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LINEAR SPACES AND PRESERVERS OF BOUNDED RANK-TWO PER-SYMMETRIC TRIANGULAR MATRICES*

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Abstract. Let $\mathbb{F}$ be a field and m, n be integers $m, n \geqslant 3$. Let $\mathcal{SM}_n(\mathbb{F})$ and $\mathcal{ST}_n(\mathbb{F})$ denote the linear space of $n \times n$ per-symmetric matrices over $\mathbb{F}$ and the linear space of $n \times n$ per-symmetric triangular matrices over $\mathbb{F}$, respectively. In this note, the structure of spaces of bounded rank-two matrices of $\mathcal{ST}_n(\mathbb{F})$ is determined. Using this structural result, a classification of bounded rank-two linear preservers $\psi : \mathcal{ST}_n(\mathbb{F}) \to \mathcal{SM}_m(\mathbb{F})$, with $\mathbb{F}$ of characteristic not two, is obtained. As a corollary, a complete description of bounded rank-two linear preservers between per-symmetric triangular matrix spaces over a field of characteristic not two is addressed.

Key words. Per-symmetric triangular matrices, Rank, Spaces of bounded rank-two matrices, Bounded rank-two linear preservers.

AMS subject classifications. 15A03, 15A04, 15A86.

1. Introduction. A linear mapping between matrix spaces is said to be rank-k non-increasing (respectively, a rank-k preserver) if it sends rank less than or equal to k matrices (respectively, if it sends rank k matrices) to matrices of the same type. Motivated by the studies of rank-one non-increasing linear mappings and rank-two non-increasing linear mappings on symmetric matrices [2, 5, 10, 11, 13] and rank-one non-increasing linear mappings on triangular matrices [3, 4], we investigate the structure of *bounded rank-two linear preservers* ψ on per-symmetric triangular matrices satisfying the condition

$$1 \leqslant \operatorname{rank} \psi(A) \leqslant 2 \quad \text{whenever} \quad 1 \leqslant \operatorname{rank} A \leqslant 2, \tag{1.1}$$

where $\operatorname{rank} A$ denotes the rank of the matrix A.

It is a known fact that the structure of rank preservers is one of the basic results and useful in the study of linear preserver problems [9, 16]. Many linear preservers problems quite often depend on or can be solved with the help of such mappings. For instance, Minc [15] deduced from rank-one linear preservers the classical theorem

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of Frobenius [6] concerning determinant linear preservers. Watkins [17] classified commutativity linear preservers by using the structure of rank-one linear preservers. In [14], rank-k non-increasing linear mappings were used by Loewy and Pierce to verify the John-Pierce conjecture [7] for certain balanced singular inertia classes. Beasley [1] showed that rank-additivity preserving linear mappings are rank-k non-increasing. For works concerning rank preservers on various matrix spaces, we refer the reader to [16, Chapter 2] and [18, Chapter 2].

Let $\mathbb{F}$ be a field and m, n be positive integers. Let $\mathcal{M}_{m,n}(\mathbb{F})$ denote the linear space of $m \times n$ matrices over $\mathbb{F}$. We abbreviate $\mathcal{M}_{n,n}(\mathbb{F})$ to $\mathcal{M}_n(\mathbb{F})$ and $\mathcal{M}_{1,n}(\mathbb{F})$ to $\mathbb{F}^n$. Given $A \in \mathcal{M}_{m,n}(\mathbb{F})$, let $A^+ := J_n A^T J_m \in \mathcal{M}_{n,m}(\mathbb{F})$, where A^T stands for the transpose of A and J_n is the element of $\mathcal{M}_n(\mathbb{F})$ with ones on the minor diagonal and zeros elsewhere. A matrix $A \in \mathcal{M}_n(\mathbb{F})$ is called *per-symmetric* if it is symmetric around the minor diagonal, i.e., $A^+ = A$. We denote by $\mathcal{SM}_n(\mathbb{F})$ the linear subspace of $\mathcal{M}_n(\mathbb{F})$ consisting of per-symmetric matrices, and $\mathcal{ST}_n(\mathbb{F}) := \mathcal{SM}_n(\mathbb{F}) \cap \mathcal{T}_n(\mathbb{F})$. Here $\mathcal{T}_n(\mathbb{F})$ stands for the linear space of $n \times n$ upper triangular matrices over $\mathbb{F}$. We shall call $\mathcal{SM}_n(\mathbb{F})$ and $\mathcal{ST}_n(\mathbb{F})$ the *per-symmetric matrix space* and the *per-symmetric triangular matrix space*, respectively.

The study of rank-k non-increasing linear mappings led naturally to the investigation of *linear spaces of bounded rank k* (i.e., linear subspaces consisting of matrices of rank at most k) and *k-spaces* (i.e., linear subspaces consisting of the zero matrix and matrices of rank k). In this note, we first give a classification of linear spaces of bounded rank-two per-symmetric matrices of $\mathcal{ST}_n(\mathbb{F})$ over an arbitrary field $\mathbb{F}$. As a corollary, a description of 2-spaces of $\mathcal{ST}_n(\mathbb{F})$ is obtained. We next deduce from the structural result of spaces of bounded rank-two per-symmetric triangular matrices a characterization of bounded rank-two linear preservers from $\mathcal{ST}_n(\mathbb{F})$ into $\mathcal{SM}_m(\mathbb{F})$, with $m, n \geqslant 3$ and $\mathbb{F}$ of characteristic not two. As an immediate consequence, a complete description of bounded rank-two linear preservers between per-symmetric triangular matrix spaces over a field of characteristic not two is addressed.

As a side remark, the structure of rank-one non-increasing linear mappings on triangular matrices is much more complicated than the one of those on symmetric matrices. Some examples of rank-one non-increasing linear mappings and rank-two non-increasing linear mappings on per-symmetric triangular matrices are given at the end of this note to indicate the aptness of condition (1.1) in arriving at our results.

In the sequel, we write $\{f_1, \ldots, f_m\}$ and $\{e_1, \ldots, e_n\}$ for the standard bases of $\mathcal{M}_{m,1}(\mathbb{F})$ and $\mathcal{M}_{n,1}(\mathbb{F})$, respectively, and let $E_{ij} := f_i \cdot e_j^T$ be the matrix unit in $\mathcal{M}_{m,n}(\mathbb{F})$ with one as the (i, j) entry and zero elsewhere. We use $\langle u_1, \ldots, u_r \rangle$ designate the linear span of the vectors $u_1, \ldots, u_r$.

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2. Preliminaries. Let $\mathbb{F}$ be a field and n be an integer such that $n \geqslant 2$. For each $\alpha \in \mathbb{F}$ and each pair of integers i, j satisfying $1 \leqslant i, j \leqslant n$ and $j \neq n+1-i$, we set

$$Z_{ij}^{\alpha} := E_{ij} + E_{ij}^{+} + \alpha E_{i,n+1-i} \in \mathcal{M}_n(\mathbb{F}) \tag{2.1}$$

and write $Z_{ij} = Z_{ij}^{0}$ for short. It is obvious that Z_{ij}^{α} is a per-symmetric triangular matrix for every $1 \leqslant i \leqslant j \leqslant n+1-i$ and $j \neq n+1-i$.

We begin with a result on the decomposition of per-symmetric triangular matrices.

LEMMA 2.1. *Let $\mathbb{F}$ be a field and n be an integer such that $n \geqslant 2$. A nonzero matrix $A \in \mathcal{ST}_n(\mathbb{F})$ is of rank k if and only if there exist an integer $0 \leqslant h \leqslant \frac{k}{2}$, scalars $\alpha_1, \ldots, \alpha_h \in \mathbb{F}$, nonzero scalars $\beta_{2h+1}, \ldots, \beta_k \in \mathbb{F}$, and an invertible matrix $P \in \mathcal{T}_n(\mathbb{F})$ such that*

$$A = P\left(\sum_{i=1}^{h} Z_{s_i\, t_i}^{\alpha_i} + \sum_{i=2h+1}^{k} \beta_i E_{p_i, n+1-p_i}\right) P^{+},$$

where $\{s_1, \ldots, s_h, n+1-t_1, \ldots, n+1-t_h, p_{2h+1}, \ldots, p_k\}$ and $\{t_1, \ldots, t_h, n+1-s_1, \ldots, n+1-s_h, n+1-p_{2h+1}, \ldots, n+1-p_k\}$ are two sets of k distinct positive integers such that $1 \leqslant s_i \leqslant t_i \leqslant n+1-s_i$ and $t_i \neq n+1-s_i$ for $i = 1, \ldots, h$, and $1 \leqslant p_i \leqslant \frac{n+1}{2}$ for $i = 2h+1, \ldots, k$; and $(\alpha_1, \ldots, \alpha_h) \neq 0$ only if $\mathbb{F}$ has characteristic two.

Proof. The proof of sufficiency is immediate. We now consider necessity.

Let $A = (a_{ij}) \in \mathcal{ST}_n(\mathbb{F})$ be a nonzero rank k matrix. We denote by $A_{(i)}$ and $A^{(j)}$ the i-th row and the j-th column of the matrix A, respectively. Let $A^{(j_0)}$ be the first nonzero column from the left of A, and let $a_{i_0\, j_0}$ be the first nonzero entry from the bottom of the column $A^{(j_0)}$. Then $a_{ij_0} = 0$ for every $i_0 + 1 \leqslant i \leqslant n$, and $a_{ij} = 0$ for every $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant j_0 - 1$, and also $1 \leqslant i_0 \leqslant j_0 \leqslant n+1-i_0$ since $A \in \mathcal{ST}_n(\mathbb{F})$. We divide our proof into the following two cases:

Case I: $j_0 = n+1-i_0$. For each $1 \leqslant s \leqslant i_0 - 1$, we apply the following elementary row and column operations on A:

$$A_{(s)} \to A_{(s)} - a_{s\,j_0}\, a_{i_0\,j_0}^{-1} A_{(i_0)} \text{ and } A^{(n+1-s)} \to A^{(n+1-s)} - a_{i_0, n+1-s}\, a_{i_0\,j_0}^{-1} A^{(j_0)}. \tag{2.2}$$

For each $1 \leqslant s \leqslant i_0 - 1$, there exists the elementary matrix $I_n - c_s E_{s\,i_0} \in \mathcal{T}_n(\mathbb{F})$ corresponding to the row operation $A_{(s)} \to A_{(s)} - c_s A_{(i_0)}$, where $c_s = a_{s\,j_0}\, a_{i_0\,j_0}^{-1} \in \mathbb{F}$. Since $A^{+} = A$, we have $a_{i_0, n+1-s} = a_{s\,j_0}$ for every $1 \leqslant s \leqslant i_0 - 1$, and so there exists an invertible matrix $P_1 \in \mathcal{T}_n(\mathbb{F})$ such that

$$P_1 A P_1^{+} = a_{i_0\,j_0} E_{i_0\,j_0} + B \tag{2.3}$$

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for some matrix $B = (b_{ij}) \in \mathcal{ST}_n(\mathbb{F})$ such that $b_{i\,j_0} = 0$ for every $1 \leqslant i \leqslant n$, $b_{i_0\,j} = 0$ for every $1 \leqslant j \leqslant n$, and $b_{ij} = 0$ for every $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant j_0 - 1$.

Case II: $j_0 \neq n+1-i_0$. Without loss of generality, we may assume $a_{i_0\,j_0} = 1 = a_{n+1-j_0,n+1-i_0}$. For each $1 \leqslant s \leqslant i_0 - 1$, we apply the following elementary row and column operations on A:

$$A_{(s)} \to A_{(s)} - a_{s\,j_0} A_{(i_0)} \quad \text{and} \quad A^{(n+1-s)} \to A^{(n+1-s)} - a_{n+1-j_0,n+1-s} A^{(n+1-i_0)},$$

and it is followed by the elementary row and column operations on A:

$$A_{(t)} \to A_{(t)} - a_{t,n+1-i_0} A_{(n+1-j_0)} \quad \text{and} \quad A^{(n+1-t)} \to A^{(n+1-t)} - a_{i_0,n+1-t} A^{(j_0)}$$

for every $1 \leqslant t \leqslant n - j_0$. We note that, for each $1 \leqslant s \leqslant i_0 - 1$ (respectively, for each $1 \leqslant t \leqslant n - j_0$), there exists the elementary matrix $I_n - a_{s\,j_0} E_{s\,i_0}$ (respectively, $I_n - a_{t,n+1-i_0} E_{t,n+1-j_0}$) in $\mathcal{T}_n(\mathbb{F})$ corresponding to the row operation $A_{(s)} \to A_{(s)} - a_{s\,j_0} A_{(i_0)}$ (respectively, $A_{(t)} \to A_{(t)} - a_{t,n+1-i_0} A_{(n+1-j_0)}$). Since $a_{n+1-j_0,n+1-s} = a_{s\,j_0}$ for every $1 \leqslant s \leqslant i_0 - 1$, and $a_{i_0,n+1-t} = a_{t,n+1-i_0}$ for every $1 \leqslant t \leqslant n - j_0$, there exists an invertible matrix $P_1 \in \mathcal{T}_n(\mathbb{F})$ such that

$$P_1 A P_1^+ = Z_{i_0 j_0}^{\alpha_1} + B \tag{2.4}$$

for some scalar $\alpha_1 \in \mathbb{F}$ and matrix $B = (b_{ij}) \in \mathcal{ST}_n(\mathbb{F})$ such that $b_{i\,j_0} = 0$ for every $1 \leqslant i \leqslant n$, $b_{i_0\,j} = 0$ for $1 \leqslant j \leqslant n$, and $b_{ij} = 0$ for every $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant j_0 - 1$.

In view of (2.3) and (2.4), if $B = 0$, then we are done. Suppose that $B \neq 0$. Let $b_{i_1\,j_1}$ be the first nonzero entry from the bottom of the first nonzero column of B counting from the left of the matrix B. Evidently, $j_1 > j_0$, $i_1 \neq i_0$ and $1 \leqslant i_1 \leqslant j_1 \leqslant n+1-i_1$. Since $b_{i\,j_0} = 0$ for all $1 \leqslant i \leqslant n$, $b_{i_0\,j} = 0$ for every $1 \leqslant j \leqslant n$, and $b_{ij} = 0$ for every $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant j_0 - 1$, by applying suitable elementary row and column operations similar to (2.2) when $j_1 = n+1-i_1$ (respectively, similar to (2.4) when $j_1 \neq n+1-i_1$), there exists an invertible matrix $P_2 \in \mathcal{T}_n(\mathbb{F})$ such that

$$P_2 B P_2^+ = a_{i_1\,j_1} E_{i_1\,j_1} + C$$

for some matrix $C \in \mathcal{ST}_n(\mathbb{F})$, and $P_2 E_{i_0\,j_0} P_2^+ = E_{i_0\,j_0}$ (respectively,

$$P_2 B P_2^+ = Z_{i_1\,j_1}^{\alpha_2} + C$$

for some scalar $\alpha_2 \in \mathbb{F}$ and matrix $C \in \mathcal{ST}_n(\mathbb{F})$, and $P_2 Z_{i_0\,j_0}^{\alpha_1} P_2^+ = Z_{i_0\,j_0}^{\alpha_1}$). If $C = 0$, then we are done. Suppose that $C \neq 0$. Since A is of rank k, by repeating a similar argument on C, there exist an integer $0 \leqslant h \leqslant \frac{k}{2}$, scalars $\alpha_1, \ldots, \alpha_h, \beta_{2h+1}, \ldots, \beta_k \in \mathbb{F}$, and an invertible matrix $Q \in \mathcal{T}_n(\mathbb{F})$ such that

$$QAQ^+ = \sum_{i=1}^{h} Z_{s_i\,t_i}^{\alpha_i} + \sum_{i=2h+1}^{k} \beta_i E_{p_i,n+1-p_i}, \tag{2.5}$$

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where $\{s_1,\ldots,s_h,n+1-t_1,\ldots,n+1-t_h,p_{2h+1},\ldots,p_k\}$ and $\{t_1,\ldots,t_h,n+1-s_1,\ldots,n+1-s_h,n+1-p_{2h+1},\ldots,n+1-p_k\}$ are two sets of k distinct positive integers such that $1\leqslant s_i\leqslant t_i\leqslant n+1-s_i$ and $t_i\neq n+1-s_i$ for $i=1,\ldots,h$, and $1\leqslant p_i\leqslant \frac{n+1}{2}$ for $i=2h+1,\ldots,k$

We denote $D=QAQ^+$. If $\mathbb{F}$ is of characteristic not two, then, for each $1\leqslant i\leqslant h$, we further perform the following elementary row and column operations on D:

$$D_{(s_i)}\to D_{(s_i)}-\frac{\alpha_i}{2}D_{(n+1-t_i)}\qquad\text{and}\qquad D^{(n+1-s_i)}\to D^{(n+1-s_i)}-\frac{\alpha_i}{2}D^{(t_i)}$$

to annihilate α_i in $Z^{\alpha_i}_{s_i\,t_i}$ as described in (2.5). Since $s_i<n+1-t_i$ for every $1\leqslant i\leqslant h$, there exists an invertible $P\in\mathcal{T}_n(\mathbb{F})$ such that

$$PAP^+=\sum_{i=1}^{h}Z_{s_i\,t_i}+\sum_{i=2h+1}^{k}\beta_iE_{p_i,n+1-p_i}.\ \square$$

As a corollary of Lemma 2.1, we notice that if $A\in\mathcal{ST}_n(\mathbb{F})$ is of rank bounded by two, then there exists an invertible matrix $P\in\mathcal{T}_n(\mathbb{F})$ such that either

$$A=P(\alpha E_{s,n+1-s}+\beta E_{t,n+1-t})P^+$$

for some $\alpha,\beta\in\mathbb{F}$ and some integers $1\leqslant s<t\leqslant\frac{n+1}{2}$, or

$$A=PZ^{\lambda}_{st}P^+$$

for some integers $1\leqslant s\leqslant t\leqslant n+1-s$ with $t\neq n+1-s$, and some scalar $\lambda\in\mathbb{F}$ with $\lambda\neq 0$ only if $\operatorname{char}\mathbb{F}=2$.

Inspired by this observation, we define

$$u\oslash v:=u\cdot v^+ + v\cdot u^+\qquad\text{and}\qquad u^2:=u\cdot u^+ \tag{2.6}$$

for every $u,v\in\mathcal{M}_{n,1}(\mathbb{F})$, where $u\cdot v^+$ denotes the usual matrix product of $u\in\mathcal{M}_{n,1}(\mathbb{F})$ and $v^+\in\mathbb{F}^n$. It can easily be verified that $(u,v)\mapsto u\oslash v$ is a symmetric bilinear map from $\mathcal{M}_{n,1}(\mathbb{F})\times\mathcal{M}_{n,1}(\mathbb{F})$ into $\mathcal{M}_n(\mathbb{F})$. We also see that

$$e_i\oslash e_j=E_{i,n+1-j}+E^+_{i,n+1-j}\qquad\text{and}\qquad e_i^2=E_{i,n+1-i}$$

for all integers $1\leqslant i,j\leqslant n$. In view of (2.1), we have

$$Z^{\alpha}_{ij}=e_i\oslash e_{n+1-j}+\alpha e_i^2$$

for every $\alpha\in\mathbb{F}$ and $1\leqslant i,j\leqslant n$ with $j\neq n+1-i$. Note that $\{e_i\oslash e_j\,|\,1\leqslant i<j\leqslant n\}\cup\{e_i^2\,|\,1\leqslant i\leqslant n\}$ and $\{e_i\oslash e_j\,|\,1\leqslant i<j\leqslant n+1-i\}\cup\{e_i^2\,|\,1\leqslant i\leqslant\frac{n+1}{2}\}$ are the standard bases of $\mathcal{SM}_n(\mathbb{F})$ and $\mathcal{ST}_n(\mathbb{F})$, respectively.

It follows immediately from (2.6) that the following elementary properties hold and their straightforward proofs are omitted. Let $u,v\in\mathcal{M}_{n,1}(\mathbb{F})$, $a,b,c\in\mathbb{F}$ and $P\in\mathcal{M}_n(\mathbb{F})$. We have

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(P1) $(u \oslash v)^+ = u \oslash v$ and $(u^2)^+ = u^2$,
(P2) $u^2 = 0 \Leftrightarrow u = 0$,
(P3) $u \oslash v = 0 \Leftrightarrow$ either $u = 0$ or $v = 0$ when $\operatorname{char}\mathbb{F} \neq 2$; and $u \oslash v = 0 \Leftrightarrow u, v$ are linearly dependent when $\operatorname{char}\mathbb{F} = 2$,
(P4) $P(u \oslash v)P^+ = (Pu) \oslash (Pv)$ and $P(u^2)P^+ = (Pu)^2$, and
(P5) $\operatorname{rank}(a(u \oslash v) + bu^2 + cv^2) \leqslant 2$; and $\operatorname{rank}(a(u \oslash v) + bu^2 + cv^2) = 2 \Leftrightarrow u, v$ are linearly independent and $a^2 \neq bc$.

Lemma 2.2. *Let $u, v, x, y \in \mathcal{M}_{n,1}(\mathbb{F})$ and $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{F}$.*

(a) *If $a_1 u \oslash v + b_1 u^2 + c_1 v^2 = a_2 x \oslash y + b_2 x^2 + c_2 y^2 \neq 0$ with $a_i^2 \neq b_i c_i$ for $i = 1, 2$, then $\langle u, v\rangle = \langle x, y\rangle$.*
(b) *If $\mathbb{F}$ has characteristic not two, then $u \oslash v = x \oslash y \neq 0$ if and only if there exists a nonzero $a \in \mathbb{F}$ such that either $u = ax$ and $v = a^{-1}y$, or $u = ay$ and $v = a^{-1}x$.*

Proof. (a) By our hypothesis, together with (2.6), we obtain

$$u \cdot (a_1 v^+ + b_1 u^+) + v \cdot (a_1 u^+ + c_1 v^+) = x \cdot (a_2 y^+ + b_2 x^+) + y \cdot (a_2 x^+ + c_2 y^+). \tag{2.7}$$

Since $a_i^2 \neq b_i c_i$ for $i = 1, 2$, we have u, v are linearly independent if and only if x, y are linearly independent. Thus, $\langle u, v\rangle = \langle x, y\rangle$ when u, v are linearly independent. If u, v are linearly dependent, assuming $u, x \neq 0$, then $v = \lambda_1 u$ and $y = \lambda_2 x$ for some scalars $\lambda_1, \lambda_2 \in \mathbb{F}$. By (2.7), we obtain $(2a_1\lambda_1 + b_1 + \lambda_1^2 c_1)u^2 = (2a_2\lambda_2 + b_2 + \lambda_2^2 c_2)x^2 \neq 0$, and so $\langle u\rangle = \langle x\rangle$. We are done.

(b) The proof of sufficiency is straightforward. We consider necessity. First note that u, v, x, y are nonzero and $\langle u, v\rangle = \langle x, y\rangle$ by (a). If u, v are linearly dependent, then $\langle u\rangle = \langle v\rangle = \langle x\rangle = \langle y\rangle$. Let $u = ax$ and $v = by$ for some nonzero scalars $a, b \in \mathbb{F}$. Then $x \oslash y = u \oslash v = ab(x \oslash y)$ implies that $b = a^{-1}$, as desired. Suppose now that u, v are linearly independent. Then x, y are linearly independent and either $\langle x\rangle \neq \langle v\rangle$ or $\langle x\rangle \neq \langle u\rangle$. We consider $\langle x\rangle \neq \langle v\rangle$ as the second case can be verified similarly. Then $x = au + bv$ and $y = cu + dv$ for some $a, b, c, d \in \mathbb{F}$ with $a \neq 0$. Then $u \oslash v = x \oslash y = (ad + bc)u \oslash v + 2acu^2 + 2bdv^2$ leads to $(ad + bc - 1)u \oslash v + 2acu^2 + 2bdv^2 = 0$. Since $u \oslash v$, u^2 and v^2 are linearly independent, we get $ad + bc = 1$ and $ac = 0 = bd$. Since $a \neq 0$, we have $c = 0$ implies that $ad = 1$ and $b = 0$. So $x = au$ and $y = a^{-1}v$. ∎

For each integer $1 \leqslant i \leqslant n$, we denote

$$\mathcal{U}_{i,n} := \left\{ (u_1, \ldots, u_i, 0, \ldots, 0)^T \in \mathcal{M}_{n,1}(\mathbb{F}) \,\middle|\, u_1, \ldots, u_i \in \mathbb{F} \right\}$$

and $\mathcal{U}_{0,n} := \{0\} \subset \mathcal{M}_{n,1}(\mathbb{F})$. When n is clear from the context, $\mathcal{U}_{i,n}$ is abbreviated to $\mathcal{U}_i$.

Lemma 2.3. *Let $u, v \in \mathcal{M}_{n,1}(\mathbb{F})$. Then the following assertions hold.*

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(a) $u^2 \in \mathcal{ST}_n(\mathbb{F})\backslash\{0\}$ *if and only if* $u \in \mathcal{U}_p\backslash\mathcal{U}_{p-1}$ *for some* $1 \leqslant p \leqslant \frac{n+1}{2}$.

(b) $u \oslash v \in \mathcal{ST}_n(\mathbb{F})\backslash\{0\}$ *if and only if either*

- **(i)** *there exist integers* $1 \leqslant p \leqslant n+1-q$ *such that* $u \in \mathcal{U}_p\backslash\mathcal{U}_{p-1}$ *and* $v \in \mathcal{U}_q\backslash\mathcal{U}_{q-1}$, *or*
- **(ii)** *there exists an integer* $\frac{n+1}{2} < q \leqslant n$ *such that* $u, v \in \mathcal{U}_q\backslash\mathcal{U}_{q-1}$ *in which* $v = \alpha u + z$ *for some* $\alpha \in \mathbb{F}\backslash\{0\}$ *and* $z \in \mathcal{U}_p\backslash\mathcal{U}_{p-1}$ *with* $1 \leqslant p \leqslant n+1-q$, *and this case holds only if* $\mathbb{F}$ *has characteristic two.*

Proof. (a) This is an immediate consequence of (2.6).

(b) Sufficiency is clear. We consider necessity. Since $u \oslash v \neq 0$, we argue in two cases:

Case A: If $u \oslash v$ is of rank one, then, by Lemma 2.1, $u \oslash v = \alpha x^2$ for some $\alpha \in \mathbb{F}\backslash\{0\}$ and $x \in \mathcal{U}_p$ with $1 \leqslant p \leqslant \frac{n+1}{2}$. Then $\operatorname{char}\mathbb{F} \neq 2$ and u, v are nonzero linearly dependent vectors such that $\langle u\rangle = \langle x\rangle = \langle v\rangle$. So $u, v \in \mathcal{U}_p$ and statement (i) holds true.

Case B: If $u \oslash v$ is of rank two, then, by Lemma 2.1, we consider two subcases:

Case B-1: $u \oslash v = \alpha x^2 + \beta y^2$ for some $\alpha, \beta \in \mathbb{F}\backslash\{0\}$ and linearly independent vectors $x, y \in \mathcal{U}_p$ with $1 \leqslant p \leqslant \frac{n+1}{2}$. By Lemma 2.2 (a), we have $\langle u, v\rangle = \langle x, y\rangle$. Then $u, v \in \mathcal{U}_p$ and statement (i) holds true.

Case B-2: $u \oslash v = x \oslash y + \lambda x^2$ for some $\lambda \in \mathbb{F}$ and linearly independent vectors $x \in \mathcal{U}_p\backslash\mathcal{U}_{p-1}$, $y \in \mathcal{U}_q\backslash\mathcal{U}_{q-1}$ with $1 \leqslant p \leqslant n+1-q \leqslant n+1-p$ and $p \neq q$. By Lemma 2.2 (a), we have $\langle u, v\rangle = \langle x, y\rangle$. Then

$$u = ax + by \quad \text{and} \quad v = cx + dy \tag{2.8}$$

for some $a, b, c, d \in \mathbb{F}$. We thus have $u \oslash v = (2ac)x^2 + (2bd)y^2 + (ad + bc)x \oslash y$, and hence,

$$(2ac - \lambda^2)x^2 + (2bd)y^2 + (ad + bc - 1)x \oslash y = 0.$$

Since $x^2, y^2, x \oslash y$ are linearly independent, we have $2bd = 0$. We first consider $\operatorname{char}\mathbb{F} \neq 2$. Then $bd = 0$ implies that either $b = 0$ or $d = 0$. It follows from (2.8) that either $u \in \mathcal{U}_p$ or $v \in \mathcal{U}_p$ with $1 \leqslant p \leqslant \frac{n+1}{2}$, and so statement (i) holds true. Next, if $\operatorname{char}\mathbb{F} = 2$, then $u \oslash v = (ad + bc)x \oslash y$. If $q \leqslant \frac{n+1}{2}$, $b = 0$, or $d = 0$, then, by (2.8), statement (i) holds. If $q > \frac{n+1}{2}$ and $b, d \neq 0$, then $1 \leqslant p < \frac{n+1}{2}$, and by (2.8), we have $u, v \in \mathcal{U}_q\backslash\mathcal{U}_{q-1}$ and $y = b^{-1}(u - ax)$. So

$$v = cx + dy = cx + b^{-1}d(u - ax) = \alpha u + z,$$

where $\alpha = b^{-1}d \in \mathbb{F}$ and $z = b^{-1}(ad + bc)x \in \mathcal{U}_p\backslash\mathcal{U}_{p-1}$. It is clear that $\alpha \neq 0$ and u, z are linearly independent vectors. Thus, statement (ii) holds. $\square$

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Let $u \in \mathcal{M}_{n,1}(\mathbb{F})$ and $\mathcal{V}$ be a subset of $\mathcal{M}_{n,1}(\mathbb{F})$. We denote

$$u \oslash \mathcal{V} := \{u \oslash v : v \in \mathcal{V}\}.$$

It is immediate that $u \oslash \mathcal{V}$ is a linear subspace of $\mathcal{SM}_n(\mathbb{F})$ when $\mathcal{V}$ is a linear subspace.

LEMMA 2.4. *Let $x \in \mathcal{M}_{n,1}(\mathbb{F})$ be nonzero. If $x \oslash \mathcal{M}_{n,1}(\mathbb{F})$ contains two linearly independent elements $u \oslash v + \alpha u^2$, $u \oslash w + \beta u^2$ for some $u, v, w \in \mathcal{M}_{n,1}(\mathbb{F})$ and $\alpha, \beta \in \mathbb{F}$, then $\langle u \rangle = \langle x \rangle$.*

Proof. Denote $A = u \oslash v + \alpha u^2$ and $B = u \oslash w + \beta u^2$. Clearly, u, x are nonzero since A, B are linearly independent. It follows from Lemma 2.2 (a) that $x \in \langle u, v\rangle$ and $x \in \langle u, w\rangle$. The result follows immediately when u, w are linearly dependent. Consider now u, w are linearly independent. Suppose that $v \notin \langle u, w\rangle$. Then $x \in \langle u, v\rangle \cap \langle u, w\rangle = \langle u\rangle$ because u, v, w are linearly independent. We next consider $v \in \langle u, w\rangle$. Then $A = a(u \oslash w) + bu^2$ for some scalars $a, b \in \mathbb{F}$. Since A, B are linearly independent, it follows that $0 \neq A - aB \in x \oslash \mathcal{M}_{n,1}(\mathbb{F})$, and thus, $u^2 \in x \oslash \mathcal{M}_{n,1}(\mathbb{F})$. Then $u^2 = x \oslash y$ for some $y \in \mathcal{M}_{n,1}(\mathbb{F})$. Since u^2 is of rank one, we have x, y are linearly dependent. If $\operatorname{char}\mathbb{F} = 2$, then $x \oslash y = 0$ by (P3), and so $u^2 = 0$, an impossibility. We thus have $\operatorname{char}\mathbb{F} \neq 2$ and $u^2 = \lambda x^2$ for some nonzero $\lambda \in \mathbb{F}$. Therefore, $\langle u\rangle = \langle x\rangle$, as required. ☐

Let $u, v, w \in \mathcal{M}_{n,1}(\mathbb{F})$. One sees immediately that u, v, w are linearly independent implies $u \oslash v, v \oslash w, w \oslash u$ are linearly independent. The converse is true if the characteristic of $\mathbb{F}$ is two. It can also be checked that if u, v, w are linearly independent and $\mathbb{F}$ has characteristic two, then each nonzero element in $\langle u \oslash v,\ v \oslash w,\ w \oslash u\rangle$ has rank two. By this observation, we next obtain a result that describes the uniqueness of $\langle u \oslash v,\ v \oslash w,\ w \oslash u\rangle$.

LEMMA 2.5. *Let $\mathbb{F}$ be a field of characteristic two and $u, v, w, x, y, z \in \mathcal{M}_{n,1}(\mathbb{F})$ be vectors such that u, v, w are linearly independent. Then $\langle u \oslash v,\ v \oslash w,\ w \oslash u\rangle = \langle x \oslash y,\ y \oslash z,\ z \oslash x\rangle$ if and only if $\langle u, v, w\rangle = \langle x, y, z\rangle$.*

Proof. We first claim that if $a, b \in \mathcal{M}_{n,1}(\mathbb{F})$ are linearly independent vectors, then

$$a \oslash b \in \langle u \oslash v,\ v \oslash w,\ w \oslash u\rangle \quad \Rightarrow \quad a, b \in \langle u, v, w\rangle\,. \tag{2.9}$$

Note that $a \oslash b = \alpha u \oslash v + \beta v \oslash w + \gamma w \oslash u$ for some $\alpha, \beta, \gamma \in \mathbb{F}$ with $(\alpha, \beta, \gamma) \neq 0$. We consider only for the case $\alpha \neq 0$ as the other cases can be proved similarly. Then $a \oslash b = (u + \beta\alpha^{-1}w) \oslash (\alpha v + \gamma w)$ implies that $\langle a, b\rangle = \left\langle u + \beta\alpha^{-1}w,\ \alpha v + \gamma w\right\rangle$ by Lemma 2.2 (a). We thus have $a, b \in \left\langle u + \beta\alpha^{-1}w,\ \alpha v + \gamma w\right\rangle \subseteq \langle u, v, w\rangle$, as claimed.

If $\langle x \oslash y,\ y \oslash z,\ z \oslash x\rangle = \langle u \oslash v,\ v \oslash w,\ w \oslash u\rangle$, then x, y, z are linearly independent. By (2.9), we have $x, y, z \in \langle u, v, w\rangle$, and so $\langle z, y, z\rangle = \langle u, v, w\rangle$. Conversely, if $\langle x, y, z\rangle = \langle u, v, w\rangle$, then $x \oslash y,\ y \oslash z,\ z \oslash x$ are linearly independent vectors contained

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in $\langle u \oslash v,\, v \oslash w,\, w \oslash u\rangle$. □

LEMMA 2.6. *Let $\mathbb{F}$ be a field, m and n be integers such that $m \geqslant n \geqslant 2$, and $P \in \mathcal{M}_{m,n}(\mathbb{F})$ be a full rank matrix. Then $PAP^+ \in \mathcal{ST}_m(\mathbb{F})$ for every $A \in \mathcal{ST}_n(\mathbb{F})$ if and only if $Pe_i \in \mathcal{U}_{p_i,m}\backslash\mathcal{U}_{p_i-1,m}$ for $i = 1,\ldots,n$ such that $1 \leqslant p_i \leqslant \frac{m+1}{2}$ for every $1 \leqslant i \leqslant \frac{n+1}{2}$, and $p_i \leqslant m+1-p_j$ for every $1 \leqslant i < j \leqslant n+1-i$. In particular, $P \in \mathcal{T}_n(\mathbb{F})$ when $m = n$.*

Proof. Denote $u_i = Pe_i$ for $i = 1,\ldots,n$. So $u_1,\ldots,u_n$ are linearly independent. Let $Pe_i \in \mathcal{U}_{p_i,m}\backslash\mathcal{U}_{p_i-1,m}$ for $i = 1,\ldots,n$. Recall that $\{e_i^2 \mid 1 \leqslant i \leqslant \frac{n+1}{2}\} \cup \{e_i \oslash e_j \mid 1 \leqslant i < j \leqslant n+1-i\}$ is a basis of $\mathcal{ST}_n(\mathbb{F})$. For each $1 \leqslant i \leqslant \frac{n+1}{2}$, by (P4) and Lemma 2.3 (a), we have $P(e_i^2)P^+ = u_i^2 \in \mathcal{ST}_m(\mathbb{F})$ since $p_i \leqslant \frac{m+1}{2}$. Again, by (P4) and Lemma 2.3 (b), $P(e_i \oslash e_j)P^+ = u_i \oslash u_j \in \mathcal{ST}_m(\mathbb{F})$ for every $1 \leqslant i < j \leqslant n+1-i$. This proves sufficiency. For necessity, we argue in the following two cases.

Case I: $m > n$. In view of Lemma 2.3 (a), $u_i^2 = P(e_i^2)P^+ \in \mathcal{ST}_m(\mathbb{F})$ for $1 \leqslant i \leqslant \frac{n+1}{2}$ implies that $1 \leqslant p_i \leqslant \frac{m+1}{2}$ for every $1 \leqslant i \leqslant \frac{n+1}{2}$. On the other hand, by Lemma 2.3 (b), $u_i \oslash u_j = P(e_i \oslash e_j)P^+ \in \mathcal{ST}_m(\mathbb{F})$ and $p_i \leqslant \frac{m+1}{2}$ for $1 \leqslant i < j \leqslant n+1-i$ leads to $p_i \leqslant m+1-p_j$ for every $1 \leqslant i < j \leqslant n+1-i$. This establishes the desired conclusion.

Case II: $m = n$. We shall show that $p_i = i$ for $i = 1,\ldots,n$ by induction on i. To begin with, note that the linear independence of $u_1,\ldots,u_n$ implies that $p_{i_0} = n$ for some $1 \leqslant i_0 \leqslant n$. By the fact that $u_1^2, u_1 \oslash u_{i_0} \in \mathcal{ST}_n(\mathbb{F})$, we conclude that $p_1 = 1$. Suppose that the inductive hypothesis holds, i.e., $p_j = j$ for $j = 1,\ldots,k$ for some $k < n$. We wish to claim that $p_{k+1} = k+1$. Since $u_1,\ldots,u_{k+1}$ are linearly independent, together with our induction hypothesis, we have $k+1 \leqslant p_{k+1} \leqslant n$. Since $u_1,\ldots,u_{n-k}$ are linearly independent, there exists an integer $1 \leqslant i_1 \leqslant n-k$ such that $n-k \leqslant p_{i1} \leqslant n$. Note that $u_{k+1} \oslash u_1,\ldots,u_{k+1} \oslash u_{n-k} \in \mathcal{ST}_n(\mathbb{F})$, and also $u_{k+1}^2 \in \mathcal{ST}_n(\mathbb{F})$ provided that $k+1 \leqslant n-k$. We consider two possibilities.

- Say $i_1 = k+1$. Then $p_{i_1} = p_{k+1}$ and $k+1 = i_1 \leqslant n-k$. We thus have $u_{k+1}^2 \in \mathcal{ST}_n(\mathbb{F})$. Hence, $k+1 \leqslant p_{k+1} \leqslant \frac{n+1}{2}$ and $n-k \leqslant p_{i_1} \leqslant \frac{n+1}{2}$. So, $k \geqslant \frac{n-1}{2}$ and thus $k = \frac{n-1}{2}$, since $k+1 \leqslant n-k$. Hence, $k+1 = \frac{n+1}{2}$. Therefore, $p_{k+1} = k+1$.
- Say $i_1 \neq k+1$. Since $i_1 \leqslant n-k$, we have $u_{k+1} \oslash u_{i_1} \in \mathcal{ST}_n(\mathbb{F})$. Then $k+1 \leqslant \frac{n+1}{2}$ or $i_i \leqslant \frac{n+1}{2}$. To see this, if $k+1 \leqslant \frac{n+1}{2}$, then we are done. Suppose that $k+1 > \frac{n+1}{2}$. Then $i_1 \leqslant n-k$ implies $k+1 \leqslant n+1-i_1$, and so $n+1-i_1 > \frac{n+1}{2}$. We thus have $i_1 < \frac{n+1}{2}$, as desired. In consequence, $p_{k+1} \leqslant \frac{m+1}{2}$ or $p_{i_1} \leqslant \frac{m+1}{2}$. By Lemma 2.3 (b), we have $p_{k+1} \leqslant n+1-p_{i_1}$. Then since $p_{i_1} \geqslant n-k$, we have $n-k \leqslant n+1-p_{k+1}$, and so $p_{k+1} \leqslant k+1$. Together with $p_{k+1} \geqslant k+1$, we conclude that $p_{k+1} = k+1$.

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By induction, we conclude that $Pe_i \in \mathcal{U}_{i,n} \backslash \mathcal{U}_{i-1,n}$ for $i = 1, \ldots, n$. It follows that $P \in \mathcal{T}_n(\mathbb{F})$. □

3. Linear spaces of bounded rank-two matrices. We recall that a linear subspace of a matrix space is a linear space of bounded rank-two matrices provided each matrix in it has rank bounded above by two. In [10], Lim classified linear spaces of bounded rank-two symmetric matrices over an infinite field of characteristic not two. Indeed, by a slight modification in the last paragraph of the proof of [10, Theorem 3, p. 49], the result holds for any field of characteristic not two. More recently, [5, Theorem 2.6] completes the work on characterization of spaces of bounded rank-two symmetric matrices over a field of characteristic two.

In this section, using the structural results of [10, Theorem 3] and [5, Theorem 2.6], we classify spaces of bounded rank-two per-symmetric triangular matrices over an arbitrary field. By treating the symmetricity on the minor diagonal, we can now rephrase [10, Theorem 3] and [5, Theorem 2.6] as follows.

Lemma 3.1. *Let $\mathbb{F}$ be a field and n be an integer such that $n \geqslant 2$. Let $\mathcal{S}$ be a linear subspace of $\mathcal{SM}_n(\mathbb{F})$. Then $\mathcal{S}$ is a linear space of bounded rank-two matrices if and only if one of the following holds:*

(I) $\mathcal{S} \subseteq \langle u^2, v^2, u \oslash v \rangle$ *for some linearly independent vectors $u, v \in \mathcal{M}_{n,1}(\mathbb{F})$.*
(II) $\mathcal{S} \subseteq u \oslash \mathcal{M}_{n,1}(\mathbb{F})$ *for some nonzero $u \in \mathcal{M}_{n,1}(\mathbb{F})$.*
(III) $\mathcal{S} = u \oslash \mathcal{V} + \langle u^2 \rangle$ *for some nonzero $u \in \mathcal{M}_{n,1}(\mathbb{F})$ and some linear subspace $\mathcal{V}$ of $\mathcal{M}_{n,1}(\mathbb{F})$; and $\mathcal{S}$ is of this form only if* $\operatorname{char}\mathbb{F} = 2$. *Here, $+$ denotes the sum of linear subspaces of $\mathcal{SM}_n(\mathbb{F})$.*
(IV) $\mathcal{S} = \langle u \oslash v_1 + \lambda_1 u^2, \ldots, u \oslash v_k + \lambda_k u^2 \rangle$ *for some linearly independent vectors $u, v_1, \ldots, v_k$ in $\mathcal{M}_{n,1}(\mathbb{F})$ and some $\lambda_1, \ldots, \lambda_k \in \mathbb{F}$ with $(\lambda_1, \ldots, \lambda_k) \neq 0$; and $\mathcal{S}$ is of this form only if* $\operatorname{char}\mathbb{F} = 2$.
(V) $\mathcal{S} = \langle u \oslash v, u \oslash w, v \oslash w \rangle$ *for some linearly independent vectors u, v, w in $\mathcal{M}_{n,1}(\mathbb{F})$; and $\mathcal{S}$ is of this form only if* $\operatorname{char}\mathbb{F} = 2$.
(VI) $\mathcal{S} \subseteq \langle u^2 + v^2, u^2 + w^2, (u + v) \oslash (u + w) \rangle$ *for some linearly independent vectors u, v, w in $\mathcal{M}_{n,1}(\mathbb{F})$; and $\mathcal{S}$ is of this form only if $|\mathbb{F}| = 2$.*

Let $\mathbb{F}$ be a field of characteristic not two. As a side remark, we notice from (2.6) that $x \oslash y + \alpha x^2 = x \oslash (y + \frac{\alpha}{2} x)$ for every $x, y \in \mathcal{M}_{n,1}(\mathbb{F})$ and $\alpha \in \mathbb{F}$, and thus, any linear space of bounded rank-two of Form (III) or (IV) in Lemma 3.1 can be simplified to Form (I) or (II) in Lemma 3.1. On the other hand, for any linearly independent vectors $u, v, w \in \mathcal{M}_{n,1}(\mathbb{F})$, $\langle u \oslash v, u \oslash w, v \oslash w \rangle$ contains rank three matrices. By a direct verification, $\operatorname{rank}(u \oslash v + u \oslash w + v \oslash w) = 3$ since

$$\det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = -2 \neq 0.$$

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In consequence, Form (V) in Lemma 3.1 is not a linear space of bounded rank-two when $\operatorname{char}\mathbb{F} \neq 2$.

LEMMA 3.2. *Let $\mathbb{F}$ be a field of characteristic two. Let $\alpha \in \mathbb{F}$ be nonzero and $u, v \in \mathcal{M}_{n,1}(\mathbb{F})$ be linearly independent vectors. Then the following assertions hold.*

(a) $u \oslash v + \alpha u^2 \in \mathcal{ST}_n(\mathbb{F})$ *if and only if* $u \in \mathcal{U}_p \backslash \mathcal{U}_{p-1}$ *and* $v \in \mathcal{U}_q \backslash \mathcal{U}_{q-1}$ *for some integers* $1 \leqslant p \leqslant \frac{n+1}{2}$ *and* $1 \leqslant q \leqslant n+1-p$.

(b) $u^2 + v^2 \in \mathcal{ST}_n(\mathbb{F})$ *if and only if* $u+v \in \mathcal{U}_p \backslash \mathcal{U}_{p-1}$ *and* $u, v \in \mathcal{U}_q \backslash \mathcal{U}_{q-1}$ *for some integers* $1 \leqslant p \leqslant \frac{n+1}{2}$ *and* $1 \leqslant q \leqslant n+1-p$.

Proof. (a) Since $\operatorname{char}\mathbb{F} = 2$, the minor diagonal of $u \oslash v$ is zero. Then $u \oslash v + \alpha u^2 \in \mathcal{ST}_n(\mathbb{F})$ with $\alpha \neq 0$ if and only if $u^2, u \oslash v \in \mathcal{ST}_n(\mathbb{F})$ if and only if $u \in \mathcal{U}_p$ for some integer $1 \leqslant p \leqslant \frac{n+1}{2}$, and $v \in \mathcal{U}_q$ for some integer $1 \leqslant q \leqslant n+1-p$ by Lemma 2.3.

(b) By noting $u^2 + v^2 = (u+v) \oslash v + (u+v)^2$ and $(u+v) \oslash v = (u+v) \oslash u$, the conclusion follows immediately from part (a). ∎

We are now in a position to provide a characterization of spaces of bounded rank-two per-symmetric triangular matrices over an arbitrary field.

THEOREM 3.3. *Let $\mathbb{F}$ be a field and n be an integer such that $n \geqslant 2$. Let $\mathcal{S}$ be a linear subspace of $\mathcal{ST}_n(\mathbb{F})$. Then $\mathcal{S}$ is a linear space of bounded rank-two matrices if and only if one of the following holds:*

(a) $\mathcal{S} \subseteq \langle u^2, v^2, u \oslash v \rangle$ *for some linearly independent vectors* $u, v \in \mathcal{U}_p$ *with* $1 \leqslant p \leqslant \frac{n+1}{2}$.

(b) $\mathcal{S} = u \oslash \mathcal{V}$ *for some nonzero* $u \in \mathcal{U}_p$ *and some linear subspace* $\mathcal{V}$ *of* $\mathcal{U}_q$ *with* $1 \leqslant p \leqslant n+1-q \leqslant n$.

(c) $\mathcal{S} = u \oslash \mathcal{V} + \langle u^2 \rangle$ *for some nonzero* $u \in \mathcal{U}_p$ *with* $1 \leqslant p \leqslant \frac{n+1}{2}$ *and some linear subspace* $\mathcal{V}$ *of* $\mathcal{U}_q$ *with* $1 \leqslant q \leqslant n+1-p \leqslant n$; *and* $\mathcal{S}$ *is of this form only if* $\operatorname{char}\mathbb{F} = 2$.

(d) $\mathcal{S} = \langle u \oslash v_1 + \lambda_1 u^2, \ldots, u \oslash v_k + \lambda_k u^2 \rangle$ *for some scalars* $\lambda_1, \ldots, \lambda_k \in \mathbb{F}$ *with* $(\lambda_1, \ldots, \lambda_k) \neq 0$, *and some linearly independent vectors* $u, v_1, \ldots, v_k$ *such that* $u \in \mathcal{U}_p$, $v_1, \ldots, v_k \in \mathcal{U}_q$ *with* $1 \leqslant p \leqslant \frac{n+1}{2}$ *and* $1 \leqslant q \leqslant n+1-p \leqslant n$; *and* $\mathcal{S}$ *is of this form only if* $\operatorname{char}\mathbb{F} = 2$.

(e) $\mathcal{S} = \langle u \oslash v, u \oslash w, v \oslash w \rangle$ *for some linearly independent vectors* $u \in \mathcal{U}_p$, $v \in \mathcal{U}_q$ *and* $w \in \mathcal{U}_r$ *such that* $1 \leqslant p, q \leqslant n+1-r \leqslant n$ *and* $p \leqslant n+1-q$; *and* $\mathcal{S}$ *is of this form only if* $\operatorname{char}\mathbb{F} = 2$.

(f) *There exist linearly independent vectors* $u, v, w \in \mathcal{M}_{n,1}(\mathbb{F})$ *such that*
- $\mathcal{S} = \langle u^2 + v^2, u^2 + w^2, (u+v) \oslash (u+w) \rangle$, *or* $\mathcal{S} = \langle u^2 + v^2, u^2 + w^2 \rangle$, *or* $\mathcal{S} = \langle x^2 + y^2, (x+z) \oslash y + (x+z)^2 \rangle$ *with* $\{x, y, z\} = \{u, v, w\}$, *where* $u+v, u+w \in \mathcal{U}_p$ *and* $u, v, w \in \mathcal{U}_q$ *for some integers* $1 \leqslant p \leqslant \frac{n+1}{2}$ *and* $1 \leqslant q \leqslant n+1-p$, *or*

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- $\circ$ *$\mathcal{S} = \langle x^2 + y^2, (u+v) \oslash (u+w) \rangle$ for a pair of distinct vectors $x, y \in \{u, v, w\}$ with $x+y \in \mathcal{U}_p$ and $u, v, w \in \mathcal{U}_q$ for some integers $1 \leqslant p \leqslant \frac{n+1}{2}$ and $1 \leqslant q \leqslant n+1-p$;*

and $\mathcal{S}$ is of this form only if $|\mathbb{F}| = 2$.

Proof. If $\mathcal{S}$ satisfies one of the statements (a) - (f) in Theorem 3.3, then $\mathcal{S}$ is a linear space of bounded rank-two matrices of $\mathcal{SM}_n(\mathbb{F})$. Moreover, by Lemmas 2.3 and 3.2, we have $\mathcal{S} \subseteq \mathcal{ST}_n(\mathbb{F})$. This proves sufficiency.

We now consider necessity. Suppose that $\mathcal{S} \neq \{0\}$. Since $\mathcal{S}$ is a linear space of bounded rank-two matrices of $\mathcal{SM}_n(\mathbb{F})$, we see that $\mathcal{S}$ satisfies one of the statements (I) - (VI) as described in Lemma 3.1. We use the notation that have been employed in Lemma 3.1 and divide our argument in the following cases.

Case I: Suppose that $\mathcal{S}$ satisfies (I) in Lemma 3.1. Let $u \in \mathcal{U}_s \backslash \mathcal{U}_{s-1}$ and $v \in \mathcal{U}_t \backslash \mathcal{U}_{t-1}$ for some integers $1 \leqslant s, t \leqslant n$. We divide our argument into the following three subcases:

- *Case I-i:* If $1 \leqslant s, t \leqslant \frac{n+1}{2}$, then $u, v \in \mathcal{U}_p$ with $p = \max\{s, t\}$. So $\mathcal{S}$ satisfies statement (a).
- *Case I-ii:* If $1 \leqslant s \leqslant n+1-t$ and $\frac{n+1}{2} < t \leqslant n$, then u^2, $u \oslash v \in \mathcal{ST}_n(\mathbb{F})$ and $v^2 \notin \mathcal{ST}_n(\mathbb{F})$. If $\mathcal{S}$ has no rank two matrices, then $\mathcal{S} = \langle u^2 \rangle$ and it satisfies statement (a). Suppose that $\mathcal{S}$ has a rank two matrix. Then $\mathcal{S} \subseteq \langle u^2, u \oslash v \rangle$ and it is of one of the following forms:
  - $\circ$ $\mathcal{S} = \langle u \oslash v \rangle = u \oslash \langle v \rangle$ and it satisfies statement (b);
  - $\circ$ $\mathcal{S} = \langle u \oslash v + au^2 \rangle$ with $a \in \mathbb{F} \backslash \{0\}$. When $\operatorname{char} \mathbb{F} = 2$, $\mathcal{S}$ satisfies statement (d); and when $\operatorname{char} \mathbb{F} \neq 2$, we get $\mathcal{S} = u \oslash \langle v + \frac{a}{2} u \rangle$ and it satisfies statement (b);
  - $\circ$ $\mathcal{S} = \langle u^2, u \oslash v \rangle$. When $\operatorname{char} \mathbb{F} = 2$, we obtain $\mathcal{S} = u \oslash \langle v \rangle + \langle u^2 \rangle$ and it satisfies statement (c); when $\operatorname{char} \mathbb{F} \neq 2$, we see that $\mathcal{S} = u \oslash \langle v, 2^{-1} u \rangle$ and it satisfies statement (b).
- *Case I-iii:* Suppose that $\frac{n+1}{2} < s, t \leqslant n$. If $\mathcal{S}$ contains no rank two matrices, then $\dim \mathcal{S} = 1$. By Lemma 2.1, we have $\mathcal{S} = \langle x^2 \rangle$ for some nonzero vector $x \in \mathcal{U}_p$ with $1 \leqslant p \leqslant \frac{n+1}{2}$. Thus, $\mathcal{S}$ satisfies statement (a). Suppose now that $\mathcal{S}$ has a rank two matrix, say A. Then $A = au^2 + bv^2 + cu \oslash v$ for some $a, b, c \in \mathbb{F}$ with $c^2 - ab \neq 0$ by (P5). On the other hand, by Lemma 2.1, there exist linearly independent vectors x, y such that either $A = \alpha x^2 + \beta y^2$ for some $\alpha, \beta \in \mathbb{F} \backslash \{0\}$ and $x, y \in \mathcal{U}_p$ with $1 \leqslant p \leqslant \frac{n+1}{2}$; or $A = x \oslash y + \alpha x^2$ for some $\alpha \in \mathbb{F} \backslash \{0\}$ and some $x \in \mathcal{U}_p$ and $y \in \mathcal{U}_q$ with $1 \leqslant p \leqslant \frac{n+1}{2}$ and $1 \leqslant q \leqslant n+1-p$. Then

  $$au^2 + bv^2 + cu \oslash v = \alpha x^2 + \beta y^2 \quad \text{ or } \quad au^2 + bv^2 + cu \oslash v = x \oslash y + \alpha x^2.$$

  In both cases, $\langle x, y \rangle = \langle u, v \rangle$ by Lemma 2.2 (a). Thus, $\langle x^2, y^2, x \oslash y \rangle =$

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$\langle u^2, v^2, u \oslash v\rangle$, and so $\mathcal{S} \subseteq \langle x^2, y^2, x \oslash y\rangle$. The result follows by a similar argument as in Cases I-i and I-ii.

Case II: If $\mathcal{S}$ satisfies (II) in Lemma 3.1, then for each nonzero $A \in \mathcal{S}$, there exists a nonzero $v_A \in \mathcal{M}_{n,1}(\mathbb{F})$ such that $A = u \oslash v_A$. Since $A \in \mathcal{ST}_n(\mathbb{F})$, it follows from Lemma 2.3 that

(i) $u \in \mathcal{U}_p \backslash \mathcal{U}_{p-1}$ and $v_A \in \mathcal{U}_{r_A} \backslash \mathcal{U}_{r_A-1}$ for some integers $1 \leqslant p \leqslant n+1-r_A$, or
(ii) $u \in \mathcal{U}_p \backslash \mathcal{U}_{p-1}$ and $v_A = \alpha_A u + z_A \in \mathcal{U}_p \backslash \mathcal{U}_{p-1}$ for some $\alpha_A \in \mathbb{F}\backslash\{0\}$ and $z_A \in \mathcal{U}_{r_A} \backslash \mathcal{U}_{r_A-1}$, where $1 \leqslant r_A \leqslant n+1-p < \frac{n+1}{2}$ and u, z_A are linearly independent, and in addition, this case holds only if $\operatorname{char} \mathbb{F} = 2$.

Notice that if (ii) holds, then $\operatorname{char} \mathbb{F} = 2$ and A can be rewritten as

$$A = u \oslash (\alpha_A u + z_A) = u \oslash z_A.$$

Consequently, in view of (i) and (ii), for each $A \in \mathcal{S}$, there exists $v_A \in \mathcal{U}_{r_A} \backslash \mathcal{U}_{r_A-1}$ with $1 \leqslant r_A \leqslant n+1-p$ such that $A = u \oslash v_A$. Accordingly, there exists a linear subspace $\mathcal{V}$ of $\mathcal{U}_q$ with $1 \leqslant q \leqslant n+1-p$ such that $\mathcal{S} = u \oslash \mathcal{V}$. Thus, $\mathcal{S}$ satisfies (b).

Case III: If $\mathcal{S}$ satisfies (III) in Lemma 3.1, then u^2, $u \oslash v \in \mathcal{ST}_n(\mathbb{F})$ for every $v \in \mathcal{V}$. It follows from Lemma 2.3 that $u \in \mathcal{U}_p$ for some $1 \leqslant p \leqslant \frac{n+1}{2}$, and for each $v \in \mathcal{V}$, there exists an integer $1 \leqslant r_v \leqslant n+1-p$ such that $v \in \mathcal{U}_{r_v}$. Consequently, $\mathcal{V}$ is a subspace of $\mathcal{U}_q$ for some integer $1 \leqslant q \leqslant n+1-p \leqslant n$. Hence, $\mathcal{S}$ satisfies (c).

Case IV: If $\mathcal{S}$ satisfies (IV) in Lemma 3.1, then $u \oslash v_i + \lambda_i u^2 \in \mathcal{ST}_n(\mathbb{F})$ for every $i = 1, \ldots, k$. Since $u, v_1, \ldots, v_k$ are linearly independent and $(\lambda_1, \ldots, \lambda_k) \neq 0$, the result follows directly from Lemma 3.2 (a) and $\mathcal{S}$ satisfies (d).

Case V: If $\mathcal{S}$ satisfies (V) of Lemma 3.1, then $u \oslash v, u \oslash w, v \oslash w \in \mathcal{ST}_n(\mathbb{F})$. In view of Lemma 2.3, each pair of elements of $\{u, v, w\}$ satisfies either (b)(i) or (b)(ii) of Lemma 2.3. If all pairs of elements of $\{u, v, w\}$ satisfy (b)(i) of Lemma 2.3, then $\mathcal{S}$ is readily seen to satisfy (e). Suppose not. We shall show that $\{u, v, w\}$ can be replaced by some other $\{x, y, z\}$ such that $\mathcal{S} = \langle x \oslash y, x \oslash z, y \oslash z\rangle$ satisfies (e). With no loss of generality, say $\{u, v\}$ satisfies (b)(ii) of Lemma 2.3. Then $u \in \mathcal{U}_q \backslash \mathcal{U}_{q-1}$ and $v = \alpha u + y \in \mathcal{U}_q \backslash \mathcal{U}_{q-1}$, where $y \in \mathcal{U}_p \backslash \mathcal{U}_{p-1}$, for some $\alpha \in \mathbb{F}\backslash\{0\}$ and integers p, q such that $1 \leqslant p \leqslant n+1-q < \frac{n+1}{2}$. Note that $\langle u, y, w\rangle = \langle u, v, w\rangle$, from which, together with Lemma 2.5, follows that $\mathcal{S} = \langle u \oslash y, u \oslash w, y \oslash w\rangle$. If $\{u, w\}$ satisfies (b)(i) of Lemma 2.3, we are done by setting $x = u$ and $z = w$. Otherwise, say $\{u, w\}$ satisfies (b)(ii) of Lemma 2.3. Then $u \in \mathcal{U}_q \backslash \mathcal{U}_{q-1}$ and $w = \beta u + z \in \mathcal{U}_q \backslash \mathcal{U}_{q-1}$, where $z \in \mathcal{U}_r \backslash \mathcal{U}_{r-1}$, for some $\beta \in \mathbb{F}\backslash\{0\}$ and an integer r such that $1 \leqslant r \leqslant n+1-q < \frac{n+1}{2}$. As before, we get that $\langle u, y, z\rangle = \langle u, y, w\rangle$, from which follows that $\mathcal{S} = \langle u \oslash y, u \oslash z, y \oslash z\rangle$ satisfies (e). Setting $u = x$, we are done.

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Case VI: Suppose $\mathcal{S}$ satisfies (VI) in Lemma 3.1. Note that

$$\begin{aligned}&\big\langle u^2+v^2,\ u^2+w^2,(u+v)\oslash(u+w)\big\rangle\\ &\qquad=\big\{\,0,u^2+v^2,u^2+w^2,v^2+w^2,(u+v)\oslash(u+w),(u+v)\oslash w+(u+v)^2,\\ &\qquad\quad(u+w)\oslash v+(u+w)^2,(v+w)\oslash u+(v+w)^2\big\}\end{aligned}$$

with

$$\begin{aligned}u^2+v^2+(u+v)\oslash(u+w)&=(u+v)\oslash w+(u+v)^2,\\ u^2+w^2+(u+v)\oslash(u+w)&=(u+w)\oslash v+(u+w)^2,\\ v^2+w^2+(u+v)\oslash(u+w)&=(v+w)\oslash u+(v+w)^2,\end{aligned}$$

and each nonzero matrix in $\mathcal{S}$ is of rank two. We argue in the following three cases:

- If $\dim\mathcal{S}=1$, then $\mathcal{S}=\langle A\rangle$ for some nonzero per-symmetric upper triangular matrix $A\in\big\langle u^2+v^2,u^2+w^2,(u+v)\oslash(u+w)\big\rangle$. By Lemma 2.1, there exist linearly independent vectors x,y such that either (i) $A=\alpha x^2+\beta y^2$ with $x,y\in\mathcal{U}_p$ for some integer $1\leqslant p\leqslant\frac{n+1}{2}$ and $\alpha,\beta\in\mathbb{F}\setminus\{0\}$; or (ii) $A=x\oslash y+\gamma x^2$ with $x\in\mathcal{U}_p$ and $y\in\mathcal{U}_q$ for some integers $1\leqslant p\leqslant\frac{n+1}{2}$ and $1\leqslant q\leqslant n+1-p$, and $\gamma\in\mathbb{F}$. Then $\mathcal{S}$ satisfies (a) when (i) holds, $\mathcal{S}$ satisfies (b) when (ii) holds with $\gamma=0$, or $\mathcal{S}$ satisfies (d) when (ii) holds with $\gamma\neq 0$.
- If $\dim\mathcal{S}=3$, then $\mathcal{S}=\big\langle u^2+v^2,u^2+w^2,(u+v)\oslash(u+w)\big\rangle$. Since u^2+v^2 and u^2+w^2 are in $\mathcal{ST}_n(\mathbb{F})$, Lemma 3.2 (b) implies that $u+v\in\mathcal{U}_{p_1}\backslash\mathcal{U}_{p_1-1}$, $u+w\in\mathcal{U}_{p_2}\backslash\mathcal{U}_{p_2-1}$, $u,v\in\mathcal{U}_{q_1}\backslash\mathcal{U}_{q_1-1}$, and $u,w\in\mathcal{U}_{q_2}\backslash\mathcal{U}_{q_2-1}$ for some integers p_1,p_2,q_1,q_2 such that $1\leqslant p_i\leqslant\frac{n+1}{2}$ and $1\leqslant q_i\leqslant n+1-p_i$ for $i=1,2$. Since $u\in(\mathcal{U}_{q_1}\backslash\mathcal{U}_{q_1-1})\cap(\mathcal{U}_{q_2}\backslash\mathcal{U}_{q_2-1})$, it is necessary that $q_1=q_2=q$ for a common q. Setting $p=\max\{p_1,p_2\}$, we note that $\mathcal{S}$ satisfies (f).
- If $\dim\mathcal{S}=2$, then one of the following holds:
  - $\mathcal{S}=\{0,u^2+v^2,u^2+w^2,v^2+w^2\}=\big\langle u^2+v^2,u^2+w^2\big\rangle$, where $u+v,u+w\in\mathcal{U}_p$ and $u,v,w\in\mathcal{U}_q$ for some integers $1\leqslant p\leqslant\frac{n+1}{2}$ and $1\leqslant q\leqslant n+1-p$;
  - $\mathcal{S}=\{0,u^2+v^2,(u+w)\oslash v+(u+w)^2,(v+w)\oslash u+(v+w)^2\}=\big\langle u^2+v^2,(u+w)\oslash v+\ (u+w)^2\big\rangle$, where $u+v,u+w\in\mathcal{U}_p$ and $u,v,w\in\mathcal{U}_q$ for some integers $1\leqslant p\leqslant\frac{n+1}{2}$ and $1\leqslant q\leqslant n+1-p$;
  - $\mathcal{S}=\{0,u^2+w^2,(v+w)\oslash u+(v+w)^2,(u+v)\oslash w+(u+v)^2\}=\big\langle u^2+w^2,(u+v)\oslash w+\ (u+v)^2\big\rangle$, where $u+v,u+w\in\mathcal{U}_p$ and $u,v,w\in\mathcal{U}_q$ for some integers $1\leqslant p\leqslant\frac{n+1}{2}$ and $1\leqslant q\leqslant n+1-p$;
  - $\mathcal{S}=\{0,v^2+w^2,(u+v)\oslash w+(u+v)^2,(u+w)\oslash v+(u+w)^2\}=\big\langle v^2+w^2,(v+u)\oslash w+\ (v+u)^2\big\rangle$, where $v+w,v+u\in\mathcal{U}_p$ and $u,v,w\in\mathcal{U}_q$ for some integers $1\leqslant p\leqslant\frac{n+1}{2}$ and $1\leqslant q\leqslant n+1-p$;
  - $\mathcal{S}=\{0,u^2+v^2,(u+v)\oslash(u+w),(u+v)\oslash w+(u+v)^2\}=\big\langle u^2+v^2,(u+v)\oslash(u+w)\big\rangle$, where $u+v\in\mathcal{U}_p$ and $u,v,w\in\mathcal{U}_q$ for some integers $1\leqslant p\leqslant\frac{n+1}{2}$ and $1\leqslant q\leqslant n+1-p$;

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- $\mathcal{S} = \{0, u^2 + w^2, (u+v) \oslash (u+w), (u+w) \oslash v + (u+w)^2\} = \langle u^2 + w^2, (u+v) \oslash (u+w)\rangle$, where $u+w \in \mathcal{U}_p$ and $u, v, w \in \mathcal{U}_q$ for some integers $1 \leqslant p \leqslant \frac{n+1}{2}$ and $1 \leqslant q \leqslant n+1-p$;
- $\mathcal{S} = \{0, v^2 + w^2, (u+v) \oslash (u+w), (v+w) \oslash u + (v+w)^2\} = \langle v^2 + w^2, (u+v) \oslash (u+w)\rangle$, where $v+w \in \mathcal{U}_p$ and $u, v, w \in \mathcal{U}_q$ for some integers $1 \leqslant p \leqslant \frac{n+1}{2}$ and $1 \leqslant q \leqslant n+1-p$.

Hence, $\mathcal{S}$ satisfies (f). $\square$

We now continue our investigation of 2-spaces of $\mathcal{ST}_n(\mathbb{F})$. We first study some examples of 2-spaces of $\mathcal{ST}_n(\mathbb{F})$.

Example 3.4. Let $\mathbb{F}$ be a field and n be an integer such that $n \geqslant 2$. Recall that $\{e_1, \ldots, e_n\}$ denotes the standard basis of $\mathcal{M}_{n,1}(\mathbb{F})$.

(a) Let $n \geqslant 2$. Then $\langle e_1 \oslash e_2 + \alpha e_1^2\rangle$ is a 1-dimensional 2-space of $\mathcal{ST}_n(\mathbb{F})$ for any $\alpha \in \mathbb{F}$.

(b) Let $n \geqslant 3$ and $\alpha, \beta, \gamma \in \mathbb{F}$ be such that $\gamma^2 \neq \alpha\beta$. Then $\langle \alpha e_1^2 + \beta e_2^2 + \gamma e_1 \oslash e_2\rangle$ is a 1-dimensional 2-space of $\mathcal{ST}_n(\mathbb{F})$.

(c) Let $n \geqslant 3$ and $\mathbb{F} = \mathbb{R}$. Then $\langle e_1 \oslash e_2, e_1 \oslash e_2 + e_1^2 - e_2^2\rangle$ is a 2-dimensional 2-space of $\mathcal{ST}_n(\mathbb{R})$. Let $A = a(e_1 \oslash e_2) + b(e_1 \oslash e_2 + e_1^2 - e_2^2) \in \langle e_1 \oslash e_2, e_1 \oslash e_2 + e_1^2 - e_2^2\rangle$ for some $a, b \in \mathbb{R}$ with $(a, b) \neq 0$. We see that A is of rank two since

$$\det \begin{bmatrix} a+b & b \\ -b & a+b \end{bmatrix} = (a+b)^2 + b^2 \neq 0.$$

(d) Let $\mathbb{F}$ be a field with four elements. Then $\operatorname{char} \mathbb{F} = 2$ and the multiplicative group of $\mathbb{F}$ is cyclic. We set $\mathbb{F} = \{0, 1, \alpha, \alpha^2\}$, where α is a primitive element of $\mathbb{F}$. We see that $\langle e_1 \oslash e_2 + e_1^2, e_1 \oslash e_2 + \alpha e_2^2\rangle$ is a 2-dimensional 2-space of $\mathcal{ST}_n(\mathbb{F})$. To proof this, let $A = \lambda_1(e_1 \oslash e_2 + e_1^2) + \lambda_2(e_1 \oslash e_2 + \alpha e_2^2) \in \langle e_1 \oslash e_2 + e_1^2, e_1 \oslash e_2 + \alpha e_2^2\rangle$ for some $\lambda_1, \lambda_2 \in \mathbb{F}$ with $(\lambda_1, \lambda_2) \neq 0$. By a direct verification, we have

$$\det \begin{bmatrix} \lambda_1 + \lambda_2 & \lambda_1 \\ \lambda_2\alpha & \lambda_1 + \lambda_2 \end{bmatrix} = \lambda_1^2 + \lambda_2^2 + \alpha\lambda_1\lambda_2 \neq 0.$$

Hence, A is of rank two.

Example 3.5. Let $\mathbb{F}$ be a field of characteristic two. Let $u \in \mathcal{U}_p \backslash \mathcal{U}_{p-1}$, $v \in \mathcal{U}_q \backslash \mathcal{U}_{q-1}$ and $w \in \mathcal{U}_r \backslash \mathcal{U}_{r-1}$ be linearly independent vectors such that $1 \leqslant p, q \leqslant n+1-r$ and $p \leqslant n+1-q$. Then $u \oslash v$, $u \oslash w$, $v \oslash w$ are linearly independent elements in $\mathcal{ST}_n(\mathbb{F})$ and each nonzero element in $\langle u \oslash v, u \oslash w, v \oslash w\rangle$ has rank two. Thus, $\langle u \oslash v, u \oslash w, v \oslash w\rangle$ is a 3-dimensional 2-space of $\mathcal{ST}_n(\mathbb{F})$. Note also that each element in $\langle u \oslash v, u \oslash w, v \oslash w\rangle$ has a zero minor diagonal.

Example 3.6. Let $\mathbb{F}$ be a field of characteristic two. Let $u \in \mathcal{U}_p \backslash \mathcal{U}_{p-1}$ and

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$v_1,\ldots,v_k \in \mathcal{U}_q\backslash\mathcal{U}_{q-1}$ be linearly independent vectors such that $1 \leqslant p \leqslant \frac{n+1}{2}$ and $1 \leqslant q \leqslant n+1-p$, and $\lambda_1,\ldots,\lambda_k \in \mathbb{F}$ be such that $(\lambda_1,\ldots,\lambda_k) \neq 0$. It is easily checked that $u \oslash v_1 + \lambda_1 u^2,\ldots,u \oslash v_k + \lambda_k u^2$ are linearly independent. Let $A \in \left\langle u \oslash v_1 + \lambda_1 u^2,\ldots,\, u \oslash v_k + \lambda_k u^2\right\rangle$ be nonzero. Then there exist $\beta_1,\ldots,\beta_k \in \mathbb{F}$ not all of which are zero such that

$$\begin{aligned} A &= \beta_1(u \oslash v_1 + \lambda_1 u^2) + \cdots + \beta_k(u \oslash v_k + \lambda_k u^2) \\ &= u \oslash (\beta_1 v_1 + \cdots + \beta_k v_k) + (\beta_1\lambda_1 + \cdots + \beta_k\lambda_k)u^2. \end{aligned}$$

Since $u, v_1,\ldots,v_k$ are linearly independent and $(\beta_1,\ldots,\beta_k) \neq 0$, we get $\beta_1 v_1 + \cdots + \beta_k v_k, u$ are linearly independent, and so $\operatorname{rank} A = 2$. Then $\left\langle u \oslash v_1 + \lambda_1 u^2,\ldots,\, u \oslash v_k + \lambda_k u^2\right\rangle$ is a k-dimensional 2-space of $\mathcal{ST}_n(\mathbb{F})$.

As an immediate consequence of Theorem 3.3, we obtain a complete description of 2-spaces of $\mathcal{ST}_n(\mathbb{F})$ over an arbitrary field $\mathbb{F}$.

Corollary 3.7. *Let $\mathbb{F}$ be a field and n be an integer such that $n \geqslant 2$. Then $\mathcal{S}$ is a 2-space of $\mathcal{ST}_n(\mathbb{F})$ if and only if one of the following holds:*

(a) $\mathcal{S} = \left\langle a_i u \oslash v + b_i u^2 + c_i v^2 \,\middle|\, i = 1,2\right\rangle$ *for some linearly independent vectors $u, v \in \mathcal{U}_p$ with $1 \leqslant p \leqslant \frac{n+1}{2}$, and some fixed scalars $a_i, b_i, c_i \in \mathbb{F}$ for $i = 1,2$ such that*

$$(\lambda_1 a_1 + \lambda_2 a_2)^2 \neq (\lambda_1 b_1 + \lambda_2 b_2)(\lambda_1 c_1 + \lambda_2 c_2)$$

for every $\lambda_1, \lambda_2 \in \mathbb{F}$ with $(\lambda_1,\lambda_2) \neq 0$.

(b) $\mathcal{S} = u \oslash \mathcal{V}$ *for some nonzero vector $u \in \mathcal{U}_p$ and some subspace $\mathcal{V}$ of $\mathcal{U}_q$ with $1 \leqslant p \leqslant n+1-q \leqslant n$, and $\mathcal{V} \cap \langle u\rangle = \{0\}$ when* $\operatorname{char}\mathbb{F} \neq 2$.

(c) $\mathcal{S} = \left\langle u \oslash v_1 + \lambda_1 u^2,\, \ldots,\, u \oslash v_k + \lambda_k u^2\right\rangle$ *for some scalars $\lambda_1,\ldots,\lambda_k \in \mathbb{F}$ with $(\lambda_1,\ldots,\lambda_k) \neq 0$, and some linearly independent vectors $u, v_1,\ldots,v_k$ such that $u \in \mathcal{U}_p$ with $1 \leqslant p \leqslant \frac{n+1}{2}$ and $v_1,\ldots,v_k \in \mathcal{U}_q$ with $1 \leqslant q \leqslant n+1-p$; and $\mathcal{S}$ is of this form only if* $\operatorname{char}\mathbb{F} = 2$.

(d) $\mathcal{S} = \langle u \oslash v,\, u \oslash w,\, v \oslash w\rangle$ *for some linearly independent vectors $u \in \mathcal{U}_p$, $v \in \mathcal{U}_q$ and $w \in \mathcal{U}_r$ such that $1 \leqslant p, q \leqslant n+1-r \leqslant n$ and $p \leqslant n+1-q$; and $\mathcal{S}$ is of this form only if* $\operatorname{char}\mathbb{F} = 2$.

(e) *There exist linearly independent vectors $u, v, w \in \mathcal{M}_{n,1}(\mathbb{F})$ such that*
- $\mathcal{S} = \left\langle u^2 + v^2,\, u^2 + w^2,\, (u+v) \oslash (u+w)\right\rangle$, *or* $\mathcal{S} = \left\langle u^2 + v^2,\, u^2 + w^2\right\rangle$, *or* $\mathcal{S} = \left\langle x^2 + y^2,\, (x+z) \oslash y + (x+z)^2\right\rangle$ *with $\{x,y,z\} = \{u,v,w\}$, where $u+v, u+w \in \mathcal{U}_p$ and $u,v,w \in \mathcal{U}_q$ for some integers $1 \leqslant p \leqslant \frac{n+1}{2}$ and $1 \leqslant q \leqslant n+1-p$; or*
- $\mathcal{S} = \left\langle x^2 + y^2, (u+v) \oslash (u+w)\right\rangle$ *for a pair of distinct vectors $x, y \in \{u,v,w\}$ with $x+y \in \mathcal{U}_p$ and $u,v,w \in \mathcal{U}_q$ for some integers $1 \leqslant p \leqslant \frac{n+1}{2}$ and $1 \leqslant q \leqslant n+1-p$,*

and $\mathcal{S}$ is of this form only if $|\mathbb{F}| = 2$.

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4. Bounded rank-two linear preservers. In this section, we characterize bounded rank-two linear preservers $\psi : \mathcal{ST}_n(\mathbb{F}) \to \mathcal{SM}_m(\mathbb{F})$, with $m, n \geqslant 3$ and $\operatorname{char}\mathbb{F} \neq 2$. We then obtain a classification of bounded rank-two linear preservers between per-symmetric triangular matrix spaces over a field of characteristic not two.

We start with the following lemma whose proof is straightforward and omitted.

LEMMA 4.1. *Let $\mathbb{F}$ be a field and $u, v, x, y, z \in \mathcal{M}_{n,1}(\mathbb{F})$. Then the following statements hold.*

(a) *If x, y are linearly independent, then the following are equivalent.*
  - **(i)** $ax^2 + by^2 + cx \oslash y \in \langle u^2, v^2, u \oslash v \rangle$ *for some $a, b, c \in \mathbb{F}$ with $ab \neq c^2$.*
  - **(ii)** $\langle x, y\rangle = \langle u, v\rangle$.
  - **(iii)** $\langle x^2, y^2, x \oslash y \rangle = \langle u^2, v^2, u \oslash v \rangle$.

(b) *If y, z are linearly independent and $x \oslash y,\ x \oslash z \in \langle u^2, v^2, u \oslash v \rangle$, then $x \in \langle y, z\rangle$.*

LEMMA 4.2. *Let $\mathbb{F}$ be a field of characteristic not two and n be an integer such that $n \geqslant 2$. Let $A = u \oslash v$ and $B = w \oslash z$ be nonzero matrices for some $u, v, w, z \in \mathcal{M}_{n,1}(\mathbb{F})$ such that u, v, w are linearly independent. If $\operatorname{rank}(A + \lambda B) \leqslant 2$ for all $\lambda \in \mathbb{F}$, then either $z \in \langle u\rangle$ or $z \in \langle v\rangle$.*

Proof. Since u, v, w are linearly independent and $\operatorname{rank}(A + B) \leqslant 2$, we have $z \in \langle u, v, w\rangle$. Let $z = au + bv + cw$ for some $a, b, c \in \mathbb{F}$. Since $A + \lambda B = 2\lambda c w^2 + u \oslash v + \lambda a(u \oslash w) + \lambda b(w \oslash v)$ has rank bounded above by two, it follows that

$$0 = \det \begin{bmatrix} 1 & \lambda a & 0 \\ \lambda b & 2\lambda c & \lambda a \\ 0 & \lambda b & 1 \end{bmatrix} = -2ab\lambda^2 + 2c\lambda \quad \text{for every } \lambda \in \mathbb{F}.$$

Since $|\mathbb{F}| \geqslant 3$, we obtain $c = 0$ and $ab = 0$. ☐

THEOREM 4.3. *Let $\mathbb{F}$ be a field of characteristic not two and m, n be integers such that $m, n \geqslant 3$. Then $\psi : \mathcal{ST}_n(\mathbb{F}) \to \mathcal{SM}_m(\mathbb{F})$ is a bounded rank-two linear preserver if and only if $m \geqslant n$ and ψ is of one of the following forms:*

(i) *There exist a nonzero vector $u \in \mathcal{M}_{m,1}(\mathbb{F})$ and a linear mapping $\varphi : \mathcal{ST}_n(\mathbb{F}) \to \mathcal{M}_{m,1}(\mathbb{F})$ such that*

$$\psi(A) = u \oslash \varphi(A) \quad \text{for all } A \in \mathcal{ST}_n(\mathbb{F}), \tag{4.1}$$

where $\varphi(A) \neq 0$ for every nonzero bounded rank-two matrix $A \in \mathcal{ST}_n(\mathbb{F})$.

(ii) *There exist a full rank matrix $P \in \mathcal{M}_{m,n}(\mathbb{F})$ and a nonzero $\lambda \in \mathbb{F}$ such that*

$$\psi(A) = \lambda P A P^{+} \quad \text{for all } A \in \mathcal{ST}_n(\mathbb{F}).$$

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(iii) *When $n = 4$, in addition to* (i) *and* (ii), *ψ also takes the form*

$$\psi(A) = P \begin{bmatrix} a_{11} & a_{12} & \alpha a_{13} + \theta(a_{14} - a_{23}) & \beta a_{14} \\ 0 & a_{22} & (2\alpha - \beta)a_{23} & \alpha a_{13} + \theta(a_{14} - a_{23}) \\ 0 & 0 & a_{22} & a_{12} \\ 0 & 0 & 0 & a_{11} \end{bmatrix} P^+$$

for all $A = (a_{ij}) \in \mathcal{ST}_4(\mathbb{F})$, where $P \in \mathcal{M}_{m,4}(\mathbb{F})$ is a full rank matrix, $\alpha, \beta \in \mathbb{F}$ are nonzero with $\beta \neq 2\alpha$, and $\theta \in \mathbb{F}$ is nonzero only if $|\mathbb{F}| = 3$.

(iv) *When $n = 3$, in addition to* (i) *and* (ii), *ψ also takes one of the following forms:*

(a) *There exist a surjective linear mapping $\phi : \mathcal{ST}_3(\mathbb{F}) \to \mathbb{F}^3$ and a full rank matrix $P \in \mathcal{M}_{m,2}(\mathbb{F})$ such that*

$$\psi(A) = P \begin{bmatrix} \phi(A)_3 & \phi(A)_1 \\ \phi(A)_2 & \phi(A)_3 \end{bmatrix} P^+ \quad \text{for all } A \in \mathcal{ST}_3(\mathbb{F}),$$

where $\phi(A)_i$ denotes the i-th component of $\phi(A) \in \mathbb{F}^3$ and $\phi(A) \neq 0$ for every nonzero bounded rank-two matrix $A \in \mathcal{ST}_3(\mathbb{F})$.

(b) *There exist a full rank matrix $P \in \mathcal{M}_{m,3}(\mathbb{F})$ and $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}$ with $\lambda_3 \neq 0$ such that either*

$$\psi(A) = P \begin{bmatrix} a_{pp} & \eta_2 a_{12} + a_{13} + \lambda_2 a_{qq} & \eta_1 a_{12} + \lambda_1 a_{qq} \\ 0 & \lambda_3 a_{qq} & \eta_2 a_{12} + a_{13} + \lambda_2 a_{qq} \\ 0 & 0 & a_{pp} \end{bmatrix} P^+$$

for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$, where $\eta_1, \eta_2 \in \mathbb{F}$ are nonzero and $\{p, q\} = \{1, 2\}$, or

$$\psi(A) = P \begin{bmatrix} a_{pp} & a_{1s} + \lambda_2 a_{qq} & \eta a_{1t} + \lambda_1 a_{qq} \\ 0 & \lambda_3 a_{qq} & a_{1s} + \lambda_2 a_{qq} \\ 0 & 0 & a_{pp} \end{bmatrix} P^+$$

for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$, where $\eta \in \mathbb{F}$ is nonzero and $\{p, q\} = \{s, t\} = \{1, 2\}$.

Proof. Sufficiency is clear. We now consider necessity. Let $\mathcal{X}_1 = e_1 \oslash \langle e_1, \ldots, e_n \rangle$ and $\mathcal{X}_2 = e_2 \oslash \langle e_1, \ldots, e_{n-1} \rangle$. By Lemma 3.1 and Theorem 3.3, together with the assumption of ψ, we see that $\psi(\mathcal{X}_1)$ and $\psi(\mathcal{X}_2)$ are spaces of bounded rank-two matrices of $\mathcal{SM}_m(\mathbb{F})$ containing linearly independent sets $\{\psi(e_1 \oslash e_1), \ldots, \psi(e_1 \oslash e_n)\}$ and $\{\psi(e_2 \oslash e_1), \ldots, \psi(e_2 \oslash e_{n-1})\}$, respectively. Thus, $m \geqslant n$. We now divide our proof into three main cases:

Case I: $n \geqslant 5$. By Lemma 3.1, we have

$$\psi(e_1^2) = u \oslash v_1 \quad \text{and} \quad \psi(e_1 \oslash e_i) = u \oslash v_i \text{ for } i = 2, \ldots, n, \tag{4.2}$$

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for some nonzero vector $u \in \mathcal{M}_{m,1}(\mathbb{F})$ and linearly independent vectors $v_1, \ldots, v_n \in \mathcal{M}_{m,1}(\mathbb{F})$, and

$$\psi(e_2^2) = x \oslash y_2 \quad \text{and} \quad \psi(e_2 \oslash e_i) = x \oslash y_i \text{ for } i = 1, 3, 4, \ldots, n-1, \tag{4.3}$$

for some nonzero vector $x \in \mathcal{M}_{m,1}(\mathbb{F})$ and linearly independent vectors $y_1, \ldots, y_{n-1} \in \mathcal{M}_{m,1}(\mathbb{F})$. We consider the following two subcases:

Case 1-A: $\langle x \rangle = \langle u \rangle$. There is no loss of generality in assuming $x = u$. For each $3 \leqslant i \leqslant \frac{n+1}{2}$, let $\mathcal{X}_i = e_i \oslash \langle e_1, e_2, e_i \rangle$. Clearly, $\psi(\mathcal{X}_i)$ is a 3-dimensional linear space of bounded rank-two matrices of $\mathcal{SM}_m(\mathbb{F})$. Then each $\psi(\mathcal{X}_i)$ can be expressed in either of the forms (I) and (II) in Lemma 3.1. Suppose that there exists $3 \leqslant i_0 \leqslant \frac{n+1}{2}$ such that $\psi(\mathcal{X}_{i_0})$ satisfies (I). Since $\psi(e_{i_0} \oslash e_1) = u \oslash v_{i_0}$, $\psi(e_{i_0} \oslash e_2) = u \oslash y_{i_0}$ and $\psi(e_{i_0}^2)$ are linearly independent elements in $\psi(\mathcal{X}_{i_0})$, v_{i_0}, y_{i_0} are linearly independent, and together with Lemma 4.1 (a), we have $\langle u^2, v_{i_0}^2, u \oslash v_{i_0} \rangle = \psi(\mathcal{X}_{i_0}) = \langle u^2, y_{i_0}^2, u \oslash y_{i_0} \rangle$. Again, by Lemma 4.1, $\langle u, v_{i_0} \rangle = \langle u, y_{i_0} \rangle$. In particular, $y_{i_0} \in \langle u, v_{i_0} \rangle$. Then $\psi(e_{i_0} \oslash e_2) = \eta_1 u^2 + \eta_2 u \oslash v_{i_0}$ and $\psi(e_{i_0}^2) = \alpha u^2 + \beta v_{i_0}^2 + \gamma u \oslash v_{i_0}$ for some $\eta_1, \eta_2, \alpha, \beta, \gamma \in \mathbb{F}$ with $\eta_1, \beta \neq 0$. By (4.2), note that

- if u, v_1 are linearly dependent, then $\psi(e_1^2) = \lambda_1 u^2$ for some $\lambda_1 \in \mathbb{F} \setminus \{0\}$;
- if u, v_1 are linearly independent, then $v_{i_0} \in \langle u, v_1 \rangle$. For, if not, then v_{i_0}, u, v_1 are linearly independent, and so $\psi(e_{i_0}^2 + e_1^2) = \alpha u^2 + \beta v_{i_0}^2 + \gamma u \oslash v_{i_0} + u \oslash v_1$ is of rank three, a contradiction. Therefore, $v_1 \in \langle u, v_{i_0} \rangle$ since $\{u, v_{i_0}\}$ is linearly independent. Thus, $\psi(e_1^2) = \varsigma_1(u \oslash v_{i_0}) + \lambda_1 u^2$ for some scalars $\varsigma_1, \lambda_1 \in \mathbb{F}$ with $\lambda_1 \neq 0$.

Accordingly, we may write generally that

$$\psi(e_1^2) = \varsigma_1(u \oslash v_{i_0}) + \lambda_1 u^2 \tag{4.4}$$

for some $\varsigma_1, \lambda_1 \in \mathbb{F}$ with $(\varsigma_1, \lambda_1) \neq 0$. We apply this argument again, with (4.2) and v_1 replaced by (4.3) and y_2, to obtain

$$\psi(e_2^2) = \varsigma_2(u \oslash v_{i_0}) + \lambda_2 u^2 \tag{4.5}$$

for some $\varsigma_2, \lambda_2 \in \mathbb{F}$ with $(\varsigma_2, \lambda_2) \neq 0$. Furthermore, since $\psi(\langle e_1^2, e_2^2, e_1 \oslash e_2 \rangle)$ has dimension three, it follows from (4.2), (4.4) and (4.5) that u, v_{i_0}, v_2 are linearly independent. Then

$$\psi(e_{i_0}^2 + (e_1^2 + e_2^2 + e_1 \oslash e_2)) = \beta v_{i_0}^2 + u \oslash v_2 + (\alpha + \lambda_1 + \lambda_2)u^2 + (\gamma + \varsigma_1 + \varsigma_2)u \oslash v_{i_0}$$

is of rank three, a contradiction. Thus, $\psi(\mathcal{X}_i)$ satisfies (II) in Lemma 3.1 for every $3 \leqslant i \leqslant \frac{n+1}{2}$. Consequently, by Lemma 2.4, for each $3 \leqslant i \leqslant \frac{n+1}{2}$, there exists a nonzero vector $z_i \in \mathcal{M}_{m,1}(\mathbb{F})$ such that

$$\psi(e_i^2) = u \oslash z_i. \tag{4.6}$$

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For $n \geqslant 6$, we will consider $\mathcal{X}_{ij} = e_i \oslash \langle e_1, e_2, e_i, e_j \rangle$ for any $3 \leqslant i \leqslant \frac{n+1}{2}$ and $i < j \leqslant n+1-i$. Clearly, $\psi(\mathcal{X}_{ij})$ is a linear space of bounded rank-two matrices of $\mathcal{SM}_m(\mathbb{F})$ containing linearly independent elements $\psi(e_i \oslash e_j)$, $\psi(e_i^2)$, $\psi(e_1 \oslash e_i)$, $\psi(e_2 \oslash e_i)$. By Lemmas 3.1 and 2.4, we obtain $\psi(\mathcal{X}_{ij}) \subseteq u \oslash \mathcal{M}_{m,1}(\mathbb{F})$. Then for each $3 \leqslant i \leqslant \frac{n+1}{2}$ and $i < j \leqslant n+1-i$, there exists a nonzero vector $v_{ij} \in \mathcal{M}_{m,1}(\mathbb{F})$ such that

$$\psi(e_i \oslash e_j) = u \oslash v_{ij}. \tag{4.7}$$

Consequently, by (4.2), (4.3), (4.6), (4.7) and the linearity of ψ, we conclude, for $n \geqslant 5$, that there exists a linear mapping $\varphi : \mathcal{ST}_n(\mathbb{F}) \to \mathcal{M}_{m,1}(\mathbb{F})$ such that

$$\psi(A) = u \oslash \varphi(A) \quad \text{for all } A \in \mathcal{ST}_n(\mathbb{F}),$$

where $\varphi(A) \neq 0$ for every nonzero bounded rank-two matrix $A \in \mathcal{ST}_n(\mathbb{F})$. Hence, (4.1) holds.

Case 1-B: $\langle x \rangle \neq \langle u \rangle$. By (4.2) and (4.3), we see that $u \oslash v_2 = \psi(e_1 \oslash e_2) = x \oslash y_1$. It follows from Lemma 2.2 (b) that $y_1 = \varsigma u$ and $x = \varsigma^{-1} v_2$ for some nonzero scalar $\varsigma \in \mathbb{F}$, because u, x are linearly independent. Then

$$\psi(e_1 \oslash e_2) = \varsigma u \oslash x. \tag{4.8}$$

Our next claim is that

$$\{u, x, v_3 \ldots, v_n\} \text{ is linearly independent.} \tag{4.9}$$

We first show that $v_1 \in \langle u, x \rangle$. Suppose that $v_1 \notin \langle u, x \rangle$. Since $\operatorname{rank} \psi(e_1^2 + \gamma e_2^2) \leqslant 2$ for all $\gamma \in \mathbb{F}$, we have either $y_2 \in \langle u \rangle$ or $y_2 \in \langle v_1 \rangle$ by Lemma 4.2. Note that $\langle y_1 \rangle = \langle u \rangle$ and $\langle y_1 \rangle \neq \langle y_2 \rangle$ implies $y_2 \in \langle v_1 \rangle$, and so $y_2 \notin \langle u, x \rangle$. Let $v_1 = \epsilon y_2$ for some nonzero scalar $\epsilon \in \mathbb{F}$. It follows that $\psi((e_1 + e_2)^2) = \epsilon u \oslash y_2 + x \oslash y_2 + \varsigma u \oslash x$ is of rank three, a contradiction. Hence, $v_1 \in \langle u, x \rangle$. Similarly, we obtain $y_2 \in \langle u, x \rangle$. By (4.2) and (4.3), since $\psi(e_1^2), \psi(e_2^2), \psi(e_1 \oslash e_2)$ are linearly independent, we obtain

$$\psi(e_1^2) = u \oslash (\theta_1 x + \vartheta_1 u) \quad \text{and} \quad \psi(e_2^2) = x \oslash (\theta_2 u + \vartheta_2 x) \tag{4.10}$$

for some scalars $\theta_1, \theta_2, \vartheta_1, \vartheta_2 \in \mathbb{F}$ with $\vartheta_1, \vartheta_2 \neq 0$. Since $\psi(e_1^2), \psi(e_1 \oslash e_2), \ldots, \psi(e_1 \oslash e_n)$ are linearly independent, it follows from (4.2), (4.8) and (4.10) that $\{\theta_1 x + \vartheta_1 u, \varsigma x, v_3, \ldots, v_n\}$ is a linearly independent set, and hence, Claim (4.9) is proved.

Let $3 \leqslant i \leqslant n-1$. Since $\operatorname{rank} \psi((e_1 + \gamma e_2) \oslash e_i) \leqslant 2$ for every $\gamma \in \mathbb{F}$, it follows from (4.9) and Lemma 4.2 that either $y_i \in \langle v_i \rangle$ or $y_i \in \langle u \rangle$. Since $u \in \langle y_1 \rangle$, we have $y_i \in \langle v_i \rangle$. Setting $w_1 = u$, $w_2 = x$, and $w_i = v_i$ for $i = 3, \ldots, n$, we thus have $\{w_1, \ldots, w_n\}$ is linearly independent by (4.9). In view of (4.2), (4.3) and (4.8), we have

$$\psi(e_1 \oslash e_2) = \varsigma w_1 \oslash w_2 \quad \text{and} \quad \psi(e_1 \oslash e_n) = w_1 \oslash w_n, \tag{4.11}$$

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and for each $3 \leqslant i \leqslant n-1$, there exists a nonzero scalar $\zeta_i \in \mathbb{F}$ such that $\psi(e_1 \oslash e_i) = w_1 \oslash w_i$ and $\psi(e_2 \oslash e_i) = \zeta_i w_2 \oslash w_i$. Moreover, since $1 \leqslant \operatorname{rank} \psi((e_1+e_2)\oslash(e_i+e_j)) \leqslant 2$ for every distinct pair $3 \leqslant i, j \leqslant n-1$, we have $\zeta_i = \zeta_j$ for any distinct integers $3 \leqslant i, j \leqslant n-1$. Consequently, there exists a nonzero scalar $\zeta \in \mathbb{F}$ such that

$$\psi(e_1 \oslash e_i) = w_1 \oslash w_i \quad \text{and} \quad \psi(e_2 \oslash e_i) = \zeta w_2 \oslash w_i \tag{4.12}$$

for all $i = 3, \ldots, n-1$.

We next claim that for each $1 \leqslant i \leqslant \frac{n+1}{2}$, there exists a nonzero scalar $\mu_i \in \mathbb{F}$ such that

$$\psi(e_i^2) = \mu_i\, w_i^2. \tag{4.13}$$

Recall that $\mathcal{X}_i = e_i \oslash \langle e_1, e_2, e_i \rangle$ for $3 \leqslant i \leqslant \frac{n+1}{2}$. Then $\psi(\mathcal{X}_i)$ is a 3-dimensional linear space of bounded rank-two matrices of $\mathcal{SM}_m(\mathbb{F})$. In view (4.12), since w_1, w_2, w_i are linearly independent, it follows from Lemma 4.1 (b) that $\psi(\mathcal{X}_i)$ is of Form (II) in Lemma 3.1. Thus, $\psi(\mathcal{X}_i) \subseteq w_i \oslash \mathcal{M}_{m,1}(\mathbb{F})$ by Lemma 2.4. For each $3 \leqslant i \leqslant \frac{n+1}{2}$, there exists a nonzero vector $z_i \in \mathcal{M}_{m,1}(\mathbb{F})$ such that $\psi(e_i^2) = w_i \oslash z_i$. We shall show that $\theta_1 = 0$. Suppose not. In view of (4.9), we have $\{w_1, \theta_1 w_2 + \vartheta_1 w_1, w_i\}$ and $\{w_1, 2\theta_1 w_2 + 2\vartheta_1 w_1 + w_i, w_i\}$ are linearly independent sets. Since $\operatorname{rank} \psi(e_1^2 + \gamma e_i^2) \leqslant 2$ and $\operatorname{rank} \psi(e_1 \oslash (e_1 + e_i) + \gamma e_i^2) \leqslant 2$ for all $\gamma \in \mathbb{F}$, it follows from (4.10), (4.12) and Lemma 4.2 that

$$z_i \in \langle w_1 \rangle \quad \text{or} \quad z_i \in \langle \theta_1 w_2 + \vartheta_1 w_1 \rangle,$$

and

$$z_i \in \langle w_1 \rangle \quad \text{or} \quad z_i \in \langle 2\theta_1 w_2 + 2\vartheta_1 w_1 + w_i \rangle .$$

We thus have $z_i \in \langle w_1 \rangle$. Therefore, $\psi(e_i^2)$, $\psi(e_1 \oslash e_i)$ are linearly dependent, a contradiction. Hence, $\theta_1 = 0$. Thus, $\psi(e_1^2) \in \langle w_1^2 \rangle$ by (4.10). Similarly, we can show that $\theta_2 = 0$ in (4.10). Consequently, Claim (4.13) holds for $i = 1, 2$. We now consider $3 \leqslant i \leqslant \frac{n+1}{2}$. Since $\operatorname{rank} \psi(e_1 \oslash (e_1 + e_i) + \gamma e_i^2) \leqslant 2$ for every $\gamma \in \mathbb{F}$, we have $\operatorname{rank}(w_1 \oslash (\mu_1 w_1 + w_i) + \gamma z_i \oslash w_i) \leqslant 2$ for every $\gamma \in \mathbb{F}$. If $z_i \notin \langle w_1, w_i \rangle$, then, by Lemma 4.2, we have either $w_i \in \langle w_1 \rangle$ or $w_i \in \langle \mu_1 w_1 + w_i \rangle$. Since w_1, w_i are linearly independent, we obtain $\mu_1 = 0$, a contradiction. Therefore, $z_i \in \langle w_1, w_i \rangle$. Furthermore, since $\operatorname{rank} \psi(e_2 \oslash (e_2 + e_i) + \gamma e_i^2) \leqslant 2$ for all $\gamma \in \mathbb{F}$, in the same manner we can show that $z_i \in \langle w_2, w_i \rangle$. Hence, $z_i \in \langle w_1, w_i \rangle \cap \langle w_2, w_i \rangle = \langle w_i \rangle$. Accordingly, Claim (4.13) is proved.

Next, we consider $n \geqslant 6$. Let $3 \leqslant i \leqslant \frac{n+1}{2}$ and $i + 1 \leqslant j \leqslant n + 1 - i$. Recall that $\mathcal{X}_{ij} = e_i \oslash \langle e_1, e_2, e_i, e_j \rangle$. Since $\psi(\mathcal{X}_{ij})$ is a linear space of bounded rank-two matrices of $\mathcal{SM}_m(\mathbb{F})$ containing linearly independent elements $\psi(e_i \oslash e_j)$, $\psi(e_i^2)$, $\psi(e_1 \oslash e_i)$,

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$\psi(e_2 \oslash e_i)$, it follows from Lemmas 3.1 and 2.4 that $\psi(\mathcal{X}_{ij}) \subseteq w_i \oslash \mathcal{M}_{m,1}(\mathbb{F})$. Then there exists a nonzero vector $z_{ij} \in \mathcal{M}_{m,1}(\mathbb{F})$ such that $\psi(e_i \oslash e_j) = w_i \oslash z_{ij}$. On the other hand, $\psi(e_j \oslash \langle e_1, e_2, e_i\rangle)$ is a linear space of bounded rank-two matrices of $\mathcal{SM}_m(\mathbb{F})$ containing linearly independent elements $\psi(e_i \oslash e_j)$, $\psi(e_1 \oslash e_j)$, $\psi(e_2 \oslash e_j)$. Since w_1, w_2, w_j are linearly independent, it follows from Lemmas 3.1, 4.1 (b) and 2.4 that $\psi(e_j \oslash \langle e_1, e_2, e_i\rangle) \subseteq w_j \oslash \mathcal{M}_{m,1}(\mathbb{F})$. Then $\psi(e_i \oslash e_j) = w_j \oslash y_{ij}$ for some nonzero vector $y_{ij} \in \mathcal{M}_{m,1}(\mathbb{F})$. Therefore, $w_i \oslash z_{ij} = \psi(e_i \oslash e_j) = w_j \oslash y_{ij}$, and so $\langle z_{ij}\rangle = \langle w_j\rangle$ and $\langle y_{ij}\rangle = \langle w_i\rangle$ by Lemma 2.2 (b). Consequently, for each $3 \leqslant i \leqslant \frac{n+1}{2}$ and $i+1 \leqslant j \leqslant n+1-i$, there exists a nonzero scalar $\eta_{ij} \in \mathbb{F}$ such that

$$\psi(e_i \oslash e_j) = \eta_{ij} w_i \oslash w_j. \tag{4.14}$$

After composing the map: $A \mapsto \mu_1^{-1}A$ for $A \in \mathcal{SM}_m(\mathbb{F})$, if necessary, we have

$$\psi(e_1^2) = w_1^2 \quad \text{and} \quad \psi(e_1 \oslash e_i) = \mu_1^{-1} w_1 \oslash w_i \tag{4.15}$$

for $i = 3, \dots, n$, and for simplicity of notation, we abbreviate $\mu_1^{-1}\varsigma$ to ς in (4.11), $\mu_1^{-1}\zeta$ to ζ in (4.12), $\mu_1^{-1}\mu_i$ to μ_i in (4.13) for $2 \leqslant i \leqslant \frac{n+1}{2}$, and $\mu_1^{-1}\eta_{ij}$ to η_{ij} in (4.14) for $3 \leqslant i \leqslant \frac{n+1}{2}$ and $i+1 \leqslant j \leqslant n+1-i$. Since $\operatorname{rank}\psi((e_1+e_2)^2 + e_k^2) \leqslant 2$ and $\operatorname{rank}\psi((e_i+e_k)^2 + e_j^2) \leqslant 2$ for any distinct integers $1 \leqslant i, j \leqslant 2$ and $3 \leqslant k \leqslant \frac{n+1}{2}$, it follows from (4.11), (4.12), (4.13) and (4.15) that $\mu_2 = \varsigma^2$, $\zeta^2 = (\mu_1^{-1}\varsigma)^2$ and $\mu_i = (\mu_1^{-1})^2$ for $3 \leqslant i \leqslant \frac{n+1}{2}$. Moreover, in view of (4.12), (4.13), (4.14) and (4.15), we have $\psi((e_1+e_i)\oslash e_j + (e_1+e_i)^2) = \mu_1^{-1}w_1 \oslash w_j + \eta_{ij} w_i \oslash w_j + w_1^2 + (\mu_1^{-1})^2 w_i^2 + \mu_1^{-1} w_1 \oslash w_i$ is of rank bounded above by two for every $3 \leqslant i \leqslant \frac{n+1}{2}$ and $i < j \leqslant n+1-i$, and hence,

$$0 = \det \begin{bmatrix} \mu_1^{-1} & \mu_1^{-1} & 1 \\ \eta_{ij} & (\mu_1^{-1})^2 & \mu_1^{-1} \\ 0 & \eta_{ij} & \mu_1^{-1} \end{bmatrix} = ((\mu_1^{-1})^2 - \eta_{ij})^2 \quad \Rightarrow \quad \eta_{ij} = (\mu_1^{-1})^2$$

for every $3 \leqslant i \leqslant \frac{n+1}{2}$ and $i < j \leqslant n+1-i$. Also, since $\zeta^2 = (\mu_1^{-1}\varsigma)^2$, we have either $\zeta = \mu_1^{-1}\varsigma$ or $\zeta = -\mu_1^{-1}\varsigma$. Suppose that $\zeta = -\mu_1^{-1}\varsigma$. Then

$$\psi((e_1+e_2+e_3)^2 - e_3^2) = w_1^2 + \varsigma^2 w_2^2 + \varsigma w_1 \oslash w_2 + \mu_1^{-1} w_1 \oslash w_3 + (-\mu_1^{-1}\varsigma) w_2 \oslash w_3$$

is of rank three, a contradiction. So $\zeta = \mu_1^{-1}\varsigma$. Consequently, by (4.11), (4.12), (4.13), (4.14) and (4.15) that $\psi(e_i^2) = (\alpha_i e_i)^2$ for all $1 \leqslant i \leqslant \frac{n+1}{2}$, and $\psi(e_i \oslash e_j) = (\alpha_i w_i) \oslash (\alpha_j w_j)$ for all $1 \leqslant i \leqslant \frac{n+1}{2}$ and $i < j \leqslant n+1-i$, where $\alpha_1 = 1$, $\alpha_2 = \varsigma$ and $\alpha_i = \mu_1^{-1}$ for $i = 3, \dots, n$. Let $P \in \mathcal{M}_{m,n}(\mathbb{F})$ be the matrix defined by $Pe_i = \alpha_i w_i$ for every $i = 1, \dots, n$. Evidently, P is of rank n since $\{w_1, \dots, w_n\}$ is linearly independent. By the linearity of ψ, we conclude that

$$\psi(A) = \lambda PAP^+ \quad \text{for all } A \in \mathcal{ST}_n(\mathbb{F}),$$

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where $\lambda = \mu_1^{-1} \in \mathbb{F}$ is nonzero. We are done.

Case II: $n = 4$. Let $\mathcal{S}_1 = e_1 \oslash \langle e_1, e_2, e_3, e_4\rangle$ and $\mathcal{S}_2 = e_2 \oslash \langle e_1, e_2, e_3\rangle$. Then $\psi(\mathcal{S}_1)$ and $\psi(\mathcal{S}_2)$ are 4-dimensional and 3-dimensional spaces of bounded rank-two matrices of $\mathcal{SM}_m(\mathbb{F})$, respectively. By Lemma 3.1, there exist a nonzero vector $u \in \mathcal{M}_{m,1}(\mathbb{F})$ and linearly independent vectors $v_1, v_2, v_3, v_4 \in \mathcal{M}_{m,1}(\mathbb{F})$ such that

$$\psi(e_1^2) = u \oslash v_1 \quad \text{and} \quad \psi(e_1 \oslash e_i) = u \oslash v_i \ \text{ for } i = 2, 3, 4, \tag{4.16}$$

and $\psi(\mathcal{S}_2)$ is either of Form (I) or Form (II) in Lemma 3.1. We claim that $\psi(\mathcal{S}_2)$ is of Form (II). Suppose to the contrary that $\psi(\mathcal{S}_2)$ is of Form (I). We argue in the following two cases:

Case II-1: $\langle v_2\rangle \neq \langle u\rangle$. By Lemma 4.1 (a), we obtain $\psi(\mathcal{S}_2) = \langle u^2, v_2^2, u \oslash v_2\rangle$. Let $\psi(e_2^2) = \mu_1 u \oslash v_2 + \mu_2 u^2 + \mu_3 v_2^2$ and $\psi(e_2 \oslash e_3) = \eta_1 u \oslash v_2 + \eta_2 u^2 + \eta_3 v_2^2$ for some $\mu_i, \eta_i \in \mathbb{F}$, $i = 1, 2, 3$. Suppose that $\mu_3 \neq 0$. Note that $\operatorname{rank} \psi(e_1^2 + e_2^2) \leqslant 2$ implies $v_1 \in \langle u, v_2\rangle$. Since v_1, v_2, v_3 are linearly independent, it follows that u, v_2, v_3 are linearly independent. In view of (4.16), we have $\psi(e_1^2) = \lambda_1 u \oslash v_2 + \lambda_2 u^2$ for some scalars $\lambda_1, \lambda_2 \in \mathbb{F}$ with $\lambda_2 \neq 0$. We set

$$\zeta = \begin{cases} 1 & \text{if } \eta_3 = 0, \\ \eta_3^{-1}\mu_3 & \text{if } \eta_3 \neq 0. \end{cases}$$

Then $\zeta \neq 0$ and $\zeta\eta_3 + \mu_3 \neq 0$, and

$$\begin{aligned} \psi(\zeta(e_1 + e_2) \oslash e_3 + (e_1 + e_2)^2) = {} & \zeta u \oslash v_3 + (\zeta\eta_3 + \mu_3)v_2^2 \\ & + (\zeta\eta_1 + \lambda_1 + \mu_1 + 1)u \oslash v_2 + (\zeta\eta_2 + \lambda_2 + \mu_2)u^2 \end{aligned}$$

is of rank three, a contradiction. Hence, $\mu_3 = 0$. Since $\psi(e_2 \oslash e_1)$, $\psi(e_2^2)$, $\psi(e_2 \oslash e_3)$ are linearly independent, it follows that $\eta_3 \neq 0$. By a similar argument, with $\psi(e_1^2)$ replaced by $\psi(e_2 \oslash e_3)$, to obtain $\psi(\zeta'(e_1 + e_2) \oslash e_3 + (e_1 + e_2)^2)$ is of rank three for $\zeta' \in \mathbb{F}$, which is impossible.

Case II-2: $\langle v_2\rangle = \langle u\rangle$. By (4.16), we have $\psi(e_1 \oslash e_2) = \alpha u^2$ for some $\alpha \in \mathbb{F}\backslash\{0\}$ and v_1, u, v_3, v_4 are linearly independent. Let $\psi(\mathcal{S}_2) = \langle x^2, v^2, x \oslash v\rangle$ for some linearly independent vectors $x, v \in \mathcal{M}_{m,1}(\mathbb{F})$. Since $\psi(e_1 \oslash e_2) \in \psi(\mathcal{S}_2)$, it follows that $\alpha u^2 = \theta_1 x^2 + \theta_2 v^2 + \theta_3 x \oslash v$ for some $\theta_1, \theta_2, \theta_3 \in \mathbb{F}$ with $(\theta_1, \theta_2, \theta_3) \neq 0$. We now show that $u \in \langle x, v\rangle$. Suppose to the contrary that $u \notin \langle x, v\rangle$. If $\theta_3 = 0$, then $\alpha u^2 - \theta_1 x^2 - \theta_2 v^2 = 0$ implies that $\alpha = \theta_1 = \theta_2 = 0$, a contradiction. Thus, $\theta_3 \neq 0$, and so $\alpha u^2 - \theta_3 x \oslash v = \theta_1 x^2 + \theta_2 v^2$, which is an impossibility. We thus have $u \in \langle x, v\rangle$. Since x, v are linearly independent, we may assume without loss of generality that u, v are linearly independent. Then $\langle u, v\rangle = \langle x, v\rangle$, so $\psi(\mathcal{S}_2) = \langle u^2, v^2, u \oslash v\rangle$ by Lemma 4.1 (a). Let $\psi(e_2^2) = a_1 u \oslash v + a_2 u^2 + a_3 v^2$ for some $a_1, a_2, a_3 \in \mathbb{F}$. Suppose that $a_3 \neq 0$. Since $\psi(e_1^2 + e_2^2) = u \oslash v_1 + a_1 u \oslash v + a_2 u^2 + a_3 v^2$ has rank bounded

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above by two, it follows that $v \in \langle u, v_1 \rangle$. Then $\langle u^2, v^2, u \oslash v \rangle = \langle u^2, v_1^2, u \oslash v_1 \rangle$, and therefore, $\psi(e_2^2) = b_1 u \oslash v_1 + b_2 u^2 + b_3 v_1^2$ for some $b_1, b_2, b_3 \in \mathbb{F}$ with $b_3 \neq 0$, because u, v are linearly independent. Let $\psi(e_2 \oslash e_3) = c_1 u \oslash v_1 + c_2 u^2 + c_3 v_1^2$ for some scalars $c_1, c_2, c_3 \in \mathbb{F}$, and let

$$\beta = \begin{cases} 1 & \text{if } c_3 = 0, \\ c_3^{-1} b_3 & \text{if } c_3 \neq 0. \end{cases}$$

Then $\beta \neq 0$ and $\beta c_3 + b_3 \neq 0$, and

$$\psi(\beta(e_1 + e_2) \oslash e_3 + (e_1 + e_2)^2) = \beta u \oslash v_3 + (\beta c_3 + b_3) v_1^2 + (\beta c_1 + b_1 + 1) u \oslash v_1 + (\beta c_2 + b_2 + \alpha) u^2$$

is of rank three, a contradiction. Then $a_3 = 0$. Since $\psi(e_2^2), \psi(e_2 \oslash e_1), \psi(e_2 \oslash e_3)$ are linearly independent, it follows that $\psi(e_2 \oslash e_3) = d_1 u \oslash v + d_2 u^2 + d_3 v^2$ for some $d_1, d_2, d_3 \in \mathbb{F}$ with $d_3 \neq 0$. Note that $\psi((e_1+e_2) \oslash e_3) = u \oslash v_3 + d_1 u \oslash v + d_2 u^2 + d_3 v^2$ has rank bounded above by two implies $v \in \langle u, v_3 \rangle$. So $\langle u^2, v^2, u \oslash v \rangle = \langle u^2, v_3^2, u \oslash v_3 \rangle$. We now apply a similar argument as above, with v_1 replaced by v_3, to obtain $\psi(\beta'(e_1 + e_2) \oslash e_3 + (e_1 + e_2)^2)$ is of rank three for some $\beta' \in \mathbb{F} \setminus \{0\}$. This leads to a contradiction.

Accordingly, $\psi(\mathcal{S}_2)$ is of Form (II). Then there exists a nonzero vector $x \in \mathcal{M}_{m,1}(\mathbb{F})$ such that

$$\psi(e_2^2) = x \oslash y_2 \quad \text{and} \quad \psi(e_2 \oslash e_i) = x \oslash y_i \ \text{ for } i = 1, 3 \tag{4.17}$$

for some linearly independent vectors $y_1, y_2, y_3 \in \mathcal{M}_{m,1}(\mathbb{F})$. We divide into two subcases:

Case A: $\langle x \rangle = \langle u \rangle$. It follows from (4.16) and (4.17), together with the linearity of ψ, that there exists a linear mapping $\varphi : \mathcal{ST}_4(\mathbb{F}) \to \mathcal{M}_{m,1}(\mathbb{F})$ such that

$$\psi(A) = u \oslash \varphi(A) \quad \text{for all } A \in \mathcal{ST}_4(\mathbb{F}),$$

where $\varphi(A) \neq 0$ for every nonzero bounded rank-two matrix $A \in \mathcal{ST}_4(\mathbb{F})$. So (4.1) holds true.

Case B: $\langle x \rangle \neq \langle u \rangle$. Note that $x \oslash y_1 = \psi(e_1 \oslash e_2) = u \oslash v_2$ implies $v_2 = \varsigma x$ and $y_1 = \varsigma u$ for some nonzero scalar $\varsigma \in \mathbb{F}$. Thus,

$$\psi(e_1 \oslash e_2) = \varsigma u \oslash x. \tag{4.18}$$

By a similar argument as in (4.9), we show that $\{u, x, v_3, v_4\}$ are linearly independent. Setting $w_1 = u$, $w_2 = \varsigma x$, $w_3 = v_3$ and $w_4 = v_4$, we thus have $\{w_1, w_2, w_3, w_4\}$ is linearly independent. In view of (4.16), (4.17) and (4.18), we have

$$\psi(e_1^2) = w_1 \oslash v_1 \quad \text{and} \quad \psi(e_1 \oslash e_i) = w_1 \oslash w_i, \ i = 2, 3, 4, \tag{4.19}$$

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and

$$\psi(e_2^2) = w_2 \oslash z_2 \quad \text{and} \quad \psi(e_2 \oslash e_3) = w_2 \oslash z_3, \tag{4.20}$$

where $z_i = \varsigma^{-1} y_i$ for $i = 2, 3$. We claim that

$$v_1 \in \langle w_1, w_2 \rangle \quad \text{and} \quad z_2 \in \langle w_1, w_2 \rangle . \tag{4.21}$$

We will only verify $v_1 \in \langle w_1, w_2 \rangle$ as the second statement can be proved similarly. Suppose, contrary to our claim, that $v_1 \notin \langle w_1, w_2 \rangle$. Since $\operatorname{rank} \psi(e_1^2 + \gamma e_2^2) \leqslant 2$ for all $\gamma \in \mathbb{F}$, it follows from (4.19), (4.20) and Lemma 4.2 that $z_2 \in \langle w_1 \rangle$ or $z_2 \in \langle v_1 \rangle$. Since y_1, y_2 are linearly independent and $w_1 \in \langle y_1 \rangle$, we conclude that $z_2 = \lambda v_1$ for some nonzero $\lambda \in \mathbb{F}$. Consequently, $\psi((e_1 + e_2)^2) = w_1 \oslash v_1 + w_1 \oslash w_2 + \lambda w_2 \oslash v_1$ is of rank three, a contradiction. Claim (4.21) is proved. By (4.19) and (4.20),

$$\psi(e_1^2) = \lambda_1 w_1^2 + \lambda_2 w_1 \oslash w_2 \quad \text{and} \quad \psi(e_2^2) = \lambda_3 w_2^2 + \lambda_4 w_1 \oslash w_2 \tag{4.22}$$

for some scalars $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{F}$ with $\lambda_1, \lambda_3 \neq 0$. Moreover, since $\operatorname{rank} \psi(e_1 \oslash e_3 + \gamma e_2 \oslash e_3) \leqslant 2$ for all $\gamma \in \mathbb{F}$, it follows from (4.19), (4.20) and Lemma 4.2 that $z_3 \in \langle w_1 \rangle$ or $z_3 \in \langle w_3 \rangle$. Since y_1, y_3 are linearly independent, we have $z_3 = \xi w_3$ for some nonzero scalar $\xi \in \mathbb{F}$. By (4.20), we have

$$\psi(e_2 \oslash e_3) = \xi w_2 \oslash w_3. \tag{4.23}$$

In view of (4.19), (4.20), (4.22) and (4.23), we see that

$$\begin{aligned}\psi((\gamma e_1 + e_2)^2 + (\gamma e_1 + e_2) \oslash e_3) = \gamma^2 \lambda_1 w_1^2 + \lambda_3 w_2^2 + (\gamma^2 \lambda_2 + \gamma + \lambda_4) w_1 \oslash w_2 \\ + \gamma w_1 \oslash w_3 + \xi w_2 \oslash w_3\end{aligned}$$

has rank bounded above by two for all $\gamma \in \mathbb{F}$. It follows that

$$\begin{aligned} 0 &= \det \begin{bmatrix} \gamma & \gamma^2\lambda_2 + \gamma + \lambda_4 & \gamma^2 \lambda_1 \\ \xi & \lambda_3 & \gamma^2\lambda_2 + \gamma + \lambda_4 \\ 0 & \xi & \gamma \end{bmatrix} \\ &= -\gamma\big(2\lambda_2 \xi \gamma^2 - (\lambda_3 - \xi(2 - \xi\lambda_1))\gamma + 2\lambda_4 \xi\big) \end{aligned} \tag{4.24}$$

for all $\gamma \in \mathbb{F}$. Since $\mathbb{F}$ is a field of characteristic not two, we conclude immediately from (4.24) that $\lambda_3 = \xi(2 - \xi\lambda_1)$ with $\xi\lambda_1 \neq 2$, and $\lambda_4 = -\lambda_2$. Moreover, if $|\mathbb{F}| \geqslant 4$, then we can deduce from (4.24) that $\lambda_2 = 0 = \lambda_4$.

Let $P \in \mathcal{M}_{m,4}(\mathbb{F})$ be the matrix defined by $Pe_i = w_i$ for $i = 1, 3, 4$, and $Pe_2 = \xi w_2$. Clearly, P is of full rank. Denote $\alpha = \xi^{-1}$, $\beta = \lambda_1$ and $\theta = \lambda_2 \xi^{-1}$. Then $\alpha, \beta \neq 0$ and $2\alpha - \beta = \lambda_3 \xi^{-2} \neq 0$. By (4.19), (4.20), (4.22), (4.23) and the linearity

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of ψ, we obtain

$$\psi(A) = P \begin{bmatrix} a_{11} & a_{12} & \alpha a_{13} + \theta(a_{14} - a_{23}) & \beta a_{14} \\ 0 & a_{22} & (2\alpha - \beta) a_{23} & \alpha a_{13} + \theta(a_{14} - a_{23}) \\ 0 & 0 & a_{22} & a_{12} \\ 0 & 0 & 0 & a_{11} \end{bmatrix} P^+$$

for all $A = (a_{ij}) \in \mathcal{ST}_4(\mathbb{F})$, where θ is nonzero only if $|\mathbb{F}| = 3$. We are done.

Case III: $n = 3$. Let $\mathcal{W} = e_1 \oslash \langle e_1, e_2, e_3 \rangle$. Then $\psi(\mathcal{W})$ is a 3-dimensional linear space of bounded rank-two matrices of $\mathcal{SM}_m(\mathbb{F})$. By Lemma 3.1, $\psi(\mathcal{W})$ is either of Form (I) or Form (II) in Lemma 3.1. Then either

$$\psi(\mathcal{W}) \subseteq u \oslash \mathcal{M}_{m,1}(\mathbb{F}) \tag{4.25}$$

for some nonzero vector $u \in \mathcal{M}_{m,1}(\mathbb{F})$; or

$$\psi(\mathcal{W}) = \left\langle u^2, v^2, u \oslash v \right\rangle \tag{4.26}$$

for some linearly independent vectors $u, v \in \mathcal{M}_{m,1}(\mathbb{F})$. We argue in the following two cases:

Case III-1: $\psi(e_2^2) \in \psi(\mathcal{W})$. We consider the following two subcases.

If (4.25) holds, then $\operatorname{Im}\psi \subseteq u \oslash \mathcal{M}_{m,1}(\mathbb{F})$. We thus obtain a linear mapping $\varphi : \mathcal{ST}_3(\mathbb{F}) \to \mathcal{M}_{m,1}(\mathbb{F})$ such that

$$\psi(A) = u \oslash \varphi(A) \quad \text{for all } A \in \mathcal{ST}_3(\mathbb{F}),$$

where $\varphi(A) \neq 0$ for every nonzero bounded rank-two matrix $A \in \mathcal{ST}_3(\mathbb{F})$. Hence, (4.1) holds.

If (4.26) holds, then $\operatorname{Im}\psi = \left\langle u^2, v^2, u \oslash v \right\rangle$. So, for each $A \in \mathcal{ST}_3(\mathbb{F})$, there exists a unique ordered triple $(\alpha_A, \beta_A, \gamma_A) \in \mathbb{F}^3$ such that $\psi(A) = \alpha_A u^2 + \beta_A v^2 + \gamma_A u \oslash v$. We define the linear mapping $\phi : \mathcal{ST}_3(\mathbb{F}) \to \mathbb{F}^3$ such that

$$\phi(A) = (\alpha_A, \beta_A, \gamma_A) \quad \text{for all } A \in \mathcal{ST}_3(\mathbb{F}).$$

Note that $\operatorname{Im}\psi = \left\langle u^2, v^2, u \oslash v \right\rangle$ and ψ preserves nonzero bounded rank-two matrices implies ϕ is surjective and $\phi(A) \neq 0$ for every nonzero bounded rank-two matrix $A \in \mathcal{ST}_3(\mathbb{F})$. Let $P \in \mathcal{M}_{m,2}(\mathbb{F})$ be the matrix defined by $Pe_1 = u$ and $Pe_2 = v$. Then P is of full rank and

$$\psi(A) = P \begin{bmatrix} \phi(A)_3 & \phi(A)_1 \\ \phi(A)_2 & \phi(A)_3 \end{bmatrix} P^+ \quad \text{for all } A \in \mathcal{ST}_3(\mathbb{F}),$$

where $\phi(A)_i$ denotes the i-th component of $\phi(A) \in \mathbb{F}^3$. We are done.

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Case III-2: $\psi(e_2^2) \notin \psi(\mathcal{W})$. Let $\mathcal{W}_1 = \langle e_1^2, e_2^2, e_1 \oslash e_2 \rangle$. Note that $\psi(\mathcal{W}_1)$ is a 3-dimensional linear space of bounded rank-two matrices of $\mathcal{SM}_m(\mathbb{F})$. By Lemma 3.1, we have either

$$\psi(\mathcal{W}_1) \subseteq x \oslash \mathcal{M}_{m,1}(\mathbb{F}) \tag{4.27}$$

for some nonzero vector $x \in \mathcal{M}_{m,1}(\mathbb{F})$; or

$$\psi(\mathcal{W}_1) = \langle x^2, y^2, x \oslash y \rangle \tag{4.28}$$

for some linearly independent vectors $x, y \in \mathcal{M}_{m,1}(\mathbb{F})$. We need to consider the following four subcases:

Case III-2-A: (4.25) and (4.27) hold. Since $\psi(\mathcal{W}) \subseteq u \oslash \mathcal{M}_{m,1}(\mathbb{F})$ contains two linearly independent elements $\psi(e_1^2) = x \oslash y_1$ and $\psi(e_1 \oslash e_2) = x \oslash y_2$ for some $y_1, y_2 \in \mathcal{M}_{m,1}(\mathbb{F})$, it follows from Lemma 2.4 that $\langle x \rangle = \langle u \rangle$. Thus, $\operatorname{Im}\psi \subseteq u \oslash \mathcal{M}_{m,1}(\mathbb{F})$, and hence, (4.1) holds true.

Case III-2-B: (4.26) and (4.28) hold. By (4.26) and (4.28), we see that

$$a_1 u^2 + a_2 v^2 + a_3 u \oslash v = \psi(e_1^2) = b_1 x^2 + b_2 y^2 + b_3 x \oslash y$$

is of rank one or rank two for some nonzero elements $(a_i), (b_i) \in \mathbb{F}^3$, and

$$c_1 u^2 + c_2 v^2 + c_3 u \oslash v = \psi(e_1 \oslash e_2) = d_1 x^2 + d_2 y^2 + d_3 x \oslash y$$

is of rank one or rank two for some nonzero elements $(c_i), (d_i) \in \mathbb{F}^3$. Therefore,

$$u \cdot (a_1 u^+ + a_3 v^+) + v \cdot (a_2 v^+ + a_3 u^+) = x \cdot (b_1 x^+ + b_3 y^+) + y \cdot (b_2 y^+ + b_3 x^+),$$
$$u \cdot (c_1 u^+ + c_3 v^+) + v \cdot (c_2 v^+ + c_3 u^+) = x \cdot (d_1 x^+ + d_3 y^+) + y \cdot (d_2 y^+ + d_3 x^+).$$

Since $\psi(e_1^2)$, $\psi(e_1 \oslash e_2)$ are linearly independent, it follows that, in each case, we obtain $\langle u, v \rangle = \langle x, y \rangle$. By Lemma 4.1 (a), $\langle u^2, v^2, u \oslash v \rangle = \langle x^2, y^2, x \oslash y \rangle$, and so $\psi(e_2^2) \in \psi(\mathcal{W})$, a contradiction.

Case III-2-C: (4.25) and (4.28) hold. Let $\psi(e_1^2) = u \oslash z_1$, $\psi(e_1 \oslash e_2) = u \oslash z_2$ and $\psi(e_1 \oslash e_3) = u \oslash z_3$ for some linearly independent vectors $z_1, z_2, z_3 \in \mathcal{M}_{m,1}(\mathbb{F})$. By (4.28), we get

$$u \oslash z_1 = a_1 x^2 + a_2 y^2 + a_3 x \oslash y,$$
$$u \oslash z_2 = b_1 x^2 + b_2 y^2 + b_3 x \oslash y$$

for some nonzero elements $(a_i), (b_i) \in \mathbb{F}^3$. Thus,

$$u \cdot z_1^+ + z_1 \cdot u^+ = x \cdot (a_1 x^+ + a_3 y^+) + y \cdot (a_2 y^+ + a_3 x^+), \tag{4.29}$$

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$$u \cdot z_2^+ + z_2 \cdot u^+ = x \cdot (b_1 x^+ + b_3 y^+) + y \cdot (b_2 y^+ + b_3 x^+). \tag{4.30}$$

We consider the following four subcases:

Subcase III-2-C-1: $\operatorname{rank}\psi(e_1^2) = \operatorname{rank}\psi(e_1 \oslash e_2) = 1$. Then $\langle z_1\rangle = \langle u\rangle = \langle z_2\rangle$. This contradicts the fact that z_1, z_2 are linearly independent.

Subcase III-2-C-2: $\operatorname{rank}\psi(e_1^2) = \operatorname{rank}\psi(e_1 \oslash e_2) = 2$. Then $\{u, z_i\}$ is linearly independent for $i = 1, 2$. It follows from (4.29) and (4.30) that $\langle u, z_1\rangle = \langle x, y\rangle = \langle u, z_2\rangle$. Since $\operatorname{rank}\psi(e_1 \oslash e_2) = 2$ and $\{z_1, z_2\}$ is linearly independent, $z_2 = \mu_1 u + \mu_2 z_1$ for some nonzero scalars $\mu_1, \mu_2 \in \mathbb{F}$. We thus have $\{z_1, u, z_3\}$ is linearly independent and $\psi(e_1 \oslash e_2) = \eta_1 u^2 + \eta_2 u \oslash z_1$, with $\eta_1 = 2\mu_1$ and $\eta_2 = \mu_2$ nonzero. Since $\left\langle x^2, y^2, x \oslash y\right\rangle = \left\langle u^2, z_1^2, u \oslash z_1\right\rangle$, we have $\psi(e_2^2) = \lambda_1 u^2 + \lambda_2 u \oslash z_1 + \lambda_3 z_1^2$ for some $(\lambda_i) \in \mathbb{F}^3$ with $\lambda_3 \neq 0$. Let $P \in \mathcal{M}_{3,m}(\mathbb{F})$ be the matrix defined by $Pe_1 = u$, $Pe_2 = z_1$ and $Pe_3 = z_3$. Then P is of full rank, and $\psi(e_1^2) = P(e_1 \oslash e_2)P^+$, $\psi(e_1 \oslash e_2) = P(\eta_1 e_1^2 + \eta_2 e_1 \oslash e_2)P^+$, $\psi(e_1 \oslash e_3) = P(e_1 \oslash e_3)P^+$ and $\psi(e_2^2) = P(\lambda_1 e_1^2 + \lambda_2 e_1 \oslash e_2 + \lambda_3 e_2^2)P^+$. By the linearity of ψ, we obtain

$$\psi(A) = P \begin{bmatrix} a_{11} & \eta_2 a_{12} + a_{13} + \lambda_2 a_{22} & \eta_1 a_{12} + \lambda_1 a_{22} \\ 0 & \lambda_3 a_{22} & \eta_2 a_{12} + a_{13} + \lambda_2 a_{22} \\ 0 & 0 & a_{11} \end{bmatrix} P^+$$

for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$. We are done.

Subcase III-2-C-3: $\operatorname{rank}\psi(e_1^2) = 1$ and $\operatorname{rank}\psi(e_1 \oslash e_2) = 2$. Then $\langle z_1\rangle = \langle u\rangle$, and so $\psi(e_1^2) = \eta u^2$ for some nonzero scalar $\eta \in \mathbb{F}$. Note that $\{u, z_2, z_3\}$ is linearly independent. By (4.30), we have $\langle u, z_2\rangle = \langle x, y\rangle$, and so $\left\langle u^2, z_2^2, u \oslash z_2\right\rangle = \left\langle x^2, y^2, x \oslash y\right\rangle$ by Lemma 4.1 (a). Thus, $\psi(e_2^2) = \lambda_1 u^2 + \lambda_2 u \oslash z_2 + \lambda_3 z_2^2$ for some $(\lambda_i) \in \mathbb{F}^3$ with $\lambda_3 \neq 0$. Let $P \in \mathcal{M}_{3,m}(\mathbb{F})$ be the matrix defined by $Pe_1 = u$, $Pe_2 = z_2$ and $Pe_3 = z_3$. Then P is of full rank and

$$\psi(A) = P \begin{bmatrix} a_{11} & a_{12} + \lambda_2 a_{22} & \eta a_{13} + \lambda_1 a_{22} \\ 0 & \lambda_3 a_{22} & a_{12} + \lambda_2 a_{22} \\ 0 & 0 & a_{11} \end{bmatrix} P^+$$

for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$. We are done.

Subcase III-2-C-4: $\operatorname{rank}\psi(e_1^2) = 2$ and $\operatorname{rank}\psi(e_1 \oslash e_2) = 1$. Then $\langle z_2\rangle = \langle u\rangle$ and $\psi(e_1 \oslash e_2) = \eta u^2$ for some nonzero scalar $\eta \in \mathbb{F}$. So $\{z_1, u, z_3\}$ is linearly independent. By (4.29), we conclude that $\left\langle u^2, z_1^2, u \oslash z_1\right\rangle = \left\langle x^2, y^2, x \oslash y\right\rangle$. Thus, $\psi(e_2^2) = \lambda_1 u^2 + \lambda_2 u \oslash z_1 + \lambda_3 z_1^2$ for some $(\lambda_i) \in \mathbb{F}^3$ with $\lambda_3 \neq 0$. Let $P \in \mathcal{M}_{3,m}(\mathbb{F})$ be the matrix

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defined by $Pe_1 = u$, $Pe_2 = z_1$ and $Pe_3 = z_3$. Then P is of full rank and

$$\psi(A) = P \begin{bmatrix} a_{11} & a_{13} + \lambda_2 a_{22} & \eta a_{12} + \lambda_1 a_{22} \\ 0 & \lambda_3 a_{22} & a_{13} + \lambda_2 a_{22} \\ 0 & 0 & a_{11} \end{bmatrix} P^+$$

for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$. We are done.

Case III-2-D: (4.26) and (4.27) hold. Let $\tau : \mathcal{ST}_3(\mathbb{F}) \to \mathcal{ST}_3(\mathbb{F})$ be the bijective linear mapping defined by

$$\tau(A) = \begin{bmatrix} a_{22} & a_{12} & a_{13} \\ 0 & a_{11} & a_{12} \\ 0 & 0 & a_{22} \end{bmatrix} \quad \text{for all } A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F}).$$

It is easily seen that τ is a bounded rank-two linear preserver such that $\tau(\mathcal{W}) = \mathcal{W}_1$ and $\tau(\mathcal{W}_1) = \mathcal{W}$. It follows from (4.26) and (4.27) that

$$(\psi \circ \tau)(\mathcal{W}) = \psi(\mathcal{W}_1) \subseteq x \oslash \mathcal{M}_{m,1}(\mathbb{F}) \quad \text{and} \quad (\psi \circ \tau)(\mathcal{W}_1) = \psi(\mathcal{W}) = \left\langle u^2, v^2, u \oslash v \right\rangle.$$

We then infer by similar arguments as in Subcase III-2-C and conclude that ψ takes one of the following forms: there exists a full rank matrix $P \in \mathcal{M}_{3,m}(\mathbb{F})$ such that

$$\psi(A) = P \begin{bmatrix} a_{22} & \eta_2 a_{12} + a_{13} + \lambda_2 a_{11} & \eta_1 a_{12} + \lambda_1 a_{11} \\ 0 & \lambda_3 a_{11} & \eta_2 a_{12} + a_{13} + \lambda_2 a_{11} \\ 0 & 0 & a_{22} \end{bmatrix} P^+$$

for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$, where $\lambda_1, \lambda_2, \lambda_3, \eta_1, \eta_2 \in \mathbb{F}$ with $\lambda_3, \eta_1, \eta_2 \neq 0$; or

$$\psi(A) = P \begin{bmatrix} a_{22} & a_{12} + \lambda_2 a_{11} & \eta a_{13} + \lambda_1 a_{11} \\ 0 & \lambda_3 a_{11} & a_{12} + \lambda_2 a_{11} \\ 0 & 0 & a_{22} \end{bmatrix} P^+$$

for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$, where $\lambda_1, \lambda_2, \lambda_3, \eta \in \mathbb{F}$ with $\lambda_3, \eta \neq 0$; or

$$\psi(A) = P \begin{bmatrix} a_{22} & a_{13} + \lambda_2 a_{11} & \eta a_{12} + \lambda_1 a_{11} \\ 0 & \lambda_3 a_{11} & a_{13} + \lambda_2 a_{11} \\ 0 & 0 & a_{22} \end{bmatrix} P^+$$

for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$, where $\lambda_1, \lambda_2, \lambda_3, \eta \in \mathbb{F}$ with $\lambda_3, \eta \neq 0$. $\square$

By Theorem 4.3, Lemma 2.3 (a) and (b) (i), and Lemma 2.6, we obtain a classification of bounded rank-two linear preservers between per-symmetric triangular matrix spaces over a field of characteristic not two.

Corollary 4.4. *Let $\mathbb{F}$ be a field of characteristic not two and m, n be integers such that $m, n \geqslant 3$. Then $\psi : \mathcal{ST}_n(\mathbb{F}) \to \mathcal{ST}_m(\mathbb{F})$ is a bounded rank-two linear preserver if and only if $m \geqslant n$ and ψ is of one of the following forms:*

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(i) *There exist a nonzero vector* $u \in \mathcal{U}_{p,m}$ *and a linear mapping* $\varphi : \mathcal{ST}_n(\mathbb{F}) \to \mathcal{U}_{q,m}$*, with* $1 \leqslant p \leqslant m+1-q \leqslant m$*, such that*

$$\psi(A) = u \oslash \varphi(A) \quad \text{for all } A \in \mathcal{ST}_n(\mathbb{F}),$$

where $\varphi(A) \neq 0$ *for every nonzero bounded rank-two matrix* $A \in \mathcal{ST}_n(\mathbb{F})$*.*

(ii) *There exist a full rank matrix* $P \in \mathcal{M}_{m,n}(\mathbb{F})$ *and a nonzero* $\lambda \in \mathbb{F}$ *such that*

$$\psi(A) = \lambda PAP^{+} \quad \text{for all } A \in \mathcal{ST}_n(\mathbb{F}),$$

where $Pe_i \in \mathcal{U}_{p_i,m} \backslash \mathcal{U}_{p_i-1,m}$ *for* $i = 1, \ldots, n$ *such that* $1 \leqslant p_i \leqslant \frac{m+1}{2}$ *for every* $1 \leqslant i \leqslant \frac{n+1}{2}$*, and* $p_i \leqslant m+1-p_j$ *for every* $1 \leqslant i < j \leqslant n+1-i$*. In particular,* $P \in \mathcal{T}_n(\mathbb{F})$ *when* $m = n$*.*

(iii) *When* $n = 4$*, in addition to* (i) *and* (ii)*,* ψ *also takes the form*

$$\psi(A) = P \begin{bmatrix} a_{11} & a_{12} & \alpha a_{13} + \theta(a_{14} - a_{23}) & \beta a_{14} \\ 0 & a_{22} & (2\alpha - \beta) a_{23} & \alpha a_{13} + \theta(a_{14} - a_{23}) \\ 0 & 0 & a_{22} & a_{12} \\ 0 & 0 & 0 & a_{11} \end{bmatrix} P^{+}$$

for all $A = (a_{ij}) \in \mathcal{ST}_4(\mathbb{F})$*, where* $\alpha, \beta, \theta \in \mathbb{F}$ *are scalars such that* α, β *are nonzero with* $\beta \neq 2\alpha$*, and* θ *is nonzero only if* $|\mathbb{F}| = 3$*, and* $P \in \mathcal{M}_{m,4}(\mathbb{F})$ *is a full rank matrix in which* $Pe_i \in \mathcal{U}_{p_i,m}$ *for* $1 \leqslant i \leqslant 4$ *with* $1 \leqslant p_i \leqslant \frac{m+1}{2}$ *for every* $1 \leqslant i \leqslant 2$*, and* $p_i \leqslant m+1-p_j$ *for every* $1 \leqslant i < j \leqslant 5-i$*. In particular,* $P \in \mathcal{T}_4(\mathbb{F})$ *when* $m = 4$*.*

(iv) *When* $n = 3$*, in addition to* (i) *and* (ii)*,* ψ *also takes one of the following forms:*

(a) *There exist a surjective linear mapping* $\phi : \mathcal{ST}_3(\mathbb{F}) \to \mathbb{F}^3$ *and a full rank matrix* $P \in \mathcal{M}_{m,2}(\mathbb{F})$ *such that*

$$\psi(A) = P \begin{bmatrix} \phi(A)_3 & \phi(A)_1 \\ \phi(A)_2 & \phi(A)_3 \end{bmatrix} P^{+} \quad \text{for all } A \in \mathcal{ST}_3(\mathbb{F}),$$

where $Pe_1, Pe_2 \in \mathcal{U}_{p,m}$ *for some integer* $1 \leqslant p \leqslant \frac{m+1}{2}$*,* $\phi(A)_i$ *is the* i*-th component of* $\phi(A) \in \mathbb{F}^3$*, and* $\phi(A) \neq 0$ *for every nonzero bounded rank-two matrix* $A \in \mathcal{ST}_3(\mathbb{F})$*.*

(b) *There exist scalars* $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}$ *with* $\lambda_3 \neq 0$ *such that either*

$$\psi(A) = P \begin{bmatrix} a_{pp} & \eta_2 a_{12} + a_{13} + \lambda_2 a_{qq} & \eta_1 a_{12} + \lambda_1 a_{qq} \\ 0 & \lambda_3 a_{qq} & \eta_2 a_{12} + a_{13} + \lambda_2 a_{qq} \\ 0 & 0 & a_{pp} \end{bmatrix} P^{+}$$

for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$*, where* $\eta_1, \eta_2 \in \mathbb{F}$ *are nonzero and* $\{p, q\} = \{1, 2\}$*; or*

$$\psi(A) = P \begin{bmatrix} a_{pp} & a_{1s} + \lambda_2 a_{qq} & \eta_1 a_{1t} + \lambda_1 a_{qq} \\ 0 & \lambda_3 a_{qq} & a_{1s} + \lambda_2 a_{qq} \\ 0 & 0 & a_{pp} \end{bmatrix} P^{+}$$

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for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$, *where* $\eta \in \mathbb{F}$ *is nonzero and* $\{p, q\} = \{s, t\} = \{1, 2\}$. *Here,* $P \in \mathcal{M}_{m,3}(\mathbb{F})$ *is a full rank matrix such that* Pe_1, $Pe_2 \in \mathcal{U}_{p,m}$ *with* $1 \leqslant p \leqslant \frac{m+1}{2}$ *and* $Pe_3 \in \mathcal{U}_{q,m}$ *with* $1 \leqslant q \leqslant m + 1 - p$. *In particular,* $P \in \mathcal{T}_3(\mathbb{F})$ *when* $m = 3$.

We end this section by giving an example of rank-one linear preserver / rank-one non-increasing linear mapping and some examples of rank-two non-increasing linear mappings on per-symmetric triangular matrices.

EXAMPLE 4.5. Let $\mathbb{F}$ be a field and m, n be integers $\geqslant 2$. Let $p := \left\lfloor \frac{n+1}{2} \right\rfloor$, where $\lfloor \cdot \rfloor$ is the floor function. Let $\psi : \mathcal{ST}_n(\mathbb{F}) \to \mathcal{SM}_m(\mathbb{F})$ be the linear mapping defined by

$$\psi(A) = \lambda P \begin{bmatrix} \phi(A_1) & \varphi(A_2) \\ 0 & \phi(A_1)^+ \end{bmatrix} P^+ \quad \text{for every } A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_1^+ \end{bmatrix} \in \mathcal{ST}_n(\mathbb{F})$$

with $A_1 \in \mathcal{T}_{p,n-p}(\mathbb{F})$ and $A_2 \in \mathcal{SM}_p(\mathbb{F})$, where $\lambda \in \mathbb{F} \backslash \{0\}$, $P \in \mathcal{M}_{m,n}(\mathbb{F})$ is of full rank, and $\phi : \mathcal{T}_{p,n-p}(\mathbb{F}) \to \mathcal{M}_{p,n-p}(\mathbb{F})$ and $\varphi : \mathcal{SM}_p(\mathbb{F}) \to \mathcal{SM}_p(\mathbb{F})$ are linear mappings. Here $\mathcal{T}_{p,n-p}(\mathbb{F}) = \mathcal{T}_p(\mathbb{F})$ when $n - p = p$, and

$$\mathcal{T}_{p,n-p}(\mathbb{F}) = \left\{ \begin{bmatrix} T \\ 0 \end{bmatrix} \in \mathcal{M}_{p,n-p}(\mathbb{F}) \,\middle|\, T \in \mathcal{T}_{p-1}(\mathbb{F}) \right\} \text{ when } n - p = p - 1.$$

It is easily verified that

- ψ is a rank-one linear preserver whenever φ is a rank-one linear preserver on $\mathcal{SM}_p(\mathbb{F})$, and
- ψ is rank-one non-increasing whenever φ is a rank-one non-increasing linear mapping on $\mathcal{SM}_p(\mathbb{F})$.

By the structural results of rank-one linear preservers and rank-one non-increasing linear mappings on symmetric matrices (see a complete result under a more general setting in [8], [12]), the structure of ψ can be established immediately.

EXAMPLE 4.6. Let $\mathbb{F}$ be a field and m, n be integers $\geqslant 2$. Let $\psi : \mathcal{ST}_n(\mathbb{F}) \to \mathcal{SM}_m(\mathbb{F})$ be the linear mapping defined by

$$\psi(A) = \lambda PAP^+$$

for every $A \in \mathcal{ST}_n(\mathbb{F})$, where $\lambda \in \mathbb{F}$ and $P \in \mathcal{M}_{m,n}(\mathbb{F})$. Clearly, ψ is rank-two non-increasing.

EXAMPLE 4.7. Let $\mathbb{F}$ be a field and n be an integer $\geqslant 2$. Let $\psi : \mathcal{ST}_n(\mathbb{F}) \to \mathcal{SM}_n(\mathbb{F})$ be the linear mapping defined by

$$\psi(A) = \operatorname{diag}(a_{11}, \ldots, a_{nn})$$

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for every $A = (a_{ij}) \in \mathcal{ST}_n(\mathbb{F})$. It is immediate to see that $\operatorname{rank}\psi(A) \leqslant 2$ whenever $\operatorname{rank} A \leqslant 2$.

EXAMPLE 4.8. Let $\mathbb{F}$ be a field and n be an integer $\geqslant 2$. Let $\psi : \mathcal{ST}_n(\mathbb{F}) \to \mathcal{SM}_n(\mathbb{F})$ be the linear mapping defined by

$$\psi(A) = \begin{bmatrix} \lambda_1 A_{11} & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 A_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_2 A_{22} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_1 A_{11} \end{bmatrix}$$

for every $A = (A_{ij}) \in \mathcal{ST}_n(\mathbb{F})$ with $A_{ij} \in \mathcal{M}_{n_i,n_j}(\mathbb{F})$ for $1 \leqslant i \leqslant j \leqslant k$. Here $\lambda_i \in \mathbb{F}$ with $\lambda_{k+1-i} = \lambda_i$ for $i = 1, \ldots, k$, and $n_1 + \cdots + n_k = n$ with $n_{k+1-i} = n_i$ for $i = 1, \ldots, k$. It is easily verified that is rank-two non-increasing.

EXAMPLE 4.9. Let $\mathbb{F}$ be a field. We define the linear mapping $\psi : \mathcal{ST}_5(\mathbb{F}) \to \mathcal{SM}_5(\mathbb{F})$ such that

$$\psi(A) = \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{12} & a_{22} & a_{23} & a_{24} & 0 \\ 0 & 0 & a_{33} & a_{23} & 0 \\ 0 & 0 & 0 & a_{22} & 0 \\ 0 & 0 & 0 & a_{12} & a_{11} \end{bmatrix}$$

for every $A = (a_{ij}) \in \mathcal{ST}_5(\mathbb{F})$. A direct verification shows that ψ satisfies $\operatorname{rank}\psi(A) \leqslant 2$ whenever $\operatorname{rank} A \leqslant 2$.

EXAMPLE 4.10. Let $\mathbb{F}$ be a field and $\psi : \mathcal{ST}_5(\mathbb{F}) \to \mathcal{SM}_5(\mathbb{F})$ be the linear mapping defined by

$$\psi(A) = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 \\ 0 & a_{23} & a_{33} & 0 & 0 \\ 0 & 0 & a_{23} & a_{22} & a_{12} \\ 0 & 0 & 0 & 0 & a_{11} \end{bmatrix}$$

for every $A = (a_{ij}) \in \mathcal{ST}_5(\mathbb{F})$. Then ψ satisfies $\operatorname{rank}\psi(A) \leqslant 2$ whenever $\operatorname{rank} A \leqslant 2$. Nevertheless, we note that ψ is not rank-one non-increasing. For example, $\psi(E_{23} + E_{24} + E_{33} + E_{34}) = E_{32} + E_{33} + E_{43}$ is of rank two.

Examples 4.6–4.10 demonstrate that the structure of rank-two non-increasing linear mappings on per-symmetric triangular matrices is complicated. This shows that condition (1.1) is a relevant assumption in our study.

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