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IMPROVED TESTS AND CHARACTERIZATIONS OF TOTALLY NONNEGATIVE MATRICES*

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Abstract. Totally nonnegative matrices, i.e., matrices having all minors nonnegative, are considered. A condensed form of the Cauchon algorithm which has been proposed for finding a parameterization of the set of these matrices with a fixed pattern of vanishing minors is derived. The close connection of this variant to Neville elimination and bidiagonalization is shown and new determinantal tests for total nonnegativity are developed which require much fewer minors to be checked than for the tests known so far. New characterizations of some subclasses of the totally nonnegative matrices as well as shorter proofs for some classes of matrices for being (nonsingular and) totally nonnegative are derived.

Key words. Totally nonnegative matrix, Totally positive matrix, Cauchon algorithm, Neville elimination, Bidiagonalization.

AMS subject classifications. 15A48.

1. Introduction. In this paper, we are concerned with totally nonnegative matrices, i.e., matrices having all their minors nonnegative. For properties of these matrices the reader is referred to the two recently published monographs [4], [17]. This paper grew out our study of the papers [14], [16]. Herein the authors apply the so-called Cauchon diagrams to parameterize the set of the totally nonnegative matrices with a fixed pattern of zero minors (if this set is nonempty), called a *totally nonnegative cell* (corresponding to this pattern). An important tool to identify such a cell is the so-called Cauchon algorithm. We derive in Subsection 3.2 a condensed form of this algorithm, hereby reducing the order of the number of the required arithmetic operations by one. We investigate in Section 4 the relationship of this algorithm to Neville elimination and bidiagonalization.

Section 5 is devoted to demonstrate the capabilities of the Cauchon algorithm. In Subsection 5.1, we derive a new determinantal test for total nonnegativity which requires the computation of significantly fewer minors than existing tests like the ones

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given in [6]. However, in contrast to these tests our criteria depend on the matrix under consideration. Furthermore, we give in Subsection 5.2 new characterizations of some subclasses of the totally nonnegative matrices; this applies to the oscillatory matrices, the pentadiagonal totally nonnegative matrices, and the almost totally positive matrices. For other classes of matrices we present shorter proofs of known criteria for being (nonsingular and) totally nonnegative; this applies to the tridiagonal matrices, the Green's matrices, and the (0,1)-matrices. Many of our results rely on the use of the so-called lacunary sequences which were investigated in [16]. We employ these sequences to relate entries of the matrix obtained by the Cauchon algorithm to minors of the original matrix. The use of the Cauchon algorithm and of the concept of the lacunary sequences allows us to present many results related to totally nonnegative matrices in a unifying and concise way similar to how it is done in [4] by employing the bidiagonalization. It is our experience, documented in Section 5, that the advantage of the use of the Cauchon algorithm compared to bidiagonalization is that it requires less effort.

Since we are mainly interested in applications to nonsingular totally nonnegative matrices we present the Cauchon algorithm only in the case of square matrices. The extension to rectangular matrices is immediate.

2. Notation and auxiliary results.

2.1. Notation. We now introduce the notation used in our paper. For nonnegative integers k, n, we denote by $Q_{k,n}$ the set of all strictly increasing sequences of k integers chosen from $\{1, 2, \ldots, n\}$. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in Q_{k,n}$, the *dispersion* of α is $d(\alpha) = \alpha_k - \alpha_1 - k + 1$. If $d(\alpha) = 0$, then the index set α is called *contiguous.* Let $A \in \mathbb{R}^{n,m}$. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in Q_{k,n}$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_l) \in Q_{l,m}$, we denote by $A[\alpha|\beta]$ the $k \times l$ submatrix of A contained in the rows indexed by $\alpha_1, \alpha_2, \ldots, \alpha_k$ and columns indexed by $\beta_1, \beta_2, \ldots, \beta_l$. We suppress the parentheses when we enumerate the indices explicitly. When $\alpha = \beta$, the principal submatrix $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$. In the special case where $\alpha = (1, 2, \ldots, k)$, we refer to the principal submatrix $A[\alpha]$ as a *leading principal submatrix* (and to $\det A[\alpha]$ as a *leading principal minor*).

A minor $\det A[\alpha|\beta]$ is called *contiguous* if both α and β are contiguous; it is called *quasi-initial* if either $\alpha = (1, 2, \ldots, k)$ and $\beta \in Q_{k,m}$ is arbitrary or $\alpha \in Q_{k,n}$ is arbitrary, while $\beta = (1, 2, \ldots, k)$. If it is in addition contiguous then it is termed *initial.*

The identity matrix is denoted by I. The n-by-n matrix whose only nonzero entry is a one in the $(i, j)^{th}$ position is denoted by E_{ij}. We reserve throughout the notation $T_n = (t_{ij})$ for the permutation matrix with $t_{i,n-i+1} = 1$, $i = 1, \ldots, n$. A matrix $A = (a_{ij}) \in \mathbb{R}^{n,n}$ is referred to as a *tridiagonal* (or *Jacobi*) and *pentadiagonal matrix*

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if $a_{ij} = 0$ whenever $|i-j| > 1$ and $|i-j| > 2$, respectively. A matrix $A \in \mathbb{R}^{n,m}$ is called *totally positive* (abbreviated TP henceforth) and *totally nonnegative* (abbreviated TN) if $\det A[\alpha|\beta] > 0$ and $\det A[\alpha|\beta] \geq 0$, respectively, for all $\alpha, \beta \in Q_{k,n}$. If A is TN and in addition nonsingular we write A is $NsTN$. If in addition A has the property that a particular minor is positive if and only if all its main diagonal entries are positive then A is called *almost totally positive* (abbreviated ATP). If A is TN and has a TP integral power it is called *oscillatory*. In passing, we note that if A is TN then so are its transpose and $A^{\#} := T_n A T_m$, see, e.g., [4, Theorem 1.4.1].

2.2. Auxiliary results. In the sequel, we will often make use of the following special case of Sylvester's Identity, see, e.g., [4, pp. 29–30], [17, p. 3].

LEMMA 2.1. *Sylvester's Identity. Partition $A \in \mathbb{R}^{n,n}$, $n \geq 3$, as follows:*

$$A = \begin{pmatrix} c & A_{12} & d \\ A_{21} & A_{22} & A_{23} \\ e & A_{32} & f \end{pmatrix},$$

where $A_{22} \in R^{n-2,n-2}$ and c, d, e, f are scalars. Define the submatrices

$$C := \begin{pmatrix} c & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad D := \begin{pmatrix} A_{12} & d \\ A_{22} & A_{23} \end{pmatrix},$$

$$E := \begin{pmatrix} A_{21} & A_{22} \\ e & A_{32} \end{pmatrix}, \quad F := \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & f \end{pmatrix}.$$

Then, if $\det A_{22} \neq 0$, the following relation holds true

$$\det A = \frac{\det C \det F - \det D \det E}{\det A_{22}}.$$

LEMMA 2.2. [Shadow property [4, Corollary 7.2.10], [17, Section 1.3]]. *Suppose that $A \in R^{n,n}$ is $NsTN$. Then $a_{ij} = 0$ implies $a_{hk} = 0$*

$$\begin{array}{ll} \text{for all } i \leq h,\ k \leq j & \text{if } j < i, \\ \text{for all } h \leq i,\ j \leq k & \text{if } i < j. \end{array}$$

LEMMA 2.3. [4, Corollary 3.8], [17, Theorem 1.13]. *All principal minors of an $NsTN$ matrix are positive.*

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3. The Cauchon algorithm and its condensed form.

3.1. Cauchon diagrams and the Cauchon algorithm. In this subsection, we first recall from [14], [16] the definition of a Cauchon diagram and of the Cauchon algorithm [1].

Definition 3.1. An $n \times n$ *Cauchon diagram* C is an $n \times n$ grid consisting of n^2 squares colored black and white, where each black square has the property that either every square to its left (in the same row) or every square above it (in the same column) is black.

We denote by C_n the set of the $n \times n$ Cauchon diagrams. We fix positions in a Cauchon diagram in the following way: For $C \in C_n$ and $i, j \in \{1, \ldots, n\}, (i, j) \in C$ if the square in row i and column j is black. Here we use the usual matrix notation for the (i, j) position in a Cauchon diagram, i.e., the square in (1, 1) position of the Cauchon diagram is in its top left corner.
For instance, for the Cauchon diagram C of Figure 1, we have $(2, 3) \notin C$, whereas $(3, 2) \in C$.

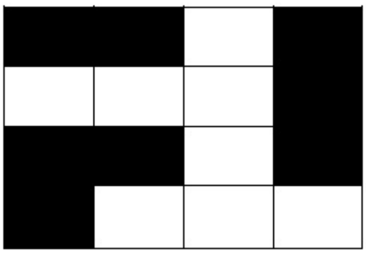

Fig. 3.1. *Example of a Cauchon diagram*

Definition 3.2. Let $A \in R^{n,n}$ and let $C \in C_n$. We say that A is a *Cauchon matrix associated with the Cauchon diagram* C if for all (i, j), $i, j \in \{1, \ldots, n\}$, we have $a_{ij} = 0$ if and only if $(i, j) \in C$. If A is a Cauchon matrix associated with an unspecified Cauchon diagram, we just say that A is a *Cauchon matrix*.

In passing, we note that every TN matrix is a Cauchon matrix [16, Lemma 2.3].

We denote by $\leq$ and $\leq_c$ the lexicographic and colexicographic order, respectively, on N^2, i.e.,

$$(g, h) \leq (i, j) :\Leftrightarrow \ (g < i) \ \text{ or } \ (g = i \,\text{and}\, h \leq j),$$

$$(g, h) \leq_c (i, j) :\Leftrightarrow \ (h < j) \ \text{ or } \ (h = j \,\text{and}\, g \leq i).$$

[1] This algorithm is called in [14] the deleting derivations algorithm (as the inverse of the restoration algorithm) and in [16] the Cauchon reduction algorithm.

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The latter one is only used in Subsection 5.1.

Set $E^\circ := \{1,\ldots,n\}^2 \setminus \{(1,1)\}$, $E := E^\circ \cup \{(n+1,2)\}$.

Let $(s,t) \in E^\circ$. Then $(s,t)^+ := \min\{(i,j) \in E \mid (s,t) \le (i,j),\ (s,t) \ne (i,j)\}$.

Algorithm 3.1. *Let $A \in R^{n,n}$. As r runs in decreasing order over the set E, we define matrices $A^{(r)} = (a_{ij}^{(r)}) \in R^{n,n}$ as follows.*

1. Set $A^{(n+1,2)} := A$.

2. For $r = (s,t) \in E^\circ$ define the matrix $A^{(r)} = (a_{ij}^{(r)})$ as follows.

(a) If $a_{st}^{(r^+)} = 0$ then put $A^{(r)} := A^{(r^+)}$.
(b) If $a_{st}^{(r^+)} \ne 0$ then put

$$a_{ij}^{(r)} := \begin{cases} a_{ij}^{(r^+)} - \frac{a_{it}^{(r^+)} a_{sj}^{(r^+)}}{a_{st}^{(r^+)}} & \text{for } i < s \text{ and } j < t, \\ a_{ij}^{(r^+)} & \text{otherwise.} \end{cases}$$

3. Set $\tilde{A} := A^{(1,2)}$; $\tilde{A}$ is called the matrix obtained from A (by the Cauchon algorithm) [2].

We conclude this subsection with some results on the application of the Cauchon algorithm to TN matrices.

Theorem 3.3. [14, Theorem B4], [16, Theorems 2.6 and 2.7], [2, Proposition 2.8]. *Let $A \in R^{n,n}$. Then the following statements hold.*
(i) If A is TN and $2 \le s$, then for all $(s,t) \in E$, $A^{(s,t)}$ is an entry-wise nonnegative Cauchon matrix and $A^{(s,t)}[1,\ldots,s-1 \mid 1,\ldots,n]$ is TN.

(ii) A is TP (TN) if and only if $\tilde{A}$ is an entry-wise positive (nonnegative) Cauchon matrix.
(iii) If A is TN then A is nonsingular if and only if $0 < \tilde{a}_{ii}$, $i = 1,\ldots,n$.

3.2. Condensed form of the Cauchon algorithm. In this subsection, we relate the entries of $A^{(k,2)}$ to the entries of $A^{(k+1,2)}$, $k = 2,\ldots,n$. This leads to a condensed form of the Cauchon algorithm which reduces the number of required arithmetic operations from $O(n^4)$ to $O(n^3)$ (for the exact number of operations, see Subsection 4.1).

[2] Note that $A^{(k,1)} = A^{(k,2)}$, $k = 1,\ldots,n-1$, and $A^{(2,2)} = A^{(1,2)}$ so that the algorithm could already be terminated when $A^{(2,2)}$ is computed.

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PROPOSITION 3.4. *For the entries of the matrices generated by Algorithm 3.1 the following relation holds true for* $k = n, \dots, 2$

$$a_{ij}^{(k,2)} = \begin{cases} \frac{\det A^{(k+1,2)}[i,k|j,u_j]}{a_{ku_j}^{(k+1,2)}} & \text{if } u_j < \infty, \\ a_{ij}^{(k+1,2)} & \text{if } u_j = \infty, \end{cases}$$

where $u_j := \min\left\{h \in \{j+1, \dots, n\} \,|\, a_{kh}^{(k+1,2)} > 0\right\}$ *(we set* $u_j := \infty$ *if this set is empty),* $j = 1, \dots, n-1$*,* $i = 1, \dots, k-1$*.*

Proof. It suffices to prove the relation for $k = n$.
Let $t^* := \max\{t \mid 0 < a_{nt}\}$; we set $t^* := 0$ if this set is empty. If $t^* < n$ then $A^{(n,t)} = A^{(n+1,2)} = A$ for all $t^* < t \le n$.
If $2 \le t^*$ we have

$$a_{ij}^{(n,t^*)} = \frac{\det A[i,n|j,t^*]}{a_{nt^*}}, \ \ i = 1, \dots, n-1, \ j = 1, \dots, t^*-1,$$

and $A^{(n,t)} = A^{(n,t^*)}$ if t decreases as long as $a_{nt} = 0$ holds.
We assume now that there are $1 < u < v$ such that $0 < a_{nu}, a_{nv}$ and $u+1 = v$ or $a_{n,u+1} = a_{n,u+2} = \cdots = a_{n,v-1} = 0$. After step $r = (n,v)$ we have

$$a_{ij}^{(n,v)} = \frac{\det A^{(r^+)}[i,n|j,v]}{a_{nv}}, \ \ i = 1, \dots, n-1, j = 1, \dots, v-1,$$

and after step (n,u) for $i = 1, \dots, n-1$, $j = 1, \dots, u-1$,

$$\begin{aligned} a_{ij}^{(n,u)} a_{nu} &= \det A^{(n,v)}[i,n|j,u] \\ &= a_{ij}^{(n,v)} a_{nu} - a_{iu}^{(n,v)} a_{nj} \\ &= \det A^{(r^+)}[i,n|j,v] a_{nu} - \det A^{(r^+)}[i,n|u,v] a_{nj} \\ &= a_{ij}^{(r^+)} a_{nu} - a_{iu}^{(r^+)} a_{nj} = \det A^{(r^+)}[i,n|j,u]. \end{aligned}$$

The claim follows now by decreasing induction on the column index. □

Based on the Proposition 3.4 we give the Cauchon algorithm in its condensed form (note that we use the upper index in a slightly more convenient form).

ALGORITHM 3.2. *Set* $A^{(n)} := A$.
For $k = n-1, \dots, 1$ *define* $A^{(k)} = (a_{ij}^{(k)}) \in \mathbb{R}^{n,n}$ *as follows:*
For $i = 1, \dots, k$*,*
for $j = 1, \dots, n-1$
set $u_j := \min\left\{h \in \{j+1, \dots, n\} \,|\, a_{kh}^{(k+1)} > 0\right\}$ *(we set* $u_j := \infty$ *if this set is empty)*

$$a_{ij}^{(k)} := \begin{cases} a_{ij}^{(k+1)} - \frac{a_{kj}^{(k+1)} a_{iu_j}^{(k+1)}}{a_{ku_j}^{(k+1)}} & \text{if } u_j < \infty, \\ a_{ij}^{(k+1)} & \text{if } u_j = \infty, \end{cases}$$

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for $i = k+1, \ldots, n,\ j = 1, \ldots, n,$ *and* $i = 1, \ldots, k,\ j = n$

$$a_{ij}^{(k)} := a_{ij}^{(k+1)}.$$

Put $\hat{A} := A^{(1)}$.

It follows from Proposition 3.4 that $\tilde{A} = \hat{A}$ holds. If A is symmetric then $\tilde{A}$ is symmetric, too, and therefore it is not necessary to consider the entries of $\tilde{A}$ above the main diagonal. The Cauchon algorithm can then be shortened in the way that after the k^{th} step the computations are continued with the submatrix $A^{(k)}[1,\ldots,k|1,\ldots,k-1]$, $k = n-1,\ldots,2$, because the k^{th} row of $A^{(k)}$ is identical with its k^{th} column.

4. Connection to Neville elimination. In this section, we show that (at least) for TP matrices the intermediate matrices of Algorithm 3.2 can be represented as matrices generated by Neville elimination.

4.1. Neville elimination. This elimination method proceeds by producing zeros in the columns of a matrix by adding to each row an appropriate multiple of the preceding one (instead of using a fixed row per column as in Gaussian elimination). We recall from [6], [9] the following description. Let $A \in \mathbb{R}^{n,n}$. The elimination procedure results in a sequence of matrices

$$A = \ddot{A}^{(1)} \to \dot{A}^{(1)} \to \ddot{A}^{(2)} \to \dot{A}^{(2)} \to \cdots \to \ddot{A}^{(n)} = \dot{A}^{(n)} = U,$$

where U is an upper triangular matrix. For each k, $1 \le k \le n$, the matrix $\dot{A}^{(k)} = (\dot{a}_{ij}^{(k)})$ has zeros below its main diagonal in the first $k-1$ columns. $\dot{A}^{(k)}$ is obtained from the matrix $\ddot{A}^{(k)}$ by shifting to the bottom the rows with a zero entry in column k. The rows are placed there in the same relative order as they appear in $\ddot{A}^{(k)}$. The matrix $\ddot{A}^{(k+1)}$ is obtained from $\dot{A}^{(k)}$ according to the following formula

$$\ddot{a}_{ij}^{(k+1)} := \begin{cases} \dot{a}_{ij}^{(k)} & \text{for } i = 1,\ldots,k, \\ \dot{a}_{ij}^{(k)} - \frac{\dot{a}_{ik}^{(k)}}{\dot{a}_{i-1,k}^{(k)}} \dot{a}_{i-1,j}^{(k)} & \text{if } \dot{a}_{i-1,k}^{(k)} \neq 0, \\ \dot{a}_{ij}^{(k)} & \text{if } \dot{a}_{i-1,k}^{(k)} = 0, \end{cases} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{ for } i = k+1,\ldots,n.$$

The number $p_{ik} := \dot{a}_{ik}^{(k)}$, $1 \le k \le i \le n$, is called the (i,k) *pivot* and the number

$$m_{ik} := \begin{cases} \frac{\dot{a}_{ik}^{(k)}}{\dot{a}_{i-1,k}^{(k)}} & \text{if } \dot{a}_{i-1,k}^{(k)} \neq 0, \\ 0 & \text{if } \dot{a}_{i-1,k}^{(k)} = 0, \end{cases} \tag{4.1}$$

is called the (i,k) *multiplier* of Neville elimination.
If all the pivots are nonzero then the entries of the intermediate matrices allow the

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following determinantal representation, see [6, formula (2.8)],

$$\text{(4.2)}\quad \dot{a}_{ij}^{(k)} = \frac{\det A[i-k+1,\ldots,i|1,\ldots,k-1,j]}{\det A[i-k+1,\ldots,i-1|1,\ldots,k-1]}, \quad i,j=k,\ldots,n,\ k=2,\ldots,n.$$

Complete Neville elimination of the matrix A consists of two steps: First Neville elimination is performed to get the upper triangular matrix U and in a second step Neville elimination is applied to U^T. The (i,k) *pivot* (respectively, *multiplier*) *of the complete Neville elimination* of A is that of the Neville elimination if $k \le i$ and the (k,i) *pivot* (respectively, *multiplier*) is that of the Neville elimination applied to U^T if $i \le k$.
Complete Neville elimination allows an efficient test of a given nonsingular matrix for total nonnegativity and total positivity (for reference in Section 5 we give in the next theorem also the respective determinantal criteria).

THEOREM 4.1. [6, Theorem 4.1], [7, Theorem 3.1], [4, Theorem 3.3.5]. *For the nonsingular matrix A the following three conditions are equivalent.*
(i) A is TP (TN).
(ii) Complete Neville elimination applied to A can be performed without exchange of rows and columns, and all of the pivots are positive (nonnegative).
(iii) All initial minors (leading principal minors) are positive (and all quasi-initial minors are nonnegative).

The Cauchon algorithm also provides an efficient test for total nonnegativity and total positivity, cf. Theorem 3.3 (ii). Complete Neville elimination and the condensed Cauchon algorithm both need the same number of arithmetic operations for a square matrix A of order n [3], viz. $(n+1)(n-1)^2$. Besides the arithmetic operations, the Cauchon algorithm requires testing whether $\tilde{A}$ is a Cauchon matrix, a test which can be implemented with quadratic complexity. However, this test is very easy in the TP and $NsTN$ cases since in the TP case we have merely to test whether $\tilde{A}$ contains only positive entries and in the $NsTN$ case we have to check whether the diagonal entries of $\tilde{A}$ are positive (due to Theorem 3.3 (iii)) and in the case of a zero entry all entries to the left of it in the same row or in the same column above it vanish. As for Neville elimination, these tests should already be performed during the run of the algorithm. In the general TN case complete Neville elimination requires in addition that the rows which have been shifted to the bottom are all zero rows, see [6, Theorem 5.4]. So the amount of work is comparable for both algorithms. However, in the next section we will derive a determinantal test for the $NsTN$ case which results from the Cauchon algorithm and which requires significantly fewer minors to be checked than the test which is based on Theorem 4.1; for a detailed discussion see Subsection 5.1.

[3] If A is symmetric then the condensed Cauchon algorithm requires $\frac{n(n-1)(4n+1)}{6}$ operations.

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4.2. Modified Neville elimination. To investigate the close relationship between both algorithms we modify the usual Neville elimination as follows: We do not produce zeros only below the main diagonal but also on it and above it below the first row which remains unchanged. In this way, we generate a sequence of matrices (here we assume that no exchange of rows is required)

$$A = A^{\dagger(1)} \to A^{\dagger(2)} \to \cdots \to A^{\dagger(n)}$$

with

$$A^{\dagger(k)}[2,\ldots,n|1,\ldots,k-1] = 0,\ k = 2,\ldots,n,$$

$$a_{ij}^{\dagger(k+1)} := \begin{cases} a_{ij}^{\dagger(k)} & \text{for } \ i = 1, \\ \left.\begin{array}{ll} a_{ij}^{\dagger(k)} - \frac{a_{ik}^{\dagger(k)}}{a_{i-1,k}^{\dagger(k)}} a_{i-1,j}^{\dagger(k)} & \text{if } \ a_{i-1,k}^{\dagger(k)} \neq 0, \\ a_{ij}^{\dagger(k)} & \text{if } \ a_{i-1,k}^{\dagger(k)} = 0, \end{array}\right\} & \text{for } \ i = 2,\ldots,n. \end{cases}$$

We call the resulting algorithm *modified Neville elimination.*

THEOREM 4.2. *Let $A \in \mathbb{R}^{n,n}$ be TP and put $B := (A^{\#})^T = (A^T)^{\#}$. We run Algorithm 3.2 on A and the modified Neville elimination on B. Then we have for $k = 1,\ldots,n$*

$$B^{\dagger(k)}[1,\ldots,n|k] = T_n A^{(n-k+1)}[n-k+1|1,\ldots,n]. \tag{4.3}$$

Proof. The entries of B are given by

$$b_{ij} = a_{n-j+1,n-i+1} \ \ i,j = 1,\ldots,n. \tag{4.4}$$

Since $B^{\dagger(1)} = B$ holds, the entries of the first column of $B^{\dagger(1)}$ are identical with the entries of the last row of A in reverse order which are the entries of the right-hand side of (4.3) so that the statement is true for $k = 1$.
To simplify the representation we write $|[\alpha \mid \beta]|$ to denote $\det A[\alpha \mid \beta] / \det A[\alpha' \mid \beta']$, where $\alpha' = (\alpha_2,\ldots,\alpha_n)$, $\beta' = (\beta_2,\ldots,\beta_n)$.
For general k, the entries of row $n-k$ of $A^{(n-k)}$ are given by, see [16, p. 376],

$$\begin{aligned} &|[\delta \mid 1,\ldots,k+1]|,\ldots,|[\delta \mid n-k-1,\ldots,n-1]|,\ |[\delta]|,\ldots, \\ &|[n-k,n-k+1,n-k+2|n-2,n-1,n]|,\ |[n-k,n-k+1|n-1,n]|,\ a_{n-k,n}, \end{aligned} \tag{4.5}$$

where $\delta := (n-k,\ldots,n)$, and similarly for row $n-k-1$ with $\epsilon := (n-k-1, n-k+1,\ldots,n)$

$$\begin{aligned} &|[\epsilon \mid 1,\ldots,k+1]|,\ldots,|[\epsilon \mid n-k-1,\ldots,n-1]|,|[\epsilon \mid n-k,\ldots,n]|,\ldots, \\ &|[n-k-1,n-k+1,n-k+2|n-2,n-1,n]|,|[n-k-1,n-k+1|n-1,n]|,\ a_{n-k-1,n}. \end{aligned}$$

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We assume that the statement is true for $k+1$. Then the $(k+2)^{th}$ column of $B^{\dagger(k+1)}$ is given by

$$
\begin{aligned}
(4.6) \qquad & a_{n-k-1,n},\ |[n-k-1, n-k+1|n-1, n]|, \\
& |[n-k-1, n-k+1, n-k+2|n-2, n-1, n]|, \ldots, |[\epsilon\ |n-k, \ldots, n]|, \\
& \ldots,\ |[\epsilon\ |n-k-1, \ldots, n-1]|, \ldots, |[\epsilon\ |1, \ldots, k+1]|.
\end{aligned}
$$

We show that then the statement is also true for $k+2$. The entries of row $n-k-1$ of $A^{(n-k-1)}$ are given by

$$
\begin{aligned}
(4.7) \qquad & |[\zeta|1, \ldots, k+2]|, \ldots, |[\zeta\ |n-k-2, \ldots, n-1]|, |[\zeta]|, \ldots, \\
& |[n-k-1, n-k, n-k+1|n-2, n-1, n]|, |[n-k-1, n-k|n-1, n]|,\ a_{n-k-1,n},
\end{aligned}
$$

where $\zeta := (n-k-1, \ldots, n)$. We apply the modified Neville elimination to $B^{\dagger(k+1)}$. Since for the lower triangular part modified Neville elimination is identical with the usual one, we may apply the determinantal representation (4.2) and obtain by (4.4) that the last $n-k-2$ entries in the $(k+2)^{th}$ column of $B^{\dagger(k+2)}$ are equal to the first $n-k-2$ entries in (4.7). The first entry of this column is $a_{n-k-1,n}$ which is identical with the last entry of (4.7). Coincidence of the second entry in that column and of the last but one of (4.7) can easily be seen from (4.5) and (4.6). Coincidence of the remaining entries above and on the main diagonal in the $(k+2)^{th}$ column of $B^{\dagger(k+2)}$ with the respective entries of (4.7) can be shown by using Sylvester's Identity, see Lemma 2.1. This completes the inductive proof. □

The extension of Theorem 4.2 to the case of TN matrices can be accomplished as follows: Firstly, we use the fact that the closure of the set of TP matrices is the set of TN matrices [18]. An alternative to the existing proofs [4, p. 62] of this fact relies on the restoration algorithm [14] which is the inverse of the Cauchon algorithm. Let A be TN. Then by Theorem 3.3 (ii) $\tilde{A}$ is a nonnegative Cauchon matrix and therefore all entries in the same row to the left or all entries in the same column above a zero entry in $\tilde{A}$ vanish, too. We replace such zero entries from the right to the left and from the bottom to the top by increasing integral powers of ϵ, $0 < \epsilon$. Call the resulting positive matrix $\tilde{A}_\epsilon$. We apply the restoration algorithm to $\tilde{A}_\epsilon$ and obtain the TP matrix A_ϵ. Since $\tilde{A}_\epsilon$ tends to $\tilde{A}$ as ϵ tends to 0, A_ϵ tends to A. So we can approximate the given TN matrix A by the TP matrix A_ϵ as closely as desired. To extend Theorem 4.2 to the TP case, we approximate the given TN matrix A by the TP matrix A_ϵ as described. Then we obtain that (after cancellation of common powers of ϵ) the denominators appearing on both sides of (4.3) do not contain ϵ. Letting ϵ tend to 0 the extension of Theorem 4.2 to the TN case follows.

4.3. Application to bidiagonalization. An important tool for the analysis of an $NsTN$ matrix A is its bidiagonalization, i.e., its factorization into a product of a

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diagonal matrix with positive diagonal entries and some matrices $L_i(t) := I + tE_{i,i-1}$ and $U_j(t) := I + tE_{j-1,j}$, where $i, j = 2, \dots, n$, see, e.g., [4, Chapter 2].

THEOREM 4.3. [4, Theorem 2.2.2 and Corollary 2.2.3]. *Any n-by-n NsTN (respectively, TP) matrix can be written as*

$$(L_n(l_k)L_{n-1}(l_{k-1})\cdots L_2(l_{k-n+2}))(L_n(l_{k-n+1})\cdots L_3(l_{k-2n+4}))\cdots(L_n(l_1))\cdot$$
$$D(U_n(u_1))(U_{n-1}(u_2)U_n(u_3))\cdots(U_2(u_{k-n+2})\cdots U_{n-1}(u_{k-1})U_n(u_k)),$$

where $k = \binom{n}{2}$; $l_i, u_j \geq 0$ *(resp.,* $l_i, u_j > 0$*) for all* $i, j \in \{1, 2, \dots, k\}$*; and* $D = diag(d_1, \dots, d_n)$ *is a diagonal matrix with all* $d_i > 0$.

It is known that the numbers l_i and u_j can be represented as multipliers of Neville elimination (4.1), see [9]. So, by the relation given in the last subsection we can obtain these multipliers also by running the condensed Cauchon algorithm with $G := (A^{\#})^T$ and get in this way a bidiagonalization of A. Specifically, we have the following relations:

(a) $L_n(l_k) = L_n(\frac{\tilde{g}_{n,n-1}}{\tilde{g}_{nn}})$, $L_{n-1}(l_{k-1}) = L_{n-1}(\frac{\tilde{g}_{n,n-2}}{\tilde{g}_{n,n-1}})$, $\dots$, $L_2(l_{k-n+2}) = L_2(\frac{\tilde{g}_{n1}}{\tilde{g}_{n2}})$, $L_n(l_{k-n+1}) = L_n(\frac{\tilde{g}_{n-1,n-2}}{\tilde{g}_{n-1,n-1}})$, $L_{k-n}(l_{k-n}) = L_{k-n}(\frac{\tilde{g}_{n-1,n-3}}{\tilde{g}_{n-1,n-2}})$, $\dots$, $L_3(l_{k-2n+4}) = L_3(\frac{\tilde{g}_{n-1,1}}{\tilde{g}_{n-1,2}})$, $\dots$, $L_n(l_1) = L_n(\frac{\tilde{g}_{21}}{\tilde{g}_{22}})$;

(b) $d_{ii} = \tilde{g}_{ii}$;

(c) $U_n(u_k) = U_n(\frac{\tilde{g}_{n-1,n}}{\tilde{g}_{nn}})$, $U_{n-1}(u_{k-1}) = U_{n-1}(\frac{\tilde{g}_{n-2,n}}{\tilde{g}_{n-1,n}})$, $\dots$, $U_2(u_{k-n+2}) = U_2(\frac{\tilde{g}_{1n}}{\tilde{g}_{2n}})$, $U_n(u_{k-n+1}) = L_n(\frac{\tilde{g}_{n-2,n-1}}{\tilde{g}_{n-1,n-1}})$, $U_{k-n}(u_{k-n}) = U_{k-n}(\frac{\tilde{g}_{n-3,n-1}}{\tilde{g}_{n-2,n-1}})$, $\dots$, $U_3(u_{k-2n+4}) = U_3(\frac{\tilde{g}_{1,n-1}}{\tilde{g}_{2,n-1}})$, $\dots$, $U_n(u_1) = U_n(\frac{\tilde{g}_{12}}{\tilde{g}_{22}})$.

By setting $\frac{0}{0} := 0$, all the above quantities are defined since in the lower (respectively, upper) half of $\tilde{G}$ if one entry vanishes then all of the entries to the left of it (respectively, above it) vanish, too.

5. Applications.

5.1. New determinantal tests for totally nonnegative matrices. We recall from [16] the definition of a lacunary sequence.

DEFINITION 5.1. Let $C \in C_n$. We say that a sequence

$$\gamma := ((i_k, j_k), k = 0, 1, \dots, p) \tag{5.1}$$

which is strictly increasing in both arguments is a *lacunary sequence with respect to* C if the following conditions hold:

1. $(i_k, j_k) \notin C$, $k = 1, \dots, p$;

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2. $(i, j) \in C$ for $i_p < i \le n$ and $j_p < j \le n$.
3. Let $s \in \{0, \ldots, p-1\}$. Then $(i, j) \in C$ if
   either for all (i, j), $i_s < i < i_{s+1}$ and $j_s < j$,
   or for all (i, j), $i_s < i < i_{s+1}$ and $j_0 \le j < j_{s+1}$
   and
   either for all (i, j), $i_s < i$ and $j_s < j < j_{s+1}$
   or for all (i, j), $i < i_{s+1}$, and $j_s < j < j_{s+1}$.

A lacunary sequence with respect to the Cauchon diagram displayed in Figure 3.1 is the sequence $((1, 1), (2, 3), (4, 4))$.

PROPOSITION 5.2. [16, Proposition 4.1], *Let $A \in \mathbb{R}^{n,m}$ and C be a Cauchon diagram. For each entry of A fix a lacunary sequence γ (5.1) with respect to C starting at this entry. Assume that either (a) or (b) below holds*

(a) The matrix A is TN and $\tilde{A}$ is associated with C.
(b) For all (i_0, j_0), we have $0 = \det A[i_0, \ldots, i_p|j_0, \ldots, j_p]$ if $(i_0, j_0) \in C$ and $0 < \det A[i_0, \ldots, i_p|j_0, \ldots, j_p]$ if $(i_0, j_0) \notin C$.

Then

$$\det A[i_0, \ldots, i_p|j_0, \ldots, j_p] = \tilde{a}_{i_0,j_0}\tilde{a}_{i_1,j_1}\ldots\tilde{a}_{i_p,j_p} \tag{5.2}$$

for all lacunary sequences γ (5.1).

We start with a determinantal test for $NsTN$ matrices.

We relate to each entry $\tilde{a}_{i_0,j_0}$ of $\tilde{A}$ a sequence γ (5.1). In contrast to [14], [16], we do not start from a fixed Cauchon diagram but sequentially construct the lacunary sequences. It is sufficient to describe the construction of the sequence from the starting pair (i_0, j_0) to the next pair (i_1, j_1) with $0 < \tilde{a}_{i_1,j_1}$ since for a given matrix A the determinantal test is performed by moving row by row from the bottom to the top row. Once we have found the next index pair (i_1, j_1) we append to (i_0, j_0) the sequence starting at (i_1, j_1). By construction, the sequence is uniquely determined.

PROCEDURE 5.1. *Construction of the sequence γ (5.1) starting at (i_0, j_0) to the next index pair (i_1, j_1) in the $NsTN$ case*

> **If** *$i_0 = n$ or $j_0 = n$ or $S := \{(i, j) \mid i_0 < i \le n,\ j_0 < j \le n,$ and $0 < \tilde{a}_{ij}\}$ is void* **then** *terminate with $p := 0$;*
> **else**
>
> *put (i_1, j_1) as the minimum of S with respect to the colexicographic order*
> *and lexicographic order if $j_0 \le i_0$ and $i_0 < j_0$, respectively;*
>
> **end if**.

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After all sequences γ starting in row i_0 are determined it is checked whether the matrix $B := A[i_0,\dots,n|1,\dots,n]$ fulfills conditions (i), (ii), and (iii) of Theorem 5.3 below (with the obvious modifications for the rectangular case). If one of the conditions is violated for any instance, the test can be terminated since A is not $NsTN$. Otherwise, $\tilde{B}$ is a Cauchon matrix, whence B is TN and we are able to apply Proposition 5.2.

THEOREM 5.3. *Let $A \in \mathbb{R}^{n,n}$. Then A is $NsTN$ if and only if*

(i) $0 < \det A[i,\dots,n],\ \ i = 1,\dots,n$*;*
(ii) $0 \le \det A[i_0,\dots,i_p|j_0,\dots,j_p]$ *for all* $i_0, j_0 = 1,\dots,n$, $i_0 \ne j_0$, *where the sequences* γ *are obtained by Procedure 5.1.*
(iii) If $\tilde{a}_{ij} = 0$ *then all entries in row* i *to the left of it vanish if* $j < i$*, respectively, in column* j *above it if* $i < j$*.*

Proof. Let A be $NsTN$. Then by Theorem 3.3 (ii), (iii) $\tilde{A}$ is a nonnegative Cauchon matrix with $0 < \tilde{a}_{ii}$, $i = 1,\dots,n$ [4]. By application of Proposition 5.2 to the lacunary sequences $\gamma = ((i,i),\dots,(n,n))$, $i = 1,\dots,n$, we obtain

$$\det A[i,\dots,n] = \tilde{a}_{ii}\cdots\tilde{a}_{nn},$$

hence (i) follows. Condition (ii) trivially holds since A is TN. Condition (iii) is a consequence of the fact that $\tilde{A}$ is a Cauchon matrix and (i). Conversely, by Theorem 3.3 (ii), (iii) it suffices to show that $\tilde{A}$ is a nonnegative Cauchon matrix with positive diagonal entries. The pairs $((n,j))$, $((i,n))$, $i,j = 1,\dots,n$, trivially form lacunary sequences, and hence, by (ii), all entries in the last row and last column are nonnegative. Furthermore, by (i) $0 < a_{nn} = \tilde{a}_{nn}$ and by (iii) if one entry in the last row vanishes then all entries left of it vanish and if one entry in the last column vanishes then all of the entries above it vanish, too. Now we construct by Procedure 5.1 row by row from the bottom to the top and in each row from the right to the left for each pair (i,j) the lacunary sequence starting at this pair, $i = n-1,\dots,1$, $j = n-1,\dots,1$. We proceed by induction on the pairs (i,j). Suppose that we have shown that $0 \le \tilde{a}_{ij}$ for all $i = i_0+1,\dots,n$, $j = 1,\dots,n$, and also for $i = i_0$ and $j = j_0+1,\dots,n$. Consider the submatrix $D := A[i_0,\dots,n|j_0,\dots,n]$; then $\tilde{D} = \tilde{A}[i_0,\dots,n|j_0,\dots,n]$ is a Cauchon matrix by (iii) and is nonnegative by the induction hypothesis with possible exception of $\tilde{d}_{11}$. We replace d_{11} by $d_{11}+t$ with a sufficiently large positive constant t to make $\tilde{d}_{11}+t$ nonnegative if necessary (put $t := 0$ if $\tilde{d}_{11}$ is nonnegative); rename the resulting matrix by D. Use of Proposition 5.2 with the lacunary sequence $((i_0,j_0),\dots,(i_p,j_p))$ associated with (i_0,j_0) according to Procedure 5.1 and application of Laplace's expansion to the related minor of D yield

$$\det A[i_0,\dots,i_p|j_0,\dots,j_p] + t\det A[i_1,\dots,i_p|j_1,\dots,j_p]$$

[4] As a consequence, the set S defined in Procedure 5.1 is not void.

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$$= \tilde{a}_{i_0,j_0} \cdot \tilde{a}_{i_1,j_1} \cdots \tilde{a}_{i_p,j_p} + t\tilde{a}_{i_1,j_1} \cdots \tilde{a}_{i_p,j_p}.$$

By construction, $((i_1, j_1), \ldots, (i_p, j_p))$ is the lacunary sequence starting at (i_1, j_1) with respect to the Cauchon diagram associated with $A[i_1, \ldots, n|j_1, \ldots, n]$. Application of Proposition 5.2 gives

$$\det A[i_1, \ldots, i_p|j_1, \ldots, j_p] = \tilde{a}_{i_1,j_1} \cdots \tilde{a}_{i_p,j_p},$$

whence

$$\det A[i_0, \ldots, i_p|j_0, \ldots, j_p] = \tilde{a}_{i_0,j_0} \cdot \tilde{a}_{i_1,j_1} \cdots \tilde{a}_{i_p,j_p}.$$

Therefore, we conclude that

$$\tilde{a}_{i_0,j_0} = \frac{\det A[i_0, \ldots, i_p|j_0, \ldots, j_p]}{\tilde{a}_{i_1,j_1} \cdots \tilde{a}_{i_p,j_p}}$$

from which it follows by (ii) that $0 \leq \tilde{a}_{i_0,j_0}$. To show that $0 < \tilde{a}_{ii}$ for all $i = n, \ldots, 1$, suppose that $1 \leq t < n$ is the largest integer such that $\tilde{a}_{tt} = 0$. Then the sequence $((t, t), (t + 1, t + 1), \ldots, (n, n))$ is a lacunary sequence and by Proposition 5.2 we have that $\det A[t, \ldots, n] = 0$, a contradiction to (i). Hence, A is $NsTN$. $\square$

If we proceed from row $i_\mu + 1$ to row i_μ we already know the determinantal entries which appear in row $i_\mu + 1$ and therefore we can easily check when $j_\mu < i_\mu$ whether all entries in the row $i_\mu + 1$ to the left of $\tilde{a}_{i_\mu+1,j_\mu+1}$ vanish. To check in the case $i_\mu < j_\mu$ whether all entries in the column $j_\mu + 1$ above $\tilde{a}_{i_\mu+1,j_\mu+1}$ vanish we have to compute in addition the minors which are associated with the positions $(s, j_\mu + 1)$, $s = 1, \ldots, i_\mu$. These minors differ in only one row index. Since a zero column stays a zero column through the performance of the Cauchon algorithm, the sign of altogether n^2 minors have to be checked (which include also trivial minors of order 1). These are significantly fewer than the number of determinants required by the determinantal test which is based on Theorem 4.1 (iii), the number of which is the number of the quasi-initial minors of A minus the number of the leading principal minors (which are twice counted), i.e.,

$$2\sum_{k=1}^{n} \binom{n}{k} - n = 2^{n+1} - n - 2.$$

However, the determinantal test based on Theorem 4.1 (iii) is independent of the matrix to be checked in contrast to the test based on Theorem 5.3 which in addition requires as a preprocessing step the computation of $\tilde{A}$. If we test a given matrix A for being TP it suffices to check n^2 fixed determinants (independent of A) for positivity. In this case, all sequences γ are running diagonally and we obtain just the numerators of the determinantal ratios which are listed in (4.5). The numerators of the entries in

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the first column and in the first row of $\tilde{A}$ are the so-called *corner minors*, i.e., minors of the form $\det A[\alpha|\beta]$ in which α consists of the first k and β consists of the last k indices or vice versa, $k = 1, \ldots, n$, see (4.5).

In the event that it is somehow known that $A \in \mathbb{R}^{n,n}$ is TN and if its corner minors are positive, then by condition (iii) of Theorem 5.3 $\tilde{A}$ does not contain any zero entry and we can conclude that A is TP. This provides a short proof of the fact that positivity of the corner minors of a TN matrix A implies that A is TP [12, Theorem 4.2], [13, Theorem D], see also [4, Theorem 3.1.10].

Now we extend the results above to the TN case and associate with each entry $\tilde{a}_{i_0,j_0}$ of $\tilde{A}$ a uniquely determined sequence γ (5.1). Again we describe only the construction of the sequence from the starting pair (i_0, j_0) to the next pair (i_1, j_1) with $0 < \tilde{a}_{i_1,j_1}$.

Procedure 5.2. *Construction of the sequence γ (5.1) starting at (i_0, j_0) to the next index pair (i_1, j_1) in the TN case*

> **If** $i_0 = n$ *or* $j_0 = n$ *or* $S := \{(i,j) \mid i_0 < i \leq n,\ j_0 < j \leq n, \text{ and } 0 < \tilde{a}_{ij}\}$ *is void* **then** *terminate with* $p := 0$;
> **else**
>   **if** $\tilde{a}_{ij_0} = 0$ *for all* $i = i_0 + 1, \ldots, n$ **then** *put* $(i_1, j_1) := \min S$ *with respect to the colexicographic order*
>   **else**
>   **put** $i' := \min\{k \mid i_0 < k \leq n \text{ such that } 0 < \tilde{a}_{kj_0}\}$,
> $J := \{k \mid j_0 < k \leq n \text{ such that } 0 < \tilde{a}_{i',k}\}$;
>   **if** J *is not void* **then** *put* $(i_1, j_1) := (i', \min J)$
>   **else** *put* $(i_1, j_1) := \min S$ *with respect to the lexicographic order;*
>   **end if**
>   **end if**
> **end if**.

As in the $NsTN$ case, we proceed row by row from the bottom to the top row. After all sequences γ starting in row i_0 are determined it is checked whether the matrix $A[i_0, \ldots, n|1, \ldots, n]$ fulfills conditions (i), (ii) of Theorem 5.4 below (with the obvious modifications for the rectangular case). Its proof is similar to the one of Theorem 5.3 and is therefore omitted.

Theorem 5.4. *Let $A \in \mathbb{R}^{n,n}$. Then A is TN if and only if*

(i) $0 \leq \det A[i_0, \ldots, i_p|j_0, \ldots, j_p]$ *for all* $i_0, j_0 = 1, \ldots, n$, *where the sequences* γ *are obtained by Procedure 5.2.*
(ii) If $\tilde{a}_{ij} = 0$ *then all entries in row* i *to the left of it or all entries in column* j *above it vanish.*

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Proof. We proceed similarly as in the proof of Theorem 5.3 and only show that each sequence constructed by Procedure 5.2 is lacunary if we pose in addition condition (ii) of Theorem 5.4. As in Procedure 5.2, we consider only the part of the sequence between two adjacent pairs. We distinguish three cases:
If $0 = \tilde{a}_{ij_0}$ for all $i = i_0+1, \ldots, n$ it follows from the choice of (i_1, j_1) that $0 = \tilde{a}_{ij}$ for all $i_0 < i$, $j_0 < j < j_1$, and if $0 < \tilde{a}_{i_0,j_0}$ we conclude by (ii) that $0 = \tilde{a}_{ij}$ for all $i_0 < i$, $j < j_0$. If the set J is not void and $j' := \min J$ it follows that $0 = \tilde{a}_{i'k}$ for $j_0 < k < j'$ and from (ii) by $0 < \tilde{a}_{i',j_0}$ that $0 = \tilde{a}_{ij}$ for all $i < i'$, $j_0 < j < j'$; if $0 < \tilde{a}_{i_0,j_0}$ we conclude from $0 = \tilde{a}_{kj_0}$ for $i_0 < k < i'$ and (ii) that $0 = \tilde{a}_{ij}$ for all $i_0 < i < i'$, $j < j_0$. Finally, if J is void then $0 = \tilde{a}_{i'j}$ for all $j_0 < j$ and since $0 < \tilde{a}_{i',j_0}$ it follows by (ii) that $0 = \tilde{a}_{ij}$ for all $i < i'$, $j_0 < j$, and therefore $i' < i_1$. By choice of (i_1, j_1) we have $0 = \tilde{a}_{ij}$ for all $i_0 < i < i_1$, $j_0 < j$.
Therefore, conditions 3. and 4. of Definition 5.1 are fulfilled in all three cases. □

Similar to the $NsTN$ case, only n^2 minors have to be checked for nonnegativity. Note that by [4, Example 3.3.1] there cannot be a specified fixed proper subset of all minors which is sufficient for testing a general matrix for being TN.

The sufficient sets of minors presented in this subsection are only of theoretical value, and they do not lead to efficient methods to check a given matrix for being $NsTN$ or TN, even when some advantage is taken in the overlap of calculation of different minors. More efficient methods are based on Theorems 3.3 (ii), (iii) and 4.1 (i),(ii). In the following subsections we apply our results to special classes of TN matrices.

5.2. Application to several subclasses of the totally nonnegative matrices.

5.2.1. Oscillatory matrices. It is known that a TN matrix is oscillatory if and only if it is nonsingular and the entries on its first sub- and superdiagonal are positive, see, e.g., [4, Theorem 2.6.5].

THEOREM 5.5. *Let $A \in \mathbb{R}^{n,n}$ be TN and $\tilde{A}$ be the matrix obtained from A by the Cauchon algorithm. Then A is oscillatory if $0 < \tilde{a}_{ij}$ whenever $|i-j| \le 1$.*

Proof. Suppose that A is TN and all entries of $\tilde{A}$ on its main diagonal and on its first sub- and superdiagonal are positive. Then the matrix A is nonsingular by Theorem 3.3 (iii). The sequences running along its sub- and superdiagonal are lacunary and by Proposition 5.2 the matrices $A[2, \ldots, n|1, \ldots, n-1]$, and $A[1, \ldots, n-1|2, \ldots, n]$ are nonsingular. By Lemma 2.3 the positivity of the entries on their main diagonals follows. □

The condition in Theorem 5.5 in not necessary as the following example shows.

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EXAMPLE 5.3. *Choose*

$$A := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 3 & 6 \end{pmatrix}.$$

Then A is oscillatory, however, $\tilde{a}_{21} = 0$.

Using the results from Subsection 4.3, we obtain from [4, Theorem 2.6.4] the following characterization of an oscillatory matrix in terms of the Cauchon algorithm.

THEOREM 5.6. *Let $A \in \mathbb{R}^{n,n}$ be TN and $\tilde{A}$ be the matrix obtained from A by the Cauchon algorithm. Then A is oscillatory if and only if $0 < \tilde{a}_{ii}$ and there is no index k such that $\tilde{a}_{ik} = 0$, $i = k+1, \dots, n$, or $\tilde{a}_{ki} = 0$, $i = k+1, \dots, n$.*

5.2.2. Tridiagonal matrices. In this subsection, we present a short proof of conditions for an entry-wise nonnegative tridiagonal matrix to be TN and $NsTN$. We have applied these conditions in [1] to find the largest amount by which the single entries of such a matrix can be perturbed without losing the property of being $TN/NsTN$.

THEOREM 5.7. [15, Lemma 6], [17, p. 100]. *Let $A \in \mathbb{R}^{n,n}$ be tridiagonal and entry-wise nonnegative.*

(a) Let $2 < n$. Then A is TN if the following conditions hold
  (i) $0 \le \det A$,
  (ii) $0 \le \det A[1, \dots, n-1]$,
  (iii) $0 < \det A[1, \dots, i]$, $i = 1, \dots, n-2$.
(b) A is $NsTN$ if and only if $0 < \det A[1, \dots, i]$, $i = 1, \dots, n$.

Proof. Let A be tridiagonal and entry-wise nonnegative and satisfy the conditions $(i)-(iii)$ of (a). Put $B := A^{\#}$, then the entries of B are given by $b_{ij} = a_{n-i+1,n-j+1}$, $i, j = 1, \dots, n$, and we have $\det B[n-i+1, \dots, n] = \det A[1, \dots, i]$, $i = 1, \dots, n$. Application of the Cauchon algorithm results in the matrix $\tilde{B} = (\tilde{b}_{ij})$ with

$$\tilde{b}_{ij} = b_{ij} \ge 0, \quad i, j = 1, \dots, n, \ i \ne j, \tag{5.3}$$

$$\tilde{b}_{ii} = \frac{\det B[i, \dots, n]}{\det B[i+1, \dots, n]} \ge 0, \quad i = 2, \dots, n, \tag{5.4}$$

$$\tilde{b}_{11} = \begin{cases} \frac{\det B}{\det B[2,\dots,n]} \ge 0 & \text{if } \det B[2, \dots, n] > 0, \\ b_{11} \ge 0 & \text{if } \det B[2, \dots, n] = 0. \end{cases} \tag{5.5}$$

Since $\tilde{B}$ is entry-wise nonnegative, by Theorem 3.3 (ii) it remains to show that $\tilde{B}$ is a Cauchon matrix. The only case we have to consider is the case $0 = \det A[1, \dots, n-1]$

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$= \det B[2,\ldots,n]$. Then by condition (i) and (5.3),

$$\begin{aligned} 0 \le \det A = \det B &= b_{11} \det B[2,\ldots,n] \,-\, b_{12}b_{21} \det B[3,\ldots,n] \\ &= -\,\tilde{b}_{12}\tilde{b}_{21} \det A[1,\ldots,n-2], \end{aligned}$$

from which it follows by condition (iii) that $\tilde{b}_{12} = 0$ or $\tilde{b}_{21} = 0$ which completes the proof of (a). If all leading principal minors of A are positive then by (5.4) and (5.5) $0 < \tilde{b}_{ii}$, $i = 1,\ldots,n$, so that $\tilde{B}$ is a Cauchon matrix. The necessity follows by Theorem 3.3 (iii). ☐

5.2.3. Pentadiagonal matrices. Pentadiagonal TN matrices are considered in [11], [13].

Theorem 5.8. *Let $2 < n$ and $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be pentadiagonal and $0 < a_{ij}$ if $|i-j| \le 2$. Then A is $NsTN$ if the following two conditions hold*

(i) $0 < \det A[i,\ldots,n]$, $i = 1,\ldots,n$,
(ii) $0 < \det A[i,\ldots,n|i-1,\ldots,n-1]$, $\det A[i-1,\ldots,n-1|i,\ldots,n]$, $i = 2, \ldots,n$.

Proof. Assume that conditions (i) and (ii) hold true. Then A is nonsingular by (i). Application of the Cauchon algorithm results in the matrix $\tilde{A}$. We show by decreasing induction on the row index i that the entries on the main diagonal and the first subdiagonal of $\tilde{A}$ are positive. By assumption, the entries $\tilde{a}_{n,n-1} = a_{n,n-1}$ and $\tilde{a}_{nn} = a_{nn}$ are positive. Suppose that the assumption holds true for the rows with index larger than i. Then the entries in the i^{th} row of $\tilde{A}$ up to position i are as follows

$$0,\ldots,a_{i,i-2}, \frac{\det A[i,\ldots,n|i-1,\ldots,,n-1]}{\det A[i+1,\ldots,n|i,\ldots,n-1]}, \frac{\det A[i,\ldots,n]}{\det A[i+1,\ldots,n]}$$

and by conditions (i), (ii) $\tilde{a}_{i,i-1}$ and $\tilde{a}_{ii}$ are positive. Similarly, one shows that the entries on the first superdiagonal of $\tilde{A}$ are positive, too. Therefore, $\tilde{A}$ is a nonnegative Cauchon matrix and we can conclude by Theorem 3.3 (ii) that A is TN. ☐

Example 5.3 shows that condition (ii) is not necessary since $0 = \det A[2,3|1,2]$.

Corollary 5.9. *Let $2 < n$ and $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be pentadiagonal and $0 < a_{ij}$ if $|i-j| \le 2$. Then A is $NsTN$ if and only if the following three conditions hold*

(i) $0 < \det A[i,\ldots,n]$, $i = 1,\ldots,n$,
(ii) $0 < \det A[i,\ldots,n|i-1,\ldots,n-1]$, $\det A[i-1,\ldots,n-1|i,\ldots,n]$, $i = 3, \ldots,n$,
(iii) $0 \le \det A[2,\ldots,n|1,\ldots,n-1]$, $\det A[1,\ldots,n-1|2,\ldots,n]$.

Proof. Suppose first that A is $NsTN$ with positive entries on its main diagonal and on its first two sub- and superdiagonals. Then (iii) trivially holds and

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(i) is satisfied by Lemma 2.3. Let k be the largest index greater than 2 for which $\det A[k,\ldots,n|k-1,\ldots,n-1]$ vanishes. Then $\tilde{a}_{k,k-1} = 0$ and by Theorem 3.3 (i) and Lemma 2.2 $0 = \tilde{a}_{k,k-2} = a_{k,k-2}$ which contradicts our assumption. The positivity of $\det A[i-1,\ldots,n-1|i,\ldots,n]$ can be shown analogously. The sufficiency follows as in the proof of Theorem 5.8. □

Condition (ii) of Theorems 5.8 and 5.9 can somewhat be relaxed. Suppose that A is a $NsTN$ pentadiagonal matrix with $0 < a_{ij}$ if $|i-j| \leq 1$. Let k be the largest index such that $\tilde{a}_{k,k-1} = 0$; then by Theorem 5.3 (iii) $\tilde{a}_{k,k-2} = a_{k,k-2}$ vanishes, too. Imagine that we want to construct a lacunary sequence starting at position $(i, i-1)$ and passing through position $(k-1, k-2)$. Since $\tilde{a}_{k,k-1} = 0$ we proceed with $\tilde{a}_{k+1,k-1}$. If this entry is zero we continue with going down, otherwise we go diagonally. If we succeed to continue with the construction of the lacunary sequence until we reach a positive entry in the last row then we obtain by Proposition 5.2 a determinant which according to the zero pattern of a pentadiagonal matrix is the product of the determinant of an upper triangular matrix which corresponds to the part of the lacunary sequence below and to the right of position $(k, k-1)$ and the determinant of a matrix which corresponds to the lacunary sequence above and to the left to this entry. Since the determinant of the upper triangular matrix is positive it suffices to check the sign of the minors $\det A[i,\ldots,k-1|i-1,\ldots,k-2]$. A similar procedure applies if there is a second largest index $k^{'}$ with $\tilde{a}_{k',k'-1} = 0$.

5.2.4. Almost totally positive matrices. The almost totally positive matrices form a class of matrices intermediate to TN and TP matrices. They are called *inner totally positive* matrices in [13]. Examples and properties of these matrices can be found in [8], [10] and the references therein.

THEOREM 5.10. *Let $A \in \mathbb{R}^{n,n}$ be TN. Then the following two properties are equivalent:*

(i) A is ATP.
(ii) The Cauchon diagrams C_A and $C_{\tilde{A}}$ associated with A and $\tilde{A}$ (obtained from A by the Cauchon algorithm), respectively, are identical and all squares on the diagonal of $C_{\tilde{A}}$ are colored white.

Proof. Let A be ATP, then A is nonsingular and by Theorem 3.3 (iii) $\tilde{A}$ has a positive diagonal, i.e., $C_{\tilde{A}}$ has a white diagonal. Suppose that C_A and $C_{\tilde{A}}$ differ by the entry in position (i_0, j_0); assume first that $0 < a_{i_0,j_0}$ whereas $\tilde{a}_{i_0,j_0} = 0$. Fix a lacunary sequence γ (5.1) with respect to $C_{\tilde{A}}$. According to the definition of a lacunary sequence, the entries $\tilde{a}_{i_k,j_k}$, $k = 1,\ldots,p$, are positive. By Proposition 5.2 it follows that $\det A[i_0,\ldots,i_p|j_0,\ldots,j_p] = 0$. The entries of A in the respective positions, i.e., a_{i_k,j_k} are positive, too, $k = 1,\ldots,p$, since a zero entry stays a zero entry through the performance of the Cauchon algorithm when it is applied to a $NsTN$ matrix. Since

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A is ATP it follows that $0 < \det A[i_0, \ldots, i_p|j_0, \ldots, j_p]$, a contradiction. The case $0 < \tilde{a}_{i_0,j_0}$ and $a_{i_0,j_0} = 0$ is excluded by the above argument of the invariance of zero entries during the performance of the Cauchon algorithm.
Now suppose that (ii) holds true. Then A is nonsingular. Fix a k-by-k submatrix $B := A[\alpha|\beta]$ of A, $1 < k$. The results in [10] show that we may restrict the discussion to contiguous α and β. If $0 < \det B$ then by Lemma 2.3 all diagonal entries of B are positive. On the other hand, if all entries of the main diagonal of B are positive then by (ii) the entries of $\tilde{A}$ in the same positions are positive. If $\alpha_k = n$ or $\beta_k = n$ these positions form a lacunary sequence. Otherwise, we fix a lacunary sequence (with respect to $C_{\tilde{A}}$) which starts at position (α_k, β_k) and append it to the sequence $((\alpha_1, \beta_1), \ldots, (\alpha_{k-1}, \beta_{k-1}))$. In both cases we obtain a lacunary sequence starting at position (α_1, β_1). By Proposition 5.2 the associated submatrix D of A has a positive determinant and since B is a principal submatrix of D it follows by Lemma 2.3 that $0 < \det B$ which completes the proof. □

5.2.5. Green's matrices. Let two sequences $c_1, \ldots, c_n$ and $d_1, \ldots, d_n$ of nonzero real numbers be given. We define the entry a_{ij} of the n-by-n matrix A by

$$a_{ij} := c_{\min\{i,j\}} d_{\max\{i,j\}}. \tag{5.6}$$

The matrix A is called a *Green's matrix* (also referred to as a *single-pair matrix* in [5, pp. 90–91]). It is known that the nonsingular Green's matrices are the inverses of symmetric tridiagonal matrices, [17, Section 4.5].

THEOREM 5.11. [5, p. 91], see also [17, Theorem 4.2]. *The Green's matrix A defined in (5.6) is TN if and only if the c_i and d_j are all of the same strict sign and the inequalities*

$$\frac{c_1}{d_1} \le \frac{c_2}{d_2} \le \cdots \le \frac{c_n}{d_n}. \tag{5.7}$$

hold true. If n' is the number of the strict inequality signs in (5.7) then $\operatorname{rank}(A) = n' + 1$.

Proof. When we compute the matrix $A^{(n-1)}$ from $A^{(n)} = A$ by Algorithm 3.2 all the entries between the main diagonal and the last row of $A^{(n-1)}$ become zero

$$a_{ij}^{(n-1)} = c_j d_i - \frac{c_j d_n c_{j+1} d_i}{c_{j+1} d_n} = 0, \text{ for all } 1 \le j < i \le n-1.$$

Furthermore, the entries on the main diagonal are already in their final form, i.e., $a_{ii}^{(n-1)} = \tilde{a}_{ii}$, $i = 1, \ldots, n$. Since A is symmetric, $\tilde{A}$ is symmetric, too, whence also the entries between the main diagonal and the last column of $\tilde{A}$ are zero. According to Theorem 3.3 (ii), A is then TN if and only if the diagonal entries of $A^{(n-1)}$ are

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nonnegative, i.e.,

$$0 \leq a_{ii}^{(n-1)} = c_i d_i - \frac{c_i d_n c_i d_{i+1}}{c_{i+1} d_n}, \quad i = 1, \ldots, n-1,$$

which is equivalent to (5.7). Suppose that

$$\frac{c_i}{d_i} = \frac{c_{i+1}}{d_{i+1}}. \tag{5.8}$$

Then row i of A is the $\frac{c_i}{c_{i+1}}$-multiple of row $i+1$. We delete in A the i^{th} row and column if (5.8) holds, $i = 1, \ldots, n-1$. The resulting matrix is denoted B and we have $rank(B) = rank(A)$. All diagonal entries of $\tilde{B}$ (obtained from B by the Cauchon algorithm) are positive and by Theorem 3.3 (iii) B is nonsingular which completes the proof. □

5.2.6. (0,1)-Matrices. In this subsection, we present a short proof of a characterization of TN (0,1)-matrices, i.e., of matrices the entries of which are only 0's and 1's.

Definition 5.12. The matrix $A \in \mathbb{R}^{m,n}$ is said to be in *double echelon form* if

(i) Each row of A has one of the following forms (an asterisk denotes a nonzero entry):
  (1) $(*, *, \ldots, *)$,
  (2) $(*, \ldots, *, 0, \ldots, 0)$,
  (3) $(0, \ldots, 0, *, \ldots, *)$, or
  (4) $(0, \ldots, 0, *, \ldots, *, 0, \ldots, 0)$.

(ii) The first and last nonzero entries in row $i+1$ are not to the left of the first and last nonzero entries in row i, respectively ($i = 1, 2, \ldots, n-1$).

Theorem 5.13. [3, Theorem 2.2], [4, Theorem 1.6.9]. *Let $A \in \mathbb{R}^{m,n}$ be a (0,1)-matrix with no zero rows or columns. Then A is TN if and only if A is in double echelon form and does not contain the submatrix*

$$B := \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Proof. By [4, Corollary 1.6.5], A must be in double echelon form. The necessity is trivial since $\det B = -1$. To prove the sufficiency we run Algortihm 3.2 on A. Then $A^{(n-1)}[1, \ldots, n-1|1, \ldots, n]$ cannot contain the entry -1 since A was supposed to be in double echelon form. So the only possible entries are 0's and 1's. The only problematic case is $\tilde{a}_{ij} = 0$ resulting from $a_{ij} = a_{nj} = a_{n,u_j} = a_{i,u_j} = 1$, where u_j is

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defined in Algorithm 3.2. Then it follows that $\tilde{a}_{ik} = 0$, $k = 1, \ldots, j-1$, or $\tilde{a}_{kj} = 0$, $k = 1, \ldots, i-1$, because otherwise $A^{(n-1)}$ would contain a submatrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

This submatrix would result from a submatrix B in the matrix A which is excluded by our assumption. We can conclude that $A^{(n-1)}$ is in double echelon form and therefore a nonnegative Cauchon matrix. It does not contain the submatrix B because otherwise $b_{13} = 0$ would result from a submatrix of A which would imply that $b_{23} = 0$, a contradiction. Now we proceed by induction and obtain that $\tilde{A}$ is a nonnegative Cauchon matrix and by Theorem 3.3 (ii) A is TN. □

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