

INEQUALITIES INVOLVING EIGENVALUES FOR DIFFERENCE OF OPERATOR MEANS*

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Abstract. In a recent paper by Hirzallah et al. [O. Hirzallah, F. Kittaneh, M. Krnić, N. Lovričević, and J. Pečarić. Eigenvalue inequalities for differences of means of Hilbert space operators. *Linear Algebra and its Applications*, 436:1516–1527, 2012.], several eigenvalue inequalities are obtained for the difference of weighted arithmetic and weighted geometric means of two positive invertible operators A and B on a separable Hilbert space under the condition that $A - B$ is compact. This paper aims to prove some general versions of eigenvalue inequalities for the difference of weighted arithmetic, weighted geometric and generalized Heinz means with better bounds under the same conditions.

Key words. Hilbert space, Positive operator, Compact operator, Operator mean, Calkin algebra, Heinz inequality.

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1. Introduction. Let $\mathbb{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on a complex separable Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. The cone of positive operators is denoted by $\mathbb{B}(\mathcal{H})_+$. As usual for selfadjoint operators $A, B \in \mathbb{B}(\mathcal{H})$, by $A \geq B$, we mean $A - B \in \mathbb{B}(\mathcal{H})_+$. We shall consider $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq 0$, the eigenvalues of a compact operator $A \in \mathbb{B}(\mathcal{H})_+$ arranged in the decreasing order and repeated according to their multiplicity. The set of eigenvalues of $X \in \mathbb{B}(\mathcal{H})$ is called the spectrum of X and is denoted by $Sp(X)$. I stands for the identity operator.

There is a vast literature on operator connections and means, for instance see [10, 11]. A one-to-one correspondence between the class of operator connections and the operator monotone functions was established by Kubo and Ando [10], as follows:

$$(1.1) \quad A\sigma B = A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}$$

for all positive operators A and B (Here $A^{1/2}$ is the positive square root of A and σ is a connection with f its representing function).

A normalized connection σ , i.e., $I\sigma I = I$, is called a mean, and accordingly the representing function $f(x)$ satisfies $f(1) = 1$. The dual $\sigma^\perp$ of a non-zero mean σ ,

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defined by $A\sigma^\perp B = (B^{-1}\sigma A^{-1})^{-1}$ for positive invertible operators A, B . We easily see from (1.1) that the representing operator monotone function of $\sigma^\perp$ is $\frac{x}{f(x)}$, where $f(x)$ is the representing function of σ .

The well known examples of operator monotone functions are $(1-\alpha)+\alpha x$, x^α , and $(\alpha x^{-1} + 1 - \alpha)^{-1}$ for $0 \leq \alpha \leq 1$. These functions represent the weighted arithmetic, weighted geometric, and weighted harmonic means and these means are denoted by ∇_α , $\#_\alpha$ and $!_\alpha$, respectively. The representing functions of $\nabla_\alpha^\perp$, $\#_\alpha^\perp$, and $!_\alpha^\perp$ are given by $((1-\alpha)x^{-1} + \alpha)^{-1}$, $x^{1-\alpha}$, and $(1-\alpha)x + \alpha$, respectively. In particular, we take $\nabla_{1/2}$, $\#_{1/2}$, and $!_{1/2}$ or simply written as ∇ , $\#$, and $!$, the arithmetic, geometric, and harmonic means, respectively. It is easy to see that ∇ and $!$ are duals of each other, while $\#$ is a self dual mean.

The notion of generalized operator Heinz mean is given in [7], i.e.,

$$H_\sigma(A, B) = \frac{A\sigma B + A\sigma^\perp B}{2},$$

for two invertible operators $A, B \in \mathbb{B}(\mathcal{H})_+$.

Observe from (1.1) that the representing function of H_σ is $\frac{f(x) + (x/f(x))}{2}$, where $f(x)$ is the representing function of the mean σ . Notice that the following inequality

$$\sqrt{x} \leq \frac{f(x) + (x/f(x))}{2} \leq \frac{1+x}{2},$$

holds true for all $x \in \mathbb{R}^+$. This leads to an interesting operator inequality,

$$A\#B \leq H_\sigma(A, B) \leq A\nabla B$$

for any two invertible operators $A, B \in \mathbb{B}(\mathcal{H})_+$.

The operator means have always been of great importance in several branches of science like electronics, electrical network theory, image scanning, radar system, etc. For detailed study of operator means, the best sources are [2, 8, 9, 10].

Hirzallah et al. [6] in 2012, proved the following results.

Let $A \geq B$ be positive invertible operators with $A - B$ compact. Then

$$(1.2) \quad \lambda_j(A\nabla_\mu B - A\#_\mu B) \geq K(\mu, \beta)\lambda_j\left(A^{-1/2}(A-B)^2A^{-1/2}\right), \quad \beta \geq 0,$$

$$(1.3) \quad \lambda_j(B\nabla_\mu A - B\#_\mu A) \leq K(\mu, \gamma)\lambda_j\left(B^{-1/2}(A-B)^2B^{-1/2}\right), \quad -1 < \gamma \leq 0,$$

where $K(\mu, t) = \frac{\mu(1-\mu)}{2(1+t)^2}$ for $\mu \in [0, 1]$ and $j = 1, 2, \dots$

In 2014, Pal, Singh and Aujla [13] generalized (1.2) and (1.3) as follows:

Let $f : (0, \infty) \rightarrow (0, \infty)$ be an operator monotone function representing a mean σ with $\mu = f'(1)$. Then, for $A \geq B$ positive invertible operators with $A - B$ compact,

$$(1.4) \quad \lambda_j(A \nabla_\mu B - A \sigma B) \geq K_f(a) \lambda_j \left(A^{-1/2} (A - B)^2 A^{-1/2} \right), \quad a \geq 0,$$

$$(1.5) \quad \lambda_j(B \nabla_\mu A - B \sigma A) \leq K_f(b) \lambda_j \left(B^{-1/2} (A - B)^2 B^{-1/2} \right), \quad -1 < b \leq 0,$$

where $K_f(t) = \frac{-f''(1)}{2(1+t)^2}$ and $j = 1, 2, \dots$

Our aim in this paper is to present a most general and different version of eigenvalue inequalities for difference of operator means. These conclude eigenvalue inequalities with better bounds, (see Remark 3.2) and subsume the existing inequalities (1.2), (1.3), (1.4) and (1.5). We also discuss several related inequalities which include a comparison of the eigenvalues of the difference of operator arithmetic, geometric, and generalized Heniz means.

2. Basic results.

LEMMA 2.1. Let $f : (0, \infty) \rightarrow (0, \infty)$ be an operator monotone function and let $f'(1) \geq 1/2$ (resp., $f'(1) \leq 1/2$). Then

$$(2.1) \quad f(1) - f'(1) + f'(1)x - f(x) \geq \frac{-3f''(1)(x-1)^2}{2((2f(1) - f'(1))x + (f(1) + f'(1)))}$$

for $x > 1$ (resp., $x < 1$). The order of the inequality (2.1) reverses for $x < 1$ (resp., $x > 1$). Moreover, when $f(x)$ is non-linear the equality holds if and only if $x = 1$.

Proof. As is well-known (see [1, 10]), $f(x)$ admits an integral representation

$$f(x) = f(0) + \beta x + \int_{(0, \infty)} \frac{x(1+\lambda)}{x+\lambda} d\nu(\lambda) = f(0) + \beta x + \int_{(0, \infty)} \phi_\lambda(x) d\nu(\lambda),$$

where $\beta \geq 0$, ν a positive measure on $(0, \infty)$ and $\phi_\lambda(x) = \frac{x(1+\lambda)}{x+\lambda}$ for $x, \lambda \in (0, \infty)$. The result holds trivially when $f(x)$ is linear. We therefore, assume that $f(x)$ is non-linear i.e. $f(x) = \phi_\lambda(x) = \frac{x(1+\lambda)}{x+\lambda}$. Since $\phi'_\lambda(1) = \frac{\lambda}{1+\lambda}$ and $\phi''_\lambda(1) = \frac{-2\lambda}{(1+\lambda)^2}$ then on taking $\mu = \frac{\lambda}{1+\lambda}$, the inequality (2.1) reduces to

$$1 - \mu + \mu x - \frac{x}{(1-\mu)x + \mu} - \frac{3\mu(1-\mu)(x-1)^2}{((2-\mu)x + (1+\mu))} \geq 0.$$

After small calculations, we see that the above inequality becomes equivalent to

$$(2.2) \quad \frac{\mu(1-\mu)(x-1)^2}{3((1-\mu)x+\mu)\left(\frac{(2-\mu)x}{3}+\frac{1+\mu}{3}\right)}(1-x)(1-2\mu) \geq 0.$$

Clearly, $\mu \leq 1$, so (2.2) holds if

$$(2.3) \quad (1-x)(1-2\mu) \geq 0.$$

The inequality (2.3) is true under the given conditions, i.e. $\mu \geq 1/2$ (*resp.*, $\mu \leq 1/2$) when $x > 1$ (*resp.*, $x < 1$). This completes the proof of (2.1).

The proof of reverse version is similar, therefore not provided. The equality conditions follow from (2.2). $\square$

REMARK 2.2. The inequality (2.1) is better than an inequality given in [7] (Lemma 2.1). However, the inequality in Lemma 2.1 in [7] is already proved better than one given in [13], (see [7], Remark 2.2).

LEMMA 2.3. *Let f be as in Lemma 2.1. Then*

$$(2.4) \quad 0 \leq f(1) - f'(1) + f'(1)x^2 - f(x^2) \leq -2f''(1)(x-1)^2$$

for $x < 1$ (*resp.*, $x > 1$). The inequality (2.4) does not hold good for $x > 1$ (*resp.*, $x < 1$). Moreover, when $f(x)$ is non-linear the equality holds if and only if $x = 1$.

Proof. First, note that $f(1) - f'(1) + f'(1)x^2 - f(x^2) \geq 0$, using Lemma 2.1. Again as in the previous lemma, we shall prove the latter inequality in (2.4) for $f(x)$ non-linear function, i.e., $f(x) = \phi_\lambda(x) = \frac{x(1+\lambda)}{x+\lambda}$. Thus the inequality on the right hand side of (2.4) reduces to prove

$$(2.5) \quad \frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda}x^2 - \frac{1+\lambda}{x^2+\lambda}x^2 - \frac{4\lambda}{(1+\lambda)^2}(x-1)^2 \leq 0.$$

On taking $\mu = \frac{\lambda}{1+\lambda}$ together with some calculations, we obtain (2.5) equivalent to

$$(2.6) \quad \frac{(1-\mu)\mu(x-1)^2}{(1-\mu)x^2+\mu}((x+1)^2 - 4((1-\mu)x^2+\mu)) \leq 0.$$

Clearly, (2.6) holds good for $\mu = 1/2$, so we assume without loss of generality $\mu \neq 1/2$.

Let

$$F(x) = (x+1)^2 - 4((1-\mu)x^2+\mu).$$

Then,

$$F'(x) = 2(x+1) - 8(1-\mu)x.$$

If $\mu > 1/2$ then for all $x < 1$,

$$F'(x) > 2(x+1) - 4x = 2(1-x) \geq 0.$$

This means $F(x)$ is an increasing function and hence $F(x) \leq F(1) = 0$. If $\mu < 1/2$ then for all $x > 1$,

$$F'(x) < 2(x+1) - 4x = 2(1-x) \leq 0.$$

This means $F(x)$ is a decreasing function and hence $F(x) \leq F(1) = 0$. This proves (2.6). Thus the inequality (2.4) holds for all $x < 1$ (resp., $x > 1$) when $f'(1) \geq 1/2$ (resp., $f'(1) \leq 1/2$).

To settle the cases $x > 1$ (resp., $x < 1$) when $f'(1) \geq 1/2$ (resp., $f'(1) \leq 1/2$), we illustrate following examples:

- (i) Take $f(x) = \frac{19x}{6x+13}$, here $f'(1) = 13/19 > 1/2$ and the inequality (2.4) does not hold good for $x > 1$, see Fig. 2.1 below.

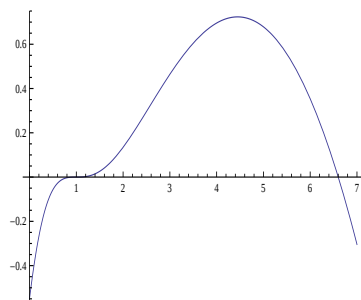


FIG. 2.1. $\frac{6}{19} + \frac{13}{19}x^2 - \frac{19x^2}{6x^2+13} - \frac{312}{361}(x-1)^2$

- (ii) Take $f(x) = \frac{5x}{3x+2}$, here $f'(1) = 2/5 < 1/2$ and the inequality (2.4) does not hold good for $0 \leq x \leq 1$, see Fig. 2.2 below.

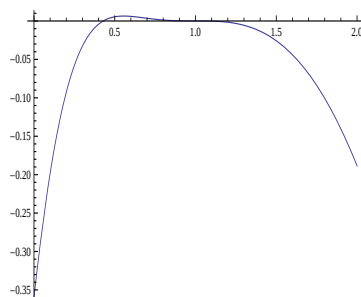


FIG. 2.2. $\frac{3}{5} + \frac{2}{5}x^2 - \frac{5x^2}{3x^2+2} - \frac{24}{25}(x-1)^2$

Finally, the equality conditions are clear from (2.6). $\square$

3. Eigenvalue inequalities for difference of operator means. To prove eigenvalue inequalities involving difference of operator means, the following facts will be used in the sequel.

An operator $A \in \mathcal{B}(\mathcal{H})$ is compact if and only if for every orthonormal set $\{e_n\}$ in $\mathcal{H}$, $\lim_{n \rightarrow \infty} \langle Ae_n, e_n \rangle \rightarrow 0$. This implies the following:

- (i) If $A, B \in \mathcal{B}(\mathcal{H})$ are positive and $A - B \in \mathbb{B}(\mathcal{H})_+$, then A compact implies B is so, (see [5] or [12, p. 59]).
- (ii) The Weyl's monotonicity principle for compact positive operators is that if $A, B \in \mathbb{B}(\mathcal{H})_+$ are compact operators such that $A - B \in \mathbb{B}(\mathcal{H})_+$, then $\lambda_j(A) \geq \lambda_j(B)$ for all $j = 1, 2, \dots$ (see [1, p. 63] or [3, p. 26]).

Since the space of compact operators is a two-sided ideal in $\mathcal{B}(\mathcal{H})$, it is easy to see that $X(A - B)Y$ is compact if $A - B$ is compact for $A, B \in \mathcal{B}(\mathcal{H})$ and X, Y are arbitrary members of $\mathcal{B}(\mathcal{H})$. Moreover, using spectral theorem in Calkin Algebra setting, we obtain $A\nabla_\mu B - A\sigma B$ compact for $A - B$ compact (for details, see [13]).

To avoid the repetitions, we shall discuss the equality cases for all the results proved henceforth at the end of this section.

THEOREM 3.1. *Let σ be an operator mean with $f(x)$ its representing function and assuming $\mu = f'(1)$. Let $A \geq B$ be positive invertible operators with $A - B$ compact. Then*

$$(3.1) \quad \lambda_j((B\nabla_\mu A) - (B\sigma A)) \geq -\frac{1}{2}f''(1)\lambda_j((B\nabla_{\frac{2-\mu}{3}}A)^{-1/2}(A-B)^2(B\nabla_{\frac{2-\mu}{3}}A)^{-1/2}),$$

$$(3.2) \quad \lambda_j((A\nabla_\mu B) - (A\sigma B)) \leq -\frac{1}{2}f''(1)\lambda_j((A\nabla_{\frac{2-\mu}{3}}B)^{-1/2}(A-B)^2(A\nabla_{\frac{2-\mu}{3}}B)^{-1/2}),$$

for $\mu \geq 1/2$ and $j = 1, 2, \dots$. The orders in the inequalities (3.1) and (3.2) are reversed for $\mu \leq 1/2$. Equality holds if and only if $A = B$.

Proof. Use Lemma 2.1 for the given operator monotone function $f(x)$ when $\mu \geq 1/2$ to obtain

$$(3.3) \quad 1 - \mu + \mu x - f(x) \geq \frac{-f''(1)}{2}(x-1) \left(\frac{2-\mu}{3}x + \frac{1+\mu}{3} \right)^{-1} (x-1), \text{ for } x > 1$$

and

$$(3.4) \quad 1 - \mu + \mu x - f(x) \leq \frac{-f''(1)}{2}(x-1) \left(\frac{2-\mu}{3}x + \frac{1+\mu}{3} \right)^{-1} (x-1), \text{ for } x < 1.$$

Replace x by $B^{-1/2}AB^{-1/2}$ in (3.3) and by $A^{-1/2}BA^{-1/2}$ in (3.4) respectively, to

obtain

$$(3.5) \quad (I\nabla_{\mu} B^{-1/2} A B^{-1/2}) - (I\sigma B^{-1/2} A B^{-1/2}) \\ \geq \frac{-f''(1)}{2} (B^{-1/2} A B^{-1/2} - I) (I\nabla_{\frac{2-\mu}{3}} B^{-1/2} A B^{-1/2})^{-1} (B^{-1/2} A B^{-1/2} - I)$$

and

$$(3.6) \quad (I\nabla_{\mu} A^{-1/2} B A^{-1/2}) - (I\sigma A^{-1/2} B A^{-1/2}) \\ \leq \frac{-f''(1)}{2} (A^{-1/2} B A^{-1/2} - I) (I\nabla_{\frac{2-\mu}{3}} A^{-1/2} B A^{-1/2})^{-1} (A^{-1/2} B A^{-1/2} - I).$$

On pre and post multiplication to both sides by $B^{1/2}$ in (3.5) and by $A^{1/2}$ in (3.6) respectively, we get

$$(B\nabla_{\mu} A) - (B\sigma A) \geq \frac{-f''(1)}{2} (A - B) (B\nabla_{\frac{2-\mu}{3}} A)^{-1} (A - B)$$

and

$$(A\nabla_{\mu} B) - (A\sigma B) \leq \frac{-f''(1)}{2} (A - B) (A\nabla_{\frac{2-\mu}{3}} B)^{-1} (A - B).$$

Since, the operators $(B\nabla_{\mu} A) - (B\sigma A)$ and $\frac{-f''(1)}{2} (A - B) (A\nabla_{\frac{2-\mu}{3}} B)^{-1} (A - B)$ are compact, so by fact (i) above, the operators $\frac{-f''(1)}{2} (A - B) (B\nabla_{\frac{2-\mu}{3}} A)^{-1} (A - B)$ and $(A\nabla_{\mu} B) - (A\sigma B)$ are compact. Using the fact (ii) and $\lambda_j(X^* X) = \lambda_j(X X^*)$, we obtain

$$\lambda_j((B\nabla_{\mu} A) - (B\sigma A)) \geq \frac{-f''(1)}{2} \lambda_j((A - B) (B\nabla_{\frac{2-\mu}{3}} A)^{-1} (A - B)) \\ = \frac{-f''(1)}{2} \lambda_j((B\nabla_{\frac{2-\mu}{3}} A)^{-1/2} (A - B)^2 (B\nabla_{\frac{2-\mu}{3}} A)^{-1/2})$$

and

$$\lambda_j((A\nabla_{\mu} B) - (A\sigma B)) \leq \frac{-f''(1)}{2} \lambda_j((A - B) (A\nabla_{\frac{2-\mu}{3}} B)^{-1} (A - B)) \\ = \frac{-f''(1)}{2} \lambda_j((A\nabla_{\frac{2-\mu}{3}} B)^{-1/2} (A - B)^2 (A\nabla_{\frac{2-\mu}{3}} B)^{-1/2}).$$

These prove (3.1) and (3.2). Similarly, the following reverse inequalities can be proved:

$$(3.7) \quad \lambda_j((B\nabla_{\mu} A) - (B\sigma A)) \leq \frac{-f''(1)}{2} \lambda_j(B\nabla_{\frac{2-\mu}{3}} A)^{-1/2} (A - B)^2 (B\nabla_{\frac{2-\mu}{3}} A)^{-1/2},$$

$$(3.8) \quad \lambda_j((A\nabla_{\mu} B) - (A\sigma B)) \geq \frac{-f''(1)}{2} \lambda_j((A\nabla_{\frac{2-\mu}{3}} B)^{-1/2} (A - B)^2 (A\nabla_{\frac{2-\mu}{3}} B)^{-1/2})$$

for $\mu \leq 1/2$ and $j = 1, 2, \dots$ $\square$

REMARK 3.2. The inequalities (3.1) and (3.8) are with better bounds than the inequalities in [7, Theorem 3.1] and in [13, Theorem 2.2]. Indeed, for $A \geq B$ positive invertible operators, we have

$$A \geq A \nabla_t B \geq B, \quad \text{for all } t \in [0, 1].$$

This implies

$$(3.9) \quad B^{-1} \geq (A \nabla_t B)^{-1} \geq A^{-1} \quad \text{for all } t \in [0, 1], \quad (\text{see [1], p.114}).$$

Observe that $-\frac{f''(1)}{2} \geq -\frac{f''(1)}{2(1+a)^2}$ for $a \geq 0$. Hence, for $j = 1, 2, \dots$, we have

$$(3.10) \quad \begin{aligned} & \frac{-f''(1)}{2} \lambda_j((A - B)(A \nabla_{\frac{2-\mu}{3}} B)^{-1}(A - B)) \\ & \geq \frac{-f''(1)}{2(1+a)^2} \lambda_j((A - B)A^{-1}(A - B)) \\ & = \frac{-f''(1)}{2(1+a)^2} \lambda_j(A^{-1/2}(A - B)^2 A^{-1/2}) \\ & = K_f(a) \lambda_j(A^{-1/2}(A - B)^2 A^{-1/2}). \end{aligned}$$

The inequalities (3.2) and (3.7) are also better similarly. In fact $\frac{-f''(1)}{2(1+b)^2} \geq -\frac{f''(1)}{2}$ for $-1 < b \leq 0$ and using (3.9), we get

$$(3.11) \quad \begin{aligned} & \frac{-f''(1)}{2} \lambda_j((A - B)(B \nabla_{\frac{2-\mu}{3}} A)^{-1}(A - B)) \\ & \leq \frac{-f''(1)}{2(1+b)^2} \lambda_j((A - B)B^{-1}(A - B)) \\ & = \frac{-f''(1)}{2(1+b)^2} \lambda_j(B^{-1/2}(A - B)^2 B^{-1/2}) \\ & = K_f(b) \lambda_j(B^{-1/2}(A - B)^2 B^{-1/2}). \end{aligned}$$

We now, present the following corollaries, in the light of the inequalities (3.10) and (3.11). One may compare these in [4] and [13] with constants $K(\mu, 0)$ and $K_{x^\mu}(0)$ respectively.

COROLLARY 3.3. (cf. [6, Theorem 4]) Let A, B and σ be as in Theorem 3.1. Then for arbitrarily fixed $\mu \in [0, 1]$ and $j = 1, 2, \dots$,

$$\begin{aligned} \lambda_j(A \nabla B - A \# B) & \geq \lambda_j(A \nabla B - H_\sigma(A, B)) \\ & \geq \frac{\mu(1-\mu)}{2} \lambda_j((A \nabla B)^{-1/2}(A - B)^2(A \nabla B)^{-1/2}). \end{aligned}$$

Proof. Note that the representing function of the generalized Heinz mean H_σ is given by $F(x) = \frac{f(x) + (x/f(x))}{2}$, when $f(x)$ is representing function of the mean σ . Since $F'(1) = 1/2$ and $F''(1) = \mu(\mu - 1)$, where $\mu = f'(1) \in [0, 1]$. Now, on replacing σ by H_σ in (3.8) and using $H_\sigma(A, B) \geq A \# B$, we obtain the desired result. $\square$

COROLLARY 3.4. (cf. [6, Theorem 4]) Let A, B, σ , and μ be as in Corollary 3.3. Then for $j = 1, 2, \dots$,

$$\lambda_j(A \nabla B - H_\sigma(A, B)) \leq \frac{\mu(1 - \mu)}{2} \lambda_j((A \nabla B)^{-1/2} (A - B)^2 (A \nabla B)^{-1/2}).$$

Proof. The proof follows similarly as in Corollary 3.3 on replacing σ by H_σ in (3.2). $\square$

COROLLARY 3.5. (cf. [4, Corollary 2.6]) Let A, B be as in Theorem 3.1 and $\mu \in [0, 1]$. Then

$$\begin{aligned} & \lambda_j((B \nabla_\mu A) - (B \#_\mu A)) \\ & \geq \frac{1}{2} \mu(1 - \mu) \lambda_j((B \nabla_{\frac{2-\mu}{3}} A)^{-1/2} (A - B)^2 (B \nabla_{\frac{2-\mu}{3}} A)^{-1/2}), \\ (3.12) \quad & \lambda_j((A \nabla_\mu B) - (A \#_\mu B)) \\ & \leq \frac{1}{2} \mu(1 - \mu) \lambda_j((A \nabla_{\frac{2-\mu}{3}} B)^{-1/2} (A - B)^2 (A \nabla_{\frac{2-\mu}{3}} B)^{-1/2}) \end{aligned}$$

for $\mu \geq 1/2$ and $j = 1, 2, \dots$. The orders are reversed for $0 \leq \mu \leq 1/2$.

Proof. Taking $f(x) = x^\mu$ in Theorem 3.1, we obtain the desired result. $\square$

The following corollary is an outcome of Corollary 3.5, which may be treated as a reverse version of Corollary 3.3.

COROLLARY 3.6. Let A, B , and σ be as in Theorem 3.1. Then for $j = 1, 2, \dots$,

$$\begin{aligned} \lambda_j((A \nabla B) - H_\sigma(A, B)) & \leq \lambda_j((A \nabla B) - (A \# B)) \\ & \leq \frac{1}{8} \lambda_j((A \nabla B)^{-1/2} (A - B)^2 (A \nabla B)^{-1/2}) \end{aligned}$$

Proof. Take $\mu = 1/2$ and using the fact that $H_\sigma(A, B) \geq A \# B$ in (3.12), we get the desired result. $\square$

THEOREM 3.7. Let A, B, f, σ , and μ be as in Theorem 3.1. Then for $j = 1, 2, \dots$,

$$(3.13) \quad 0 \leq \lambda_j((B \nabla_\mu A) - (B \sigma A)) \leq -4f''(1) \lambda_j((A \nabla B) - (A \# B)); \quad \mu \leq \frac{1}{2}$$

and

$$(3.14) \quad 0 \leq \lambda_j((A\nabla_\mu B) - (A\sigma B)) \leq -4f''(1)\lambda_j((A\nabla B) - (A\#B)); \quad \mu \geq \frac{1}{2}.$$

Equality holds if and only if $A = B$.

Proof. We prove the inequality (3.13) only, since the proof of (3.14) is similar. Replace x^2 by x in Lemma 2.3, to get

$$(3.15) \quad 0 \leq f(1) - f'(1) + f'(1)x - f(x) \leq -2f''(1)(1 + x - 2\sqrt{x}).$$

Choose $f'(1) = \mu \leq 1/2$. Since $B^{-1/2}AB^{-1/2} \geq I$, so, we replace x by $B^{-1/2}AB^{-1/2}$ in (3.15) to obtain

$$(3.16) \quad \begin{aligned} & (I\nabla_\mu B^{-1/2}AB^{-1/2}) - (I\sigma B^{-1/2}AB^{-1/2}) \\ & \leq -4f''(1)((I\nabla B^{-1/2}AB^{-1/2}) - (I\#B^{-1/2}AB^{-1/2})). \end{aligned}$$

Pre and post multiplication by $B^{1/2}$ to both sides in the inequality (3.16) lead to

$$(B\nabla_\mu A) - (B\sigma A) \leq -4f''(1)((A\nabla B) - (A\#B)).$$

Finally, the operator $A\nabla B - A\#B$ is compact, so in view of the facts (i) and (ii), we obtain the desired result. $\square$

COROLLARY 3.8. Let A, B be as in Theorem 3.1 and $\mu \in [0, 1]$. Then for $j = 1, 2, \dots$,

$$\begin{aligned} \lambda_j(B\nabla_\mu A - B\#_\mu A) & \leq 4\mu(1 - \mu)\lambda_j(A\nabla B - A\#B) \\ & \leq 2\mu(A\nabla B - A\#B); \quad \mu \leq \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} \lambda_j(A\nabla_\mu B - A\#_\mu B) & \leq 4\mu(1 - \mu)\lambda_j(A\nabla B - A\#B) \\ & \leq 2(1 - \mu)\lambda_j(A\nabla B - A\#B); \quad \frac{1}{2} \leq \mu. \end{aligned}$$

Equality holds if and only if $A = B$.

Proof. The result follows on taking $f(x) = x^\mu$ in Theorem 3.7, and using the fact that $2\mu(1 - \mu)$ is the harmonic mean between μ and $1 - \mu$. $\square$

COROLLARY 3.9. Let A, B, σ , and μ be as in Corollary 3.3. Then for $j = 1, 2, \dots$,

$$\begin{aligned} \lambda_j(A\nabla B - H_\sigma(A, B)) & \leq 4\mu(1 - \mu)\lambda_j(A\nabla B - A\#B) \\ & \leq 2 \max\{\mu, 1 - \mu\}\lambda_j(A\nabla B - A\#B). \end{aligned}$$

Proof. Replace σ by H_σ in Theorem 3.7, to get the desired result. $\square$

Finally, we shall discuss the equality conditions in (3.1), (3.2), (3.7), (3.8), (3.13) and (3.14). We recall a result from [3, p. 26], which says, if $A, B \in \mathcal{B}(\mathcal{H})$ are positive invertible compact operators with $A \geq B$, then $A = B$ if and only if $\lambda_j(A) = \lambda_j(B)$, $j = 1, 2, \dots$. The equality in (3.1) leads to

$$\lambda_j((B\nabla_\mu A) - (B\sigma A)) = -\frac{1}{2}f''(1)\lambda_j((B\nabla_{\frac{2-\mu}{3}}A)^{-1/2}(A-B)^2(B\nabla_{\frac{2-\mu}{3}}A)^{-1/2}),$$

for $j = 1, 2, \dots$ and $1/2 \leq \mu \leq 1$. Then, we have

$$\begin{aligned} & (1-\mu)I + \mu B^{-\frac{1}{2}}AB^{-\frac{1}{2}} - f(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}) \\ &= -\frac{1}{2}f''(1)((I - B^{-1/2}AB^{-1/2})(I\nabla_{\frac{2-\mu}{3}}B^{-1/2}AB^{-1/2})^{-1}(I - B^{-1/2}AB^{-1/2})) \end{aligned}$$

which is equivalently written as

$$1 + \mu(t-1) - f(t) = -\frac{3}{2}f''(1)\frac{(t-1)^2}{1 + \mu + (2-\mu)t}$$

for all $t \in Sp(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})$. Hence it follows from the equality condition in the Lemma 2.1 that $Sp(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}) = \{1\}$, i.e. $A = B$. The converse is trivial.

The proofs for equality condition in other cases are similar.

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