - 25q - 25r + 31 = 0$. If $q = 2$, or $q = 3$, it follows that r is not a prime number. Then we can get $\Delta \neq 0$, and it means $\alpha_k \neq 0$. For $q \geq 7$, $19qr - 25q - 25r + 31 = q(19r - 25) - 25r + 31 \geq 7(19r - 25) - 25r + 31 = 108r - 144 > 0$. Thus, $\Delta < 0$ and $\alpha_k < 0$.

(d) $p \geq 7$. If $q = 2$, then $\Delta = -p^3r + p^3 + 3p^2r - 2p^2 - 2r + 1 = -p^2[p(r-1) - 3r + 2] - 2r + 1$. Since $p \geq 7$, we have $p(r-1) - 3r + 2 \geq 7(r-1) - 3r + 2 = 4r - 5 > 0$. Thus, $\Delta < 0$. Using the same arguments to the case $r = 2$, we can get that $\Delta < 0$. For $q \geq 3$ and $r \geq 3$, we have

$$2p^2qr + p^3r + p^3q - p^3qr = p^2qr(2 - \frac{p}{3}) + p^3[\frac{q}{3}(3 - r) + \frac{r}{3}(3 - q)] < 0.$$

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Thus, $\Delta < 0$. Hence, we conclude that $\Delta < 0$, and therefore, $\alpha_k < 0$.

For $i = 2, j = 3$, we have

$$\alpha_{e,k} = \beta_{e,5}(p^4, p^2q, p^3r, pqr, p^4qr) = \frac{\Delta}{(p^4qr)^e},$$

where $\Delta = 1 - q^er^e - p^{2e}r^e - p^eq^e - p^{3e} + (p^{2e}q^er^e + p^eq^er^e + p^{3e}r^e + p^{3e}q^e - p^{3e}q^er^e)$.

It is easy to see that $\Delta < 0$ when $e \geq 2$. Now let $e = 1$, then $\Delta = 1 - qr - p^2r - pq - p^3 + (p^2qr + pqr + p^3r + p^3q - p^3qr)$. Consider the following three cases:

(a') $p = 2$. In this case, $\Delta = -(3qr - 6q - 4r + 7) = -((3q - 4)(r - 2) - 1) < 0$. It means that $\alpha_k < 0$.

(b') $p = 3$. $\Delta = -26qr + 24q + 18r - 26$. We have $13qr - 12q - 9r + 13 = q(13r - 12) - 9r + 13 \geq 2(13r - 12) - 9r + 13 = 17r - 11 > 0$. Thus, $\Delta < 0$ and $\alpha_k < 0$.

(c') $p \geq 5$. In this case, we have

$$p^2qr + pqr + p^3r + p^3q - p^3qr = pqr(p + 1 - \frac{p^2}{3}) + p^3[\frac{q}{3}(3 - r) + \frac{r}{3}(3 - q)].$$

Clearly, $p + 1 - \frac{p^2}{3} = \frac{1}{3}[-(p - \frac{3}{2})^2 + \frac{21}{4}] < 0$. If both $q \neq 2$ and $r \neq 2$, then $p^2qr + pqr + p^3r + p^3q - p^3qr < 0$. It leads to $\Delta < 0$.

If $q = 2$, then $\Delta = -p(p^2r - p^2 - pr - 2r) - 2r - 2p - 1 = -p[p(p(r-1) - r) - 2r] - 2r - 2p - 1$. We have $p\big(p(r - 1) - r\big) - 2r \geq 5(4r - 5) - 2r = 18r - 25 > 0$. Then $\Delta < 0$. If $r = 2$, then $\Delta = -p[p(p(q - 1) - 2q) - q] - 2p^2 - 2q + 1$. Since $p(q-1)-2q \geq 5q-5-2q = 3q-5 > 0$, we observe that $p[p(q-1)-2q]-q \geq 14q-25 > 0$. Then we conclude that $\Delta < 0$ and $\alpha_k < 0$.

For $i = 3, j = 2$, we have $\alpha_{e,k} = \beta_{e,5}(p^4, p^3q, p^2r, pqr, p^4qr)$. Using the same arguments, one has $\alpha_{e,k} < 0$.

Finally, if $pqr, p^2qr, p^3qr \notin R_k$, we have $R_k \subseteq \{p^4, pq, p^2q, p^3q, pr, p^2r, p^3r\}$. At this moment, $R_k = \{p^4, p^iq, p^jr\}$, $i, j \in \{1, 2, 3\}$. It can be confirmed that $|M^{(3)}| \leq 2$. Then from Lemma 2.6 and (3.1), we can easily deduce that $\alpha_k \neq 0$. This ends the proof of Lemma 3.3. ∎

Lemma 3.4. *If $p, q, r \notin R_k$ and p^4, $qr \in R_k$, then $\alpha_{e,k} \neq 0$.*

Proof. Since $p^4 \in R_k$ and $qr \in R_k$, we have $R_k \subseteq \{p^4, pq, p^2q, p^3q, pr, p^2r, p^3r, qr\}$. Thus, we can obtain that $R_k = \{p^iq, qr, p^4\}$, $i \in \{1, 2, 3\}$, or $R_k = \{p^jr, qr, p^4\}$, $j \in \{1, 2, 3\}$, or $R_k = \{p^iq, p^jr, qr, p^4\}$, $i, j \in \{1, 2, 3\}$. In the former two cases, we deduce that $|M^{(3)}| = 3$. Then the desired result follows from Lemma 2.6 and (3.1).

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For the latter case, $R_k = \{p^i q, p^j r, qr, p^4\}$, $i, j \in \{1, 2, 3\}$. Therefore we have

$$\begin{aligned}\alpha_{e,k} =& \beta_{e,5}(p^4, qr, p^i q, p^j r, p^4 qr)\\ =& \frac{1}{(p^4 qr)^e} - \frac{1}{(p^4)^e} - \frac{1}{q^e r^e} - \frac{1}{(p^i q)^e} - \frac{1}{(p^j r)^e} + \frac{1}{p^{ie}} + \frac{1}{p^{je}} + \frac{1}{q^e} + \frac{1}{r^e} - 1\\ =& \frac{\Delta}{(p^4 qr)^e},\end{aligned}$$

where $\Delta = 1 - q^e r^e - p^{4e} - p^{(4-i)e} r^e - p^{(4-j)e} q^e + p^{(4-i)e} q^e r^e + p^{(4-j)e} q^e r^e + p^{4e} r^e + p^{4e} q^e - p^{4e} q^e r^e$.

For $i = 1$, $j = 1$, we have

$$\Delta = (1 - p^e)[(1 + p^e + p^{2e} + p^{3e}) + p^{3e}(\frac{q^e r^e}{2} - (q^e + r^e)) + q^e r^e (p^e(\frac{p^{2e}}{2} - p^e - 1) - 1)].$$

It is easy to see that $\alpha_{e,k} < 0$ when $e \geq 2$. Now we return to the case $e = 1$. Let $A = 1 + p + p^2 + p^3 + p^3(\frac{qr}{2} - (q + r)) + qr(p(\frac{p^2}{2} - p - 1) - 1)$. We consider the following cases.

(a) If $p = 2$, then $A = (q - 8)(r - 8) - 49$. We have $A = 0$ if and only if $q - 8 = 1$ and $r - 8 = 49$, or $q - 8 = 49$ and $r - 8 = 1$, or $q - 8 = 7$ and $r - 8 = 7$. For q and r are distinct primes, clearly this is a contradiction. Thus, $A \neq 0$, hence $\alpha_k \neq 0$.

(b) If $p \geq 3$, we have $p(\frac{p^2}{2} - p - 1) - 1 = p(\frac{(p-1)^2 - 3}{2}) - 1 > 0$. Now we return to $\frac{qr}{2} - (q+r)$. Suppose that $(q-2)(r-2) - 4 \geq 0$. Then $\frac{qr}{2} - (q+r) = \frac{1}{2}((q-2)(r-2) - 4) \geq 0$. It leads to $\alpha_k < 0$. If $(q-2)(r-2) - 4 < 0$, we have $(q-2)(r-2) = 0, 1, 2, 3$. For q and r are distinct primes, we can obtain $(q-2)(r-2) = 0$ or 3. When $(q-2)(r-2) = 3$, we have $q = 3$ and $r = 5$, or $q = 5$ and $r = 3$. Then

$$A = 8p^3 - 14p^2 - 14p - 14 = p(8p^2 - 14p - 14) - 14 \geq 3 \times (8 \times 3^2 - 14 \times 3 - 14) - 14 > 0.$$

It leads to $\alpha_k < 0$. When $(q - 2)(r - 2) = 0$, we can solve the equation and get $q = 2$, or $r = 2$. If $q = 2$, $A = (r - 1)p^3 + (1 - 2r)p^2 + (1 - 2r)p + (1 - 2r) = p[(r - 1)p^2 + (1 - 2r)p + (1 - 2r)] + (1 - 2r)$. If $p = 3$, at the moment, $A = 27(r - 1) + 9(1 - 2r) + 3(1 - 2r) + (1 - 2r) = r - 14 \neq 0$. Thus, $\alpha_k \neq 0$. If $p \geq 5$, define the function $h(x) := (r-1)x^2 + (1-2r)x + (1-2r)$. Then $h^{'}(x) = 2(r-1)x + (1-2r)$. We can obtain that the derivative $h^{'}(x) > 0$ if $x > \frac{2r-1}{2(r-1)}$. It is easy to see $5 > \frac{2r-1}{2(r-1)}$ and $h(5) = 13r - 19 > 0$. Hence, $A = ph(p) + (1 - 2r) \geq 5h(5) + (1 - 2r) = 63r - 94 > 0$. So $\alpha_k < 0$.

For $i = 1$, $j = 2$, we have $R_k = \{p^4, qr, pq, p^2 r\}$. Then

$$\alpha_{e,k} = \beta_{e,5}(p^4, qr, pq, p^2 r, p^4 qr) = \frac{\Delta}{(p^4 qr)^e}.$$

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We still have

$$\begin{aligned}\Delta &= 1 - q^e r^e - p^{4e} - p^{3e} r^e - p^{2e} q^e + p^{3e} q^e r^e + p^{2e} q^e r^e + p^{4e} r^e + p^{4e} q^e - p^{4e} q^e r^e \\ &= (1-p^e)[p^e(p^{2e}(1-q^e)(1-r^e) + p^e(1-q^e) + (1-q^e r^e)) + (1-q^e r^e)].\end{aligned}$$

Consider the function $h(x) = (1-q^e)(1-r^e)x^2 - (q^e-1)x + 1 - q^e r^e$. If $x > \frac{1}{2(r^e-1)}$, then $h'(x) > 0$. Assume that $e \geq 2$, we have $p^e \geq 4$ and $h(4) > 0$. This leads to $p^e h(p^e) + 1 - q^e r^e \geq 4h(4) + 1 - q^e r^e = 4[16(1-q^e)(1-r^e) - 4(q^e-1) + 1 - q^e r^e] + 1 - q^e r^e = 59q^e r^e - 80q^e - 64r^e + 85 \geq 59 \times 4q^e + 59 \times 4r^e - 80q^e - 64r^e - 16 \times 59 + 85 = 156q^e + 172r^e - 859 > 0$. This proves that $\Delta < 0$ and $\alpha_{e,k} < 0$.

Now let $e = 1$. If $p = 2$, we consider the function $k(p) = p^3(1-q)(1-r) + p^2(1-q) + p(1-qr) + (1-qr)$. Then $k(2) = 5qr - 12q - 8r + 15$. We can prove $k(2) \neq 0$. Suppose that $k(2) = 0$. If $q = 3$, it is easy to see $7r = 21$. For q, r are distinct prime number, this is clearly a contradiction. Thus, $k(2) \neq 0$. Similarly, by using the same arguments as in the cases $q = 5$, $q = 7$, we can show $k(2) \neq 0$. If $q \geq 11$, $k(2) = 4q(r-3) + r(q-8) + 15 > 0$. Thus, we conclude that $k(2) \neq 0$. It means that $\Delta \neq 0$ if $p = 2$, and therefore, $\alpha_k \neq 0$. If $p \geq 3$, then consider the function $h(x) := (1-q)(1-r)x^2 - (q-1)x + 1 - qr$. We have $h(3) = 8qr - 12q - 9r + 13 \geq 8(2q+2r-4) - 12q - 9r + 13 = 4q + 7r - 19 > 0$. Since $h'(x) > 0$ for $x > \frac{1}{2(r-1)}$, then $ph(p) + 1 - qr \geq 3h(3) + 1 - qr = 23qr - 36q - 27r + 40 \geq 23(2q+2r) - 92 - 36q - 27r + 40 = 10q + 19r - 52 > 0$. Therefore, we obtain $\Delta = (1-p)[ph(p) + 1 - qr] < 0$ and $\alpha_k < 0$.

For $i = 2$, $j = 1$, $R_k = \{p^4, qr, p^2q, pr\}$. We have

$$\alpha_{e,k} = \beta_{e,5}(p^4, qr, p^2q, pr, p^4qr) = \frac{\Delta}{(p^4qr)^e},$$

where

$$\Delta = (1-p^e)[p^e(p^{2e}(1-q^e)(1-r^e) + p^e(1-r^e) + (1-q^e r^e)) + (1-q^e r^e)].$$

By the same arguments as in the case $i = 1$, $j = 2$, we can show that $\alpha_{e,k} \neq 0$.

For $i = 2$, $j = 2$, $R_k = \{p^4, qr, p^2q, p^2r\}$. Then

$$\alpha_{e,k} = \beta_{e,5}(p^4, qr, p^2q, p^2r, p^4qr) = \frac{\Delta}{(p^4qr)^e},$$

where

$$\Delta = (1-p^e)[(p^{3e}(q^e-1)(r^e-1) - p^e q^e r^e) + p^{2e}(q^e r^e - q^e - r^e + 1) + p^e + 1 - q^e r^e].$$

For $\frac{q^e}{q^e-1} \leq 2$, $\frac{r^e}{r^e-1} \leq 2$ and $q \neq r$, we observe $\frac{q^e r^e}{(q^e-1)(r^e-1)} < 2^2 \leq p^{2e}$. Therefore $p^{2e}(q^e-1)(r^e-1) - q^e r^e > 0$, and it leads to $p^{3e}(q^e-1)(r^e-1) - p^e q^e r^e > 0$. Since $p^{2e}(q^e r^e - q^e - r^e + 1) + p^e + 1 - q^e r^e \geq 4(q^e r^e - q^e - r^e + 1) + 2 + 1 - q^e r^e$

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$= 3q^er^e - 4q^e - 4r^e + 7 \geq 6q^e + 6r^e - 12 - 4q^e - 4r^e + 7 = 2q^e + 2r^e - 5 > 0$, we have $\Delta < 0$ and $\alpha_{e,k} < 0$.

For $i = 3$, $j = 3$, $R_k = \{p^4, qr, p^3q, p^3r\}$. In this case, one has

$$\alpha_{e,k} = \beta_{e,5}(p^4, qr, p^3q, p^3r, p^4qr) = \frac{\Delta}{(p^4qr)^e},$$

where $\Delta = (1 - p^e)[(q^er^e + 1 - q^e - r^e)p^e(p^{2e} + p^e + 1) + 1 - q^er^e]$.

We consider $A = (q^er^e + 1 - q^e - r^e)p^e(p^{2e} + p^e + 1) + 1 - q^er^e$. For $p^e \geq 2$ and $q^er^e + 1 - q^e - r^e > 0$, we have $A \geq 14(q^er^e - q^e - r^e + 1) + 1 - q^er^e = 13q^er^e - 14q^e - 14r^e + 15 \geq 12q^e + 12r^e - 37 > 0$. This proves that $\Delta < 0$ and $\alpha_{e,k} < 0$.

For $i = 1$, $j = 3$, $R_k = \{p^4, qr, pq, p^3r\}$. It follows that

$$\alpha_{e,k} = \beta_{e,5}(p^4, qr, pq, p^3r, p^4qr) = \frac{\Delta}{(p^4qr)^e}$$

with $\Delta = (1 - p^e)[(q^e - 1)p^e((r^e - 1)p^{2e} - p^e - 1) + 1 - q^er^e]$.

Assume that $e \geq 2$. Then it is easy to show that $\Delta < 0$, and hence, $\alpha_{e,k} < 0$.

Now we return to $e = 1$. Let $A(p) = (q - 1)p((r - 1)p^2 - p - 1) + 1 - qr$.

If $p = 2$, $A(2) = 7qr - 14q - 8r + 15$. Suppose $A(2) = 0$, this means $7qr - 14q - 8r + 15 = 0$. For $r \neq 2$, then $q = \frac{8r-15}{7r-14} = 1 + \frac{r-1}{7(r-2)}$. Since $0 < \frac{r-1}{7(r-2)} < 1$ for $r > 2$, this contradicts with that q is a prime number. Thus, we can deduce $A(2) \neq 0$, and therefore $\Delta \neq 0$. It leads to $\alpha_{e,k} \neq 0$. If $p \geq 3$, let the function $h(x) = (r-1)x^2 - x - 1$. One has $h'(x) = 2(r-1)x - 1$. It is obvious that $h'(x) > 0$ if $x > \frac{1}{2(r-1)}$. Since $h(3) > 0$, then $A(p) = (q-1)ph(p) + 1 - qr \geq 3(q-1)h(3) + 1 - qr = 3(q-1)(9r-13) + 1 - qr = 26qr - 39q - 27r + 40 \geq 52q + 52r - 104 - 27r - 39q + 40 = 13q + 25r - 64 > 0$. So we have $\Delta < 0$. It leads to $\alpha_k < 0$.

For $i = 3$, $j = 1$, $R_k = \{p^4, qr, p^3q, pr\}$. We have

$$\alpha_{e,k} = \beta_{e,5}(p^4, qr, p^3q, pr, p^4qr) = \frac{\Delta}{(p^4qr)^e},$$

where

$$\Delta = (1 - p^e)[(r^e - 1)p^e((q^e - 1)p^{2e} - p^e - 1) + 1 - q^er^e].$$

We can still prove that $\alpha_{e,k} < 0$ by using similar arguments as $i = 1$, $j = 3$.

For $i = 2$, $j = 3$, $R_k = \{p^4, qr, p^2q, p^3r\}$. In this case, we have

$$\alpha_{e,k} = \beta_{e,5}(p^4, qr, p^2q, p^3r, p^4qr) = \frac{\Delta}{(p^4qr)^e},$$

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where

$$\Delta = (1-p^e)[p^e((q^e r^e + 1 - q^e - r^e)p^{2e} + (q^e r^e + 1 - q^e - r^e)p^e + 1 - q^e) + 1 - q^e r^e].$$

Let $h(x) = (q^e r^e + 1 - q^e - r^e)x^2 + (q^e r^e + 1 - q^e - r^e)x + 1 - q^e$, and thereby $h'(x) = 2(q^e r^e + 1 - q^e - r^e)x + q^e r^e + 1 - q^e - r^e$. It is easy to see that $h'(x) > 0$ if $x > -\frac{1}{2}$. And $p^e \geq 2$ implies $h(p^e) \geq h(2) = 6q^e r^e - 7q^e - 6r^e + 7 \geq 12q^e + 12r^e - 24 - 7q^e - 6r^e + 7 = 5q^e + 6r^e - 17 > 0$. Then $p^e h(p^e) + 1 - q^e r^e \geq 2h(2) + 1 - q^e r^e = 11q^e r^e - 14q^e - 12r^e + 15 \geq 8q^e + 10r^e - 29 > 0$. It follows that $\Delta < 0$ and $\alpha_{e,k} < 0$.

For $i = 3$, $j = 2$, $R_k = \{p^4, qr, p^3q, p^2r\}$. We have

$$\alpha_{e,k} = \beta_{e,5}(p^4, qr, p^3q, p^2r, p^4qr) = \frac{\Delta}{(p^4qr)^e},$$

where

$$\Delta = (1-p^e)[p^e((q^e r^e + 1 - q^e - r^e)p^{2e} + (q^e r^e + 1 - q^e - r^e)p^e + 1 - r^e) + 1 - q^e r^e].$$

By using similar arguments as the case $i = 2$, $j = 3$, the result will be observed. This finishes the proof of Lemma 3.4. $\square$

Lemma 3.5. *If p, q, r, p^2, p^3 and $p^4 \notin R_k$, then $\alpha_{e,k} \neq 0$.*

Proof. The proof of the lemma is rather complicated. We proceed the proof by considering two cases.

Case 1. $qr \notin R_k$. For $p, q, r, p^2, p^3, p^4 \notin R_k$, one has

$$R_k \subseteq \{pq, p^2q, p^3q, p^4q, pr, p^2r, p^3r, p^4r, pqr, p^2qr, p^3qr\}.$$

Assume pqr, p^2qr and $p^3qr \notin R_k$. Then $R_k \subseteq \{pq, p^2q, p^3q, p^4q, pr, p^2r, p^3r, p^4r\}$. Thus, $R_k = \{p^iq, p^jr\}$, i, $j \in \{1, 2, 3, 4\}$. This contradicts $m \geq 3$. Then R_k contains exactly one of pqr, p^2qr and p^3qr.

If $p^3qr \in R_k$, then $R_k = \{p^4q, p^4r, p^3qr\}$. It follows that

$$\alpha_{e,k} = \beta_{e,4}(p^4q, p^4r, p^3qr, p^4qr) = \frac{1}{(p^4qr)^e}(1-p^e)(1-q^e)(1-r^e) < 0.$$

If $p^2qr \in R_k$, then $R_k \subseteq \{p^3q, p^4q, p^3r, p^4r, p^2qr\}$. Thus, $R_k = \{p^iq, p^jr, p^2qr\}$, i, $j \in \{3, 4\}$. In [20], Hong, Shum and Sun proved that $\beta_{e,4}(p^lq, p^gr, pqr, p^3qr) < 0$ for l, $g \in \{2, 3\}$. For i, $j \in \{3, 4\}$, we have

$$\alpha_{e,k} = \beta_{e,4}(p^iq, p^jr, p^2qr, p^4qr) = \frac{1}{p^e}\beta_{e,4}(p^lq, p^gr, pqr, p^3qr) < 0$$

with $l, g \in \{2, 3\}$.

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If $pqr \in R_k$, then $R_k \subseteq \{p^2q, p^3q, p^4q, p^2r, p^3r, p^4r, pqr\}$. In this case, $R_k = \{p^iq, p^jr, pqr\}$, $i, j \in \{2, 3, 4\}$. It follows that

$$\begin{aligned}
\alpha_{e,k} &= \beta_{e,4}(p^iq, p^jr, pqr, p^4qr)\\
&= \frac{1}{(p^4qr)^e} - \frac{1}{(p^iq)^e} - \frac{1}{(p^jr)^e} - \frac{1}{(pqr)^e} + \frac{1}{(pq)^e} + \frac{1}{(pr)^e} + \frac{1}{p^{e\cdot min\{i,j\}}} - \frac{1}{p^e}\\
&\leq (\frac{1}{(p^4qr)^e} - \frac{1}{(p^iq)^e}) - \frac{1}{(p^jr)^e} - \frac{1}{(pqr)^e} + \frac{1}{(pq)^e} + \frac{1}{(pr)^e} + \frac{1}{p^{2e}} - \frac{1}{p^e}\\
&= (\frac{1}{(p^4qr)^e} - \frac{1}{(p^iq)^e}) - \frac{1}{(p^jr)^e} - \frac{1}{(pqr)^e} + \frac{1}{p^e}(\frac{1}{q^e} + \frac{1}{r^e} + \frac{1}{p^e} - 1).
\end{aligned}$$

It is easy to see that $\alpha_{e,k} < 0$ when $e \geq 2$. Let $e = 1$. Then

$$\alpha_k \leq \frac{1}{p^4qr} - \frac{1}{p^iq} - \frac{1}{p^jr} - \frac{1}{pqr} + \frac{1}{p}(\frac{1}{q} + \frac{1}{r} + \frac{1}{p} - 1).$$

If $\{p, q, r\} \neq \{2, 3, 5\}$, then $\frac{1}{q} + \frac{1}{r} + \frac{1}{p} - 1 \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{7} - 1 = -\frac{1}{42}$. This shows that $\alpha_k < 0$. If $\{p, q, r\} = \{2, 3, 5\}$, we have $\frac{1}{q} + \frac{1}{r} + \frac{1}{p} - 1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - 1 = \frac{1}{30}$. At this time, $pqr = 30$ and $-\frac{1}{pqr} + \frac{1}{p}(\frac{1}{q} + \frac{1}{r} + \frac{1}{p} - 1) = -\frac{1}{30} + \frac{1}{30p} < 0$. Hence, $\alpha_k < 0$.

Case 2. $qr \in R_k$. Since $p, q, r, p^2, p^3, p^4 \notin R_k$, we have

$$R_k \subseteq \{pq, p^2q, p^3q, p^4q, pr, p^2r, p^3r, p^4r, qr\}.$$

From $m \geq 3$, one has $R_k = \{p^iq, p^jr, qr\}$, $i, j \in \{1, 2, 3, 4\}$. Thus,

$$\alpha_{e,k} = \frac{1}{(p^4qr)^e} - \frac{1}{(qr)^e} - \frac{1}{(p^iq)^e} - \frac{1}{(p^jr)^e} + \frac{1}{q^e} + \frac{1}{r^e} + \frac{1}{p^{e\cdot\min(i,j)}} - 1. \tag{3.2}$$

For $i, j \in \{1, 2, 3\}$, by (3.1) and the Hong-Shum-Sun theorem [20], we have

$$\alpha_{e,k} = \beta_{e,4}(p^iq, p^jr, qr, p^4qr) < \beta_{e,4}(p^iq, p^jr, qr, p^3qr) < 0.$$

For $i = 1, j = 4$, or $i = 4, j = 1$, by (3.2) we have

$$\begin{aligned}
\alpha_{e,k} &= \frac{1}{(p^4qr)^e} - \frac{1}{(qr)^e} - \frac{1}{(pq)^e} - \frac{1}{(p^4r)^e} + \frac{1}{q^e} + \frac{1}{r^e} + \frac{1}{p^e} - 1\\
&= \frac{1}{(p^4qr)^e}[(1 - p^e)(1 - q^e)(p^{3e}(1 - r^e) + p^{2e} + p + 1)] < 0,
\end{aligned}$$

or

$$\alpha_{e,k} = \frac{1}{(p^4qr)^e}[(1 - p^e)(1 - r^e)(p^{3e}(1 - q^e) + p^{2e} + p + 1)] < 0.$$

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For $i = 2, j = 4$, or $i = 4, j = 2$, by (3.2) we have

$$\begin{aligned}\alpha_{e,k} &= \frac{1}{(p^4qr)^e} - \frac{1}{(qr)^e} - \frac{1}{(p^2q)^e} - \frac{1}{(p^4r)^e} + \frac{1}{q^e} + \frac{1}{r^e} + \frac{1}{p^{2e}} - 1\\ &= \frac{(1-p^e)(1-q^e)(1-r^e)}{(pqr)^e} + \frac{(1-p^e)(1-q^e)(p^{2e}+p^e+1-p^{2e}r^e)}{(p^4qr)^e} < 0,\end{aligned}$$

or

$$\alpha_{e,k} = \frac{(1-p^e)(1-q^e)(1-r^e)}{(pqr)^e} + \frac{(1-p^e)(1-r^e)(p^{2e}+p^e+1-p^{2e}q^e)}{(p^4qr)^e} < 0.$$

For $i = 3, j = 4$, or $i = 4, j = 3$, by (3.2) we have

$$\begin{aligned}\alpha_{e,k} &= \frac{1}{(p^4qr)^e} - \frac{1}{(qr)^e} - \frac{1}{(p^3q)^e} - \frac{1}{(p^4r)^e} + \frac{1}{q^e} + \frac{1}{r^e} + \frac{1}{p^{3e}} - 1\\ &= \frac{(1-p^e)(1-q^e)(1-r^e)}{(pqr)^e} + \frac{(1-p^e)(1-q^e)[1+(1-r^e)(p^e+p^{2e})]}{(p^4qr)^e} < 0,\end{aligned}$$

or

$$\alpha_{e,k} = \frac{(1-p^e)(1-q^e)(1-r^e)}{(pqr)^e} + \frac{(1-p^e)(1-r^e)[1+(1-q^e)(p^e+p^{2e})]}{(p^4qr)^e} < 0.$$

For $i = 4$, $j = 4$, by (3.2) we have

$$\begin{aligned}\alpha_{e,k} &= \frac{1}{(p^4qr)^e} - \frac{1}{(qr)^e} - \frac{1}{(p^4q)^e} - \frac{1}{(p^4r)^e} + \frac{1}{q^e} + \frac{1}{r^e} + \frac{1}{p^{4e}} - 1\\ &= \frac{(p^{4e}-1)(q^e+r^e-q^er^e-1)}{(p^4qr)^e} < 0.\end{aligned}$$

Hence, the proof of Lemma 3.5 is complete. ◻

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let $S = \{x_1, \ldots, x_n\}$ be a gcd-closed set. By [11], we know that the determinant of power LCM matrix $([x_i, x_j]^e)$ defined on S equals the product $\prod_{k=1}^n x_k^{2e}\alpha_{e,k}$ with $\alpha_{e,k}$ being defined as in (1.2). Clearly, if there exists an element $x_k \in S$ $(1 \le k \le n)$ such that $\alpha_{e,k} = 0$, then the power LCM matrix $([x_i, x_j]^e)$ on S is singular. The assumption tells us that each element x_k of S satisfies that $\omega(x_k) \le 2$ or $x_k = p^lqr$ with $l \le 4$ being a positive integer and p, q and r being distinct prime numbers. First Lemma 2.7 gives us that if $x_k \in S$ such that $\omega(x_k) \le 2$, then $\alpha_{e,k} \ne 0$. Now we turn our attention to the case that $x_k = p^lqr$ with $l \le 4$. It is divided into the following two cases.

Case 1. $1 \le l \le 3$. It is easy to see that except for the case that $e = 1$ and 270, 520$\in S$, the power LCM matrix $([x_i, x_j]^e)$ on S is nonsingular by Theorem 1.2.

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Case 2. $l = 4$. Note that we use $R_k = \{y_1, y_2, \ldots, y_m\}$ to denote the set of all the greatest-type divisors of x_k in S. Thus,

$$R_k \subseteq \{p, p^2, p^3, p^4, q, pq, p^2q, p^3q, p^4q, r, pr, p^2r, p^3r, p^4r, qr, pqr, p^2qr, p^3qr\}.$$

If $m \leq 2$, from Lemma 2.4 and (3.1) we can obtain that $\alpha_{e,k} \neq 0$. Then we can only concentrate on the case $m \geq 3$. Consider the following five subcases:

Subcase 2.1. At least one of p, q and r is in R_k. By Lemma 3.1 we have $\alpha_{e,k} \neq 0$.

Subcase 2.2. p, q, $r \notin R_k$, and either $p^2 \in R_k$ or $p^3 \in R_k$. By Lemma 3.2, $\alpha_{e,k} \neq 0$.

Subcase 2.3. p, q, r, $qr \notin R_k$ and $p^4 \in R_k$. If $e \geq 2$ or $x_k \notin \{810, 1040\}$, then Lemma 3.3 tells us that $\alpha_{e,k} \neq 0$. If $e = 1$ and $x_k = 810$ or 1040, then by Lemma 3.3 we know that there exists a gcd-closed set $S = \{x_1, \ldots, x_k\}$, where $1 \leq x_1 < \cdots < x_{k-1} < x_k$, such that $\alpha_{e,k} = 0$.

Subcase 2.4. $p, q, r \notin R_k$ and p^4, $qr \in R_k$. Then $\alpha_{e,k} \neq 0$ by Lemma 3.4.

Subcase 2.5. p, q, r, p^2, p^3 and $p^4 \notin R_k$. Then $\alpha_{e,k} \neq 0$ by Lemma 3.5.
It follows from the above five subcases that if $l = 4$, then except for the case that $e = 1$ and 810, 1040 $\in S$, the power LCM matrix $([x_i, x_j]^e)$ on S is nonsingular.

This completes the proof of Theorem 1.3. $\square$

4. Application. In this section, we give an application of our main result. First we need a known definition.

Definition 4.1. [14] Let $S = \{x_1, \ldots, x_n\}$ be a set of n distinct positive integers. Let $m = \operatorname{lcm}(S)$ denote the least common multiple of all elements in S. Then the reciprocal set of S, denoted by mS^{-1}, is defined by $mS^{-1} = \{\frac{m}{x_1}, \ldots, \frac{m}{x_n}\}$.

Lemma 4.2. [14] *Let $S = \{x_1, \ldots, x_n\}$ be a set of n distinct positive integers. Let e be a real number and $m = \operatorname{lcm}(S)$. Then,*

$$([x_i, x_j]^e) = \frac{1}{m^e} \cdot \operatorname{diag}(x_1^e, \ldots, x_n^e) \cdot \left(\left[\frac{m}{x_i}, \frac{m}{x_j}\right]^e\right) \cdot \operatorname{diag}(x_1^e, \ldots, x_n^e).$$

Lemma 4.3. [20] *Let $S = \{x_1, \ldots, x_n\}$ be a set of n distinct positive integers. Then, S is lcm closed if and only if the reciprocal set $\operatorname{lcm}(S)S^{-1}$ is gcd closed.*

Theorem 4.4. *Let $e \geq 1$ be an integer and S be an lcm-closed set such that each element x of the reciprocal set $\operatorname{lcm}(S)S^{-1}$ satisfies that $\omega(x) \leq 2$ or $x = p^lqr$ with $1 \leq l \leq 4$ being a positive integer and p, q and r being distinct prime numbers. Then*

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except for the case that $e = 1$ *and 270, 520, 810, 1040* $\in \mathrm{lcm}(S)S^{-1}$*, the power LCM matrix* $([x_i, x_j]^e)$ *on* S *is nonsingular.*

Proof. By Lemmas 4.1 to 4.2 and Theorem 1.3, the desired result follows immediately. □

COROLLARY 4.5. *Let* $e \geq 1$ *be an integer and* S *be an odd lcm-closed set such that each element* x *of the reciprocal set* $\mathrm{lcm}(S)S^{-1}$ *satisfies that* $\omega(x) \leq 2$ *or* $x = p^l qr$ *with* $1 \leq l \leq 4$ *being a positive integer and* p*,* q *and* r *being distinct prime numbers. Then the power LCM matrix* $([x_i, x_j]^e)$ *on* S *is nonsingular.*

Proof. This corollary follows immediately from Theorem 4.3. □

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