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HERMITIAN OCTONION MATRICES AND NUMERICAL RANGES*

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Abstract. Notions of numerical ranges and joint numerical ranges of octonion matrices are introduced. Various properties of hermitian octonion matrices related to eigenvalues and convex cones, such as the convex cone of positive semidefinite matrices, are described. As an application, convexity of joint numerical ranges of 2×2 hermitian matrices is characterized. Another application involves existence of a matrix with a high eigenvalue multiplicity in a given real vector subspace of hermitian matrices.

Key words. Octonion matrix, Positive semidefinite, Numerical range, Joint numerical range.

AMS subject classifications. 15A33, 15A60, 15B57.

1. Introduction. Denote by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ the reals, complexes, real quaternions, and real octonions, respectively. Thus, $\mathbb{O}$ is spanned as a real vector space by elements $c_0, \dots, c_7$, with multiplication given by $c_0 = 1$ (the multiplicative unity) and by the following multiplication table:

$c_i c_j$	c_1	c_2	c_3	c_4	c_5	c_6	c_7
c_1	-1	c_4	c_7	$-c_2$	c_6	$-c_5$	$-c_3$
c_2	$-c_4$	-1	c_5	c_1	$-c_3$	c_7	$-c_6$
c_3	$-c_7$	$-c_5$	-1	c_6	c_2	$-c_4$	c_1
c_4	c_2	$-c_1$	$-c_6$	-1	c_7	c_3	$-c_5$
c_5	$-c_6$	c_3	$-c_2$	$-c_7$	-1	c_1	c_4
c_6	c_5	$-c_7$	c_4	$-c_3$	$-c_1$	-1	c_2
c_7	c_3	c_6	$-c_1$	c_5	$-c_4$	$-c_2$	-1

See, for example, [3] for more information on octonions. It is easily seen from the table that

$$c_i c_j = c_k \implies c_{i+1} c_{j+1} = c_{k+1}, \qquad \forall \ i, j, k \in \{1, \dots, 7\}, \tag{1.1}$$

where the indices are understood modulo 7. We identify xc_0, $x \in \mathbb{R}$, with x. The real part of $x = \sum_{j=0}^{7} x_j c_j \in \mathbb{O}$, where $x_j \in \mathbb{R}$, is $\mathfrak{R}(x) := x_0$, and the vector part

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is $\mathfrak{V}(x) := \sum_{j=1}^{7} x_j c_j$. The norm of $x \in \mathbb{O}$ is defined by $|x| = \sqrt{x_0^2 + \cdots + x_7^2}$. Note that $|xy| = |x| \cdot |y|$ for all $x, y \in \mathbb{O}$. The *conjugation* map in $\mathbb{O}$ is defined by $x^* = \mathfrak{R}(x) - \mathfrak{V}(x)$; it has the standard properties of a conjugation: the map $x \mapsto x^*$ is real linear, involutory, antimultiplicative, and $|x|^2 := x^*x = xx^*$ is real nonnegative and equal zero only if $x = 0$.

It is well known that $\mathbb{O}$ is an alternative algebra (every two elements generate an associative subalgebra) but it is not associative. In fact, the subalgebra of $\mathbb{O}$ generated by any two elements (not both zero) is *- isomorphic to either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and the subalgebra of $\mathbb{O}$ generated by one nonzero element is *-isomorphic to $\mathbb{R}$ or to $\mathbb{C}$.

Octonions of the form

$$(x, y, z) := (xy)z - x(yz), \quad x, y, z \in \mathbb{O},$$

are *associators*, and those of the form $[x, y] := xy - yx$, $x, y \in \mathbb{O}$, are *commutators*. The following fact will be used repeatedly:

Proposition 1.1. *The following three sets of octonions coincide:* (1) *the set of associators;* (2) *the set of commutators;* (3) *the set*

$$\{x \in \mathbb{O} : \mathfrak{R}(x) = 0\}. \tag{1.2}$$

Proof. It is easy to see that the commutators have zero real parts, and the identity

$$6(x, y, z) = [[x, y], z] + [[y, z], x] + [[z, x]y], \quad \forall \ x, y, z \in \mathbb{O},$$

shows that associators have zero real parts as well. Conversely, we have

$$(c_1c_2)c_3 = -c_6, \quad c_1(c_2c_3) = c_6, \tag{1.3}$$

so c_6 is an associator, and using the property (1.1) it follows from (1.3) that every c_j, $j = 1, \ldots, 7$, is an associator. The linearity (over the reals) of the associator as function of each of the variables x, y, z shows that (1.2) consists of associators. The proof for commutators is analogous. □

Conjugation extends to $m \times n$ matrices $A = [a_{i,j}]_{i,j=1}^{m,n} \in \mathbb{O}^{m\times n}$ with entries on $\mathbb{O}$: $A^* = [a_{j,i}^*]_{j,i=1}^{n,m} \in \mathbb{O}^{n\times m}$. A matrix $A \in \mathbb{O}^{n\times n}$ is *hermitian* if $A = A^*$ and *skewhermitian* if $A = -A^*$. The *norm* of a vector $x \in \mathbb{O}^{n\times 1}$ is defined by

$$\|x\| = \sqrt{x^*x} = \sqrt{x_1^*x_1 + \cdots + x_n^*x_n},$$

where $x_1, \ldots, x_n$ are the components of x.

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To avoid trivialities, we assume everywhere in the paper that n is an integer greater than or equal to 2. We define the numerical range $W(A)$ of a matrix $A \in \mathbb{O}^{n\times n}$, as follows. Let

$$K = \{(i,j) : 1 \le i < j \le n, \quad i,j \text{ integers}\},$$

$$L = \{(i,j) : 1 \le j < i \le n, \quad i,j \text{ integers}\}.$$

Fix a subset $K_0 \subseteq K$, and let

$$L_0 = \{(i,j) : (j,i) \in K \setminus K_0\} \subseteq L.$$

(K may be empty.) Then for $A = [a_{i,j}]_{i,j=1}^n \in \mathbb{O}^{n\times n}$ and $x = [x_1 \;\; x_2 \;\; \cdots \;\; x_n]^T \in \mathbb{O}^{n\times 1}$, we let

$$\begin{aligned} x^*Ax := &\sum_{(i,j)\in K_0} x_i^*(a_{i,j}x_j) + \sum_{i=1}^n x_i^* a_{i,i} x_i + \sum_{(i,j)\in K\setminus K_0} (x_i^* a_{i,j})x_j \\ &+ \sum_{(i,j)\in L_0} x_i^*(a_{i,j}x_j) + \sum_{(i,j)\in L\setminus L_0} (x_i^* a_{i,j})x_j \;\in\; \mathbb{O}. \end{aligned}$$

Note that $\sum_{i=1}^n x_i^* a_{i,i} x_i$ is defined unambiguously, because the factors in each term $x_i^* a_{i,i} x_i$ belong to the subalgebra generated by x_i and $a_{i,i}$, and therefore this subalgebra is associative. Define

$$W(A) := \{x^*Ax : x_1^*x_1 + \cdots + x_n^*x_n = 1\}.$$

Generally, $W(A)$ depends on the choice of K_0, but we suppress this dependence in the notation.

Clearly, $W(A)$ is compact and (pathwise) connected.

It is well known that the numerical ranges in the context of quaternion matrices are generally not convex; essentially the same example (taken from [1]) shows non-convexity of octonion numerical ranges. Indeed,

$$W(c_1 \oplus I_{n-1}) = \{\cos\theta + (\sin\theta)x : 0 \le \theta \le \pi/2, \quad x \in \mathsf{S}\},$$

where S is the set of octonions with zero real parts and norm 1. Clearly, $W(c_1 \oplus I_{n-1})$ is not convex.

Some elementary properties of numerical ranges are listed next:

Proposition 1.2. *Let $A \in \mathbb{O}^{n\times n}$. Then:*

(a) $W(A) \subseteq \mathbb{R}$ *if and only if A is hermitian.*

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(b) $W(A)$ *is contained in* $\{x \in \mathbb{O} : \Re(x) = 0\}$ *if and only if* A *is skewhermitian.*
(c) $W(A) = \{0\}$ *if and only if* $A = 0$.
(d) $W(aA + \alpha I) = aW(A) + \alpha$ *for every* $a, \alpha \in \mathbb{R}$.
(e) $W(A^*) = \{\lambda^* : \lambda \in W(A)\}$.
(f) $W(A)$ *is a singleton if and only if* $A = aI$, *where* a *is real.*

Proof. (c) follows from (a) and (b), and parts (d) and (e) are evident from the definition. The "if" parts of (a), (b), and (f) follow easily from the definition, and for the proof of the "only if" parts of (a), (b), and (f), it suffices to consider the case $n = 2$. Thus, let $A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$, where $a_{i,j} \in \mathbb{O}$.

Assume that $W(A)$ is real. Then, examining x^*Ax for

$$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{1.4}$$

we see that $a_{1,1}$, $a_{2,2}$, and $a_{1,2} + a_{2,1}$ are real. Thus, $a_{1,2} = \Re(a_{1,2}) + q$, $a_{2,1} = \Re(a_{2,1}) - q$ for some $q \in \mathbb{O}$ with real part zero. Note that q^2 is real and negative (unless $q = 0$). Since $[1 \;\; q^*]A \begin{bmatrix} 1 \\ q \end{bmatrix}$ is real, a basic algebra shows that $q(\Re(a_{1,2}) - \Re(a_{2,1}))$ is real. So, unless $q = 0$, we have $a_{2,1} = a_{1,2}^*$. If $q = 0$, then $a_{1,2}$ and $a_{2,1}$ are real, and for any $v \in \mathbb{O} \setminus \{0\}$ with $\Re(v) = 0$ we have that $-va_{2,1} + va_{1,2} = 0$, and hence, $a_{1,2} = a_{2,1}$.

Next, assume that every element in $W(A)$ has zero real part. Examining x^*Ax for x given by (1.4), we find that $a_{1,1}$, $a_{2,2}$, and $a_{1,2} + a_{2,1}$ have zero real parts. Thus,

$$a_{1,2} = s + w_2, \quad a_{2,1} = -s + w_1,$$

where s is real and $\Re(w_1) = \Re(w_2) = 0$. For every $v \in \mathbb{O}$ with $\Re(v) = 0$ we have $\Re(v^*a_{2,2}v) = 0$ and

$$\begin{aligned} 0 &= \Re\left([1 \;\; v^*] \begin{bmatrix} a_{1,1} & s + w_2 \\ -s + w_1 & a_{2,2} \end{bmatrix} \begin{bmatrix} 1 \\ v \end{bmatrix} \right) \\ &= \Re\left(a_{1,1} + 2sv - vw_1 + w_2v + v^*a_{2,2}v \right), \end{aligned}$$

and it follows that

$$\Re(-vw_1 + w_2v) = 0. \tag{1.5}$$

Taking $v = c_j$, $j = 1, 2, \ldots, 7$ in (1.5), we conclude that $w_1 = w_2$.

Finally, assume that $W(A) = \{\lambda\}$, $\lambda \in \mathbb{O}$. Then, by taking $x \in \mathbb{O}$ with $|x| = 1$ we have $x^{-1}a_{1,1}x = x^*a_{1,1}x = \lambda$, so the similarity orbit of $a_{1,1}$ is a singleton. This

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happens if and only if $a_{1,1}$ is real, as the similarity orbit of $a \in \mathbb{O}$ consists of all octonions b such that $\mathfrak{R}(a) = \mathfrak{R}(b)$ and $|\mathfrak{V}(a)| = |\mathfrak{V}(b)|$ (see, e.g. [13, Theorem 3.1]). Thus, λ is real. Subtracting λI from A we may assume without loss of generality that $\lambda = 0$. But then $A = 0$ by (c), and we are done. □

Note that for hermitian A, the real number x^*Ax is independent of the choice of K_0. Indeed, temporarily label x^*Ax with respect to K_0 and K_0' (another choice for K_0) as $x^*Ax_{K_0}$ and $x^*Ax_{K_0'}$, respectively. Then $x^*Ax_{K_0} - x^*Ax_{K_0'}$ is a sum of associators. It follows that $x^*Ax_{K_0} - x^*Ax_{K_0'}$ has zero real part. On the other hand, $x^*Ax_{K_0} - x^*Ax_{K_0'}$ is real by Proposition 1.2. Thus, $x^*Ax_{K_0} - x^*Ax_{K_0'} = 0$.

In view of this observation, we assume $K_0 = K$ when dealing with the function x^*Ax, for an hermitian matrix A.

In the next section, we present preliminaries on hermitian octonion matrices, with special emphasis on 2×2 and 3×3 matrices, and on convex cones of hermitian matrices related to the convex cone of positive semidefinite matrices. Our main results Theorems 3.3 and 4.1 are stated and proved in Sections 3 and 4, respectively.

We say that a (necessarily hermitian) matrix $A \in \mathbb{O}^{n\times n}$ is *positive definite* if $x^*Ax > 0$ for every nonzero $x \in \mathbb{O}^{n\times n}$, and *positive semidefinite* if $x^*Ax \geq 0$ for every $x \in \mathbb{O}^{n\times n}$. Negative definite and negative semidefinite matrices are defined analogously. We denote by $\mathsf{W}^{n\times n}$ the real vector space of $n \times n$ octonion hermitian matrices.

2. Preliminaries on octonion hermitian matrices.

2.1. Numerical ranges and eigenvalues. We begin with a general result relating the form x^*Ax and eigenvalues. All eigenvalues of octonion hermitian matrices in this paper are understood as *right* eigenvalues.

Theorem 2.1. *Let $A = [a_{p,q}]_{p,q=1}^n \in \mathsf{W}^{n\times n}$. If*

$$\mu := \max_{x\in\mathbb{O}^{n\times 1}\,:\,x^*x=1} x^*Ax \quad \text{and} \quad \mu = y^*Ay \quad \text{for} \quad y \in \mathbb{O}^{n\times 1},\ y^*y = 1,$$

then y is an eigenvector of A with eigenvalue μ.

Proof. Write

$$x = [x_1 \quad x_2 \quad \cdots \quad x_n]^T \in \mathbb{O}^{n\times 1}, \quad \text{where} \quad x_j = \sum_{k=0}^{7} x_{j,k}c_k, \quad x_{i,k} \in \mathbb{R};$$

analogously,

$$y = [y_1 \quad y_2 \quad \cdots \quad y_n]^T \in \mathbb{O}^{n\times 1}, \quad \text{where} \quad y_j = \sum_{k=0}^{7} y_{j,k}c_k, \quad y_{i,k} \in \mathbb{R}.$$

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Consider the real valued function

$$F(x_{j,k},\lambda)=x^*Ax+\lambda(x^*x-1),$$

where $x_{j,k}$ are real variables and λ is a real parameter (the Lagrange multiplier). We then have

$$\frac{\partial F}{\partial x_{j,k}}|_{x=y}=0,\quad \forall\ \ j=1,2,\dots,n\ \text{ and }\ \forall\ \ k=0,\dots,7.$$

A computation shows that

$$\begin{aligned}\frac{\partial F}{\partial x_{j,k}}=&-c_ka_{j,j}x_j+x_j^*a_{j,j}c_k+\sum_{i=1}^{j-1}(x_i^*(a_{i,j}c_k))+\sum_{i=j+1}^{n}(-c_k)(a_{j,i}x_i)\\&+\sum_{i=1}^{j-1}((-c_k)a_{j,i})x_i+\sum_{i=j+1}^{n}(x_i^*a_{i,j})c_k+2\lambda x_{j,k}\end{aligned}\tag{2.1}$$

for $k=1,2,\dots,7$, and

$$\begin{aligned}\frac{\partial F}{\partial x_{j,0}}=&c_0a_{j,j}x_j+x_j^*a_{j,j}c_0+\sum_{i=1}^{j-1}(x_i^*(a_{i,j}c_0))+\sum_{i=j+1}^{n}c_0(a_{j,i}x_i)\\&+\sum_{i=1}^{j-1}(c_0a_{j,i})x_i+\sum_{i=j+1}^{n}(x_i^*a_{i,j})c_0+2\lambda x_{j,0}\end{aligned}\tag{2.2}$$

for $k=0$. The left hand side of (2.1) is equal to twice the real part of

$$-c_ka_{j,j}x_j+\sum_{i=1}^{j-1}((-c_k)a_{j,i})x_i+\sum_{i=j+1}^{n}(-c_k)(a_{j,i}x_i)+\lambda x_{j,k},\tag{2.3}$$

or, what is the same, twice the real part of

$$-c_k\sum_{i=1}^{n}a_{j,i}x_i+\lambda x_{j,k}.\tag{2.4}$$

Letting

$$\sum_{i=1}^{n}a_{j,i}y_i=\sum_{k=0}^{7}\beta_k^{(j)}c_k,\quad \beta_k^{(j)}\in\mathbb{R},$$

it follows that

$$\beta_k^{(j)}+\lambda y_{j,k}=0,\quad k=1,2,\dots,7.$$

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Similarly, (2.2) leads to $\beta_0^{(j)} + \lambda y_{j,0} = 0$. Thus,

$$\sum_{i=1}^{n} a_{j,i} y_i = -\lambda \sum_{k=0}^{7} y_{j,k} c_k = -\lambda y_j.$$

Since this equality holds for $j = 1, 2, \ldots, n$, $Ay = \lambda y$. Now $y^*(Ay) = -\lambda$, and since the difference $y^*Ay - y^*(Ay)$ is a sum of associators, hence it must have zero part, we find that

$$\mu = y^*Ay = y^*(Ay) = -\lambda,$$

and the theorem is proved. □

A analogous result holds with maximum replaced by minimum, with the same proof.

Proposition 2.2. *If $A \in \mathsf{W}^{n\times n}$ and if $y \in \mathbb{O}^{n\times 1}$, is an eigenvector of A corresponding to a real eigenvalue λ, and $y^*y = 1$, then $y^*Ay = \lambda$.*

Indeed, $y^*(Ay) = \lambda$, and since the difference $y^*Ay - y^*(Ay)$ is a sum of associators, hence it must have zero part, we find that $y^*Ay = y^*(Ay)$.

Corollary 2.3. *$A \in \mathsf{W}^{n\times n}$ is positive semidefinite, resp., positive definite, if and only if all real eigenvalues of A are nonnegative, resp., positive.*

Indeed, if A is positive semidefinite, then clearly all real eigenvalues of A are nonnegative. Conversely, if all real eigenvalues of A are nonnegative, then by (the analog of) Theorem 2.1,

$$\min_{x\in\mathbb{O}^{n\times 1}\,:\,x^*x=1} x^*Ax \geq 0.$$

Thus, A is positive semidefinite. The proof for positive definiteness is analogous.

Note that hermitian octonion matrices may have nonreal eigenvalues, for example (taken from [6])

$$\begin{bmatrix} 1 & -c_1 \\ c_1 & 1 \end{bmatrix} \begin{bmatrix} c_2 \\ c_6 \end{bmatrix} = \begin{bmatrix} c_2 \\ c_6 \end{bmatrix} (1 + c_4 c_6).$$

2.2. Real matrix representation. We introduce a real matrix representation of octonions. Note that such representation cannot faithfully represent multiplication, because multiplication of octonions is non-associative. However, as it turns out it can represent matrix-vector multiplication. Such representations have been developed in the literature, see [13].

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Define the real linear isomorphism $\nu : \mathbb{O} \to \mathbb{R}^{8\times 1}$ by

$$\nu\left(\sum_{j=0}^{7} x_k c_k\right) = [x_0 \quad x_1 \quad \cdots \quad x_7]^T,$$

and extend it to vectors (using the same notation):

$$\nu : \mathbb{O}^{n\times 1} \to \mathbb{R}^{8n\times 1}, \quad \nu\left(\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}\right) = \begin{bmatrix} \nu(y_1) \\ \vdots \\ \nu(y_n) \end{bmatrix}, \quad \text{where } y_1, \dots, y_n \in \mathbb{O}.$$

Note that $\|\nu(x)\| = \|x\|$ for all $x \in \mathbb{O}^{n\times 1}$. Also, let the matrices $C_j \in \mathbb{R}^{8\times 8}$ for $j = 1, 2, \dots, 7$ be defined by the property that

$$C_j \nu(c_k) = \nu(c_j c_k), \quad k = 0, 1, \dots, 7.$$

The multiplication table for octonions shows that C_j's are skewsymmetric and invertible, in fact $C_j^2 = -I$. We now define the map $\chi : \mathbb{O} \to \mathbb{R}^{8\times 8}$ by

$$\chi\left(\sum_{k=1}^{7} x_k c_k\right) = x_0 I + \sum_{k=1}^{7} x_k C_k,$$

and extend (using the same notation) to the map on matrices

$$\chi : \mathbb{O}^{m\times n} \to \mathbb{R}^{8m\times 8n}, \quad \chi([a_{i,j}]_{i,j=1}^{m,n}) = [\chi(a_{i,j})]_{i,j=1}^{m,n}, \quad \text{where } a_{i,j} \in \mathbb{O}.$$

Proposition 2.4. *The map χ is real linear, injective, and $\chi(A^*) = (\chi(A))^T$ for every $A \in \mathbb{O}^{m\times n}$.*

The maps ν and χ have the following properties with respect to matrix-vector multiplication:

Lemma 2.5. *We have $\nu(Ax) = \chi(A)\nu(x)$ for every $A \in \mathbb{O}^{m\times n}$ and every $x \in \mathbb{O}^{n\times 1}$, and*

$$\Re(y^*(Ax)) = \Re((y^*A)x) = (\nu(y))^T \chi(A)\nu(x), \quad \forall\ x \in \mathbb{O}^{n\times 1},\ y \in \mathbb{O}^{m\times 1},\ A \in \mathbb{O}^{m\times n}. \tag{2.5}$$

Proof. It suffices to consider the case $m = n = 1$. Using the definition of the C_j's, we have by linearity

$$\nu(Ax) = \chi(A)\nu(x) \tag{2.6}$$

for every $A \in \mathbb{O}$ and every $x \in \mathbb{O}$.

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Because of linearity, it suffices to prove (2.5) for the case when $x = c_i$, $y = c_j$, $A = c_k$ for $i, j, k = 0, 1, \ldots, 7$. In view of (2.6), equality (2.5) reads

$$\mathfrak{R}(y^*(Ax)) = (\nu(y))^T \nu(Ax), \tag{2.7}$$

and it remains to prove

$$\mathfrak{R}(c_j^* c_\ell) = (\nu(c_j))^T \nu(c_\ell), \quad \forall \ \ j, \ell = 0, 1, \ldots, 7. \tag{2.8}$$

Clearly, both sides of (2.8) are zero if $j \neq \ell$. And for $j = \ell$ both sides are equal to 1. Thus, (2.5) is proved. □

In particular (taking $m = n$ and $x = y$ in (2.5)):

Lemma 2.6. *If $A \in \mathbb{O}^{n\times n}$ is hermitian, then*

$$x^* A x = (\nu(x))^T \chi(A) \nu(x), \quad \forall \ \ x \in \mathbb{O}^{n\times 1}.$$

In particular, A is positive, resp., negative, definite or positive, resp., negative, semi-definite if and only if $\chi(A)$ is such.

2.3. Cones of hermitian matrices. Let $\mathsf{K}^{n\times n}$ be the convex cone of positive semidefinite $n \times n$ octonion matrices. Clearly $\mathsf{K}^{n\times n}$ is closed and pointed (by Proposition 1.2(c)). Introduce the real valued inner product on $\mathsf{W}^{n\times n}$ by the formula

$$\langle A, B\rangle := \mathfrak{R}(\mathrm{Trace}\,(AB)) = \mathfrak{R} \sum_{i,j=1}^{n} a_{i,j} b_{i,j}^* = \mathfrak{R} \sum_{i,j=1}^{n} a_{i,j} b_{j,i},$$

where $A = [a_{i,j}]_{i,j=1}^n$, $B = [b_{i,j}]_{i,j=1}^n \in \mathsf{W}^{n\times n}$. It has all the usual properties of the inner product; in particular $\langle A, B\rangle = \langle B, A\rangle$, and $\langle A, A\rangle \geq 0$ with equality $\langle A, A\rangle = 0$ only if $A = 0$. Let $\mathsf{K}_*^{n\times n}$ be the dual cone of $\mathsf{K}^{n\times n}$ with respect to $\langle\cdot,\cdot\rangle$. In other words,

$$\mathsf{K}_*^{n\times n} = \{A \in \mathsf{W}^{n\times n} \ : \ \langle A, B\rangle \geq 0 \quad \text{for all} \quad B \in \mathsf{K}^{n\times n}\}.$$

Clearly, $\mathsf{K}_*^{n\times n}$ is closed and convex.

Note that matrices of the form uu^*, $u \in \mathbb{O}^{n\times 1}$, belong to $\mathsf{K}_*^{n\times n}$. Indeed, letting $u_1, \ldots, u_n$ be the components of u, for every $A = [a_{i,j}]_{i,j=1}^n \in \mathsf{K}^{n\times n}$ we have

$$\begin{aligned} \langle A, uu^*\rangle = \sum_{i,j=1}^{n} \mathfrak{R}(a_{i,j}(u_j u_i^*)) \ &= \ \sum_{i,j=1}^{n} \mathfrak{R}((a_{i,j}u_j)u_i^*) \\ = \sum_{i,j=1}^{n} \mathfrak{R}(u_i^*(a_{i,j}u_j)) \ &= \ \mathfrak{R}(u^*(Au)) = u^*Au \geq 0. \end{aligned}$$

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Note also that

$$(uu^*)u = u(u^*u), \quad u \in \mathbb{O}^{n\times 1},$$

which implies that u^*u is a eigenvalue of uu^*, with the eigenvector u (if $u \neq 0$).

Let $\mathsf{L}^{n\times n}$ be the closed cone generated by all matrices of the form uu^*, $u \in \mathbb{O}^{n\times 1}$, i.e.,

$$\mathsf{L}^{n\times n} := \text{closure of } \{\lambda_1 u_1 u_1^* + \cdots + \lambda_s u_s u_s^* : \lambda_1, \ldots, \lambda_s > 0, \ u_1, \ldots, u_s \in \mathbb{O}^{n\times 1}\}.$$

Clearly, $\mathsf{L}^{n\times n}$ is convex. Also, it follows from the equality $\langle A, uu^*\rangle = u^*Au$, where $A \in \mathsf{W}^{n\times n}$, that $\mathsf{K}^{n\times n} = \mathsf{L}_*^{n\times n}$, the dual cone of $\mathsf{L}^{n\times n}$. Also, we have

$$\mathsf{L}^{n\times n} = \mathsf{L}_{**}^{n\times n} = \mathsf{K}_*^{n\times n}, \tag{2.9}$$

where the first equality follows from the general properties of cones (see, e.g., [4]).

Lemma 2.7. *We have*

$$\mathsf{K}^{2\times 2} = \mathsf{K}_*^{2\times 2}. \tag{2.10}$$

Moreover, the matrices uu^*, where $u \in \mathbb{O}^{2\times 1} \setminus \{0\}$, constitute the set of nonzero extremal points in the convex cone (2.10). *Also, $\mathsf{K}_*^{3\times 3} \supseteq \mathsf{K}^{3\times 3}$, $\mathsf{K}_*^{3\times 3} \neq \mathsf{K}^{3\times 3}$.*

Proof. If $n \leq 3$, then there is a spectral decomposition for $A \in \mathsf{W}^{n\times n}$:

$$A = \sum_{k=1}^{n} \lambda_k (v_k v_k^*), \tag{2.11}$$

where λ_k are real eigenvalues of A and v_k are normalized (quaternion) eigenvectors of A such that $(v_k v_k^*)v_j = 0$ for $j \neq k$ ([7, Theorems 1O2, 1O3]). If in addition A is positive semidefinite, then we must have $\lambda_k \geq 0$ (Corollary 2.3). Since $v_k v_k^* \in \mathsf{K}_*^{n\times n}$, it follows that

$$\mathsf{K}^{n\times n} \subseteq \mathsf{K}_*^{n\times n}. \tag{2.12}$$

The following example shows that $\mathsf{K}_*^{3\times 3} \neq \mathsf{K}^{3\times 3}$. Let $u = [-c_1 \quad -c_2 \quad -c_3]^T$, and let

$$A := \begin{bmatrix} 1 & -c_4 & -c_7 \\ c_4 & 1 & -c_5 \\ c_7 & c_5 & 1 \end{bmatrix} = uu^*.$$

Then

$$\Re\left([-c_5 \quad c_7 \quad -c_4]\left(A\begin{bmatrix} c_5 \\ -c_7 \\ c_4 \end{bmatrix}\right)\right) = \Re\left([-c_5 \quad c_7 \quad -c_4]\begin{bmatrix} -c_5 \\ c_7 \\ -c_4 \end{bmatrix}\right) = -3,$$

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thus A is not positive semidefinite.

Now assume $n = 2$. Then the real eigenvalues of any matrix of the form uu^*, $u \in \mathbb{O}^{2\times 1}$, are zero and u^*u [7, Lemma 1O2]. Thus, uu^* is positive semidefinite by Corollary 2.3, and $\mathsf{K}_*^{2\times 2} \subseteq \mathsf{K}^{2\times 2}$. Together with (2.12), this shows equality

$$\mathsf{K}_*^{2\times 2} = \mathsf{K}^{2\times 2}. \tag{2.13}$$

Next, we show that every matrix uu^* is extremal in (2.10). Suppose $u \in \mathbb{O}^{2\times 1}\backslash\{0\}$ and

$$uu^* = \sum_{j=1}^{m} \lambda_j u_j u_j^*, \quad \lambda_j > 0, \quad u_j \in \mathbb{O}^{2\times 1}. \tag{2.14}$$

Replacing u_j with $u_j\sqrt{\lambda_j}$, we may assume without loss of generality that $\lambda_j = 1$, $j = 1, \ldots, n$. Then, for any fixed j, $uu^* - u_j u_j^*$ is positive semidefinite. We may consider uu^* and $u_j u_j^*$ as matrices over a subalgebra of $\mathbb{O}$ which is isomorphic to $\mathbb{H}$ and contains the $(1,2)$ entries of uu^* and $u_j u_j^*$. Now use the fact that over the quaternions, the positive semidefiniteness of $uu^* - u_j u_j^*$ implies that $u_j u_j^* = \mu_j uu^*$ for some $\mu_j \in [0,1]$. Therefore uu^* is extremal in (2.10).

Conversely, let $A \in \mathsf{K}^{2\times 2} \setminus \{0\}$ be extremal in (2.10). Write a spectral decomposition (2.11) for A, and we may assume without loss of generality that λ_1 and λ_2 are both nonzero. Since A is extremal, we must have $A = \mu_1 v_1 v_1^* = \mu_2 v_2 v_2^*$ for some positive μ_1, μ_2, and so A has the required form. □

In connection with the proof of Lemma 2.7, note that at present it is unknown whether or not every $A \in \mathsf{W}^{n\times n}$ admits a spectral decomposition (2.11) if $n \geq 4$. More information about eigenvalues of octonion hermitian matrices can be found in [6], [7], [11], [13], and [8].

We conclude this section with a characterization of interior points in $\mathsf{K}_*^{n\times n}$.

Proposition 2.8. *The following statements are equivalent for* $A \in \mathsf{W}^{n\times n}$*:*

(1) *A belongs to the interior of $\mathsf{K}_*^{n\times n}$;*
(2) $\mathfrak{R}\langle A, B\rangle > 0, \quad \forall \ B \in \mathsf{K}^{n\times n} \setminus \{0\}$;
(3) $\mathfrak{R}\langle A, B\rangle > 0, \quad \forall \ B \in \mathsf{K}_1^{n\times n} := \{X \in \mathsf{K}^{n\times n} : \mathrm{Trace}\,(X) = 1\}$.

Proof. Note that the trace of any positive semidefinite matrix is nonnegative, and it is equal to zero only when the matrix is zero. Therefore, by scaling B we see that (2) and (3) are equivalent. If (3) holds then (1) holds as well because of compactness of $\mathsf{K}_1^{n\times n}$, which can be easily verified. Assume now (1) holds, but $\mathfrak{R}\langle A, B\rangle = 0$ for some $B \in \mathsf{K}_1^{n\times n}$. Then for $\epsilon > 0$ we have $\mathfrak{R}\langle A - \epsilon B, B\rangle < 0$. Thus, $A - \epsilon B \notin \mathsf{K}_*^{n\times n}$, a contradiction to (1). □

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3. Joint numerical ranges. For a p-tuple of hermitian matrices $A_1, \dots, A_p \in \mathbb{O}^{n\times n}$, define the *joint numerical range* by

$$WJ(A_1, \dots, A_p) = \{(x^*A_1x, \dots, x^*A_px) \in \mathbb{R}^p : x = [x_1 \quad \dots \quad x_n]^T \in \mathbb{O}^{n\times 1} \text{ is such that } x^*x := x_1^*x_1 + \cdots + x_n^*x_n = 1\}.$$

Joint numerical ranges and their generalizations for tuples of hermitian matrices have been extensively studied in the context of real, complex, and quaternion matrices, see. e.g., [2], [10] and [12].

The following easily verified property of joint numerical ranges will be used:

Proposition 3.1. *Let $A_1, \dots, A_p, B_1, \dots, B_r \in \mathsf{W}^{n\times n}$. Assume that for some $r \times p$ real matrix S we have*

$$S \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_p \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_r \end{bmatrix}.$$

Then

$$W(B_1, \dots, B_r) = SW(A_1, \dots, A_p) := \{Sz : z \in W(A_1, \dots, A_p)\},$$

where the elements of $W(A_1, \dots, A_p)$ and $W(B_1, \dots, B_r)$ are understood as real column vectors.

Lemma 2.6 allows us to obtain easily a convexity result for joint numerical ranges of two hermitian matrices:

Theorem 3.2. *Let $A_1, A_2 \in \mathsf{W}^{n\times n}$. Then $WJ(A_1, A_2)$ is convex.*

For the proof just observe that by Lemma 2.6, $WJ(A_1, A_2)$ is the real joint numerical range of two real symmetric $8n \times 8n$ matrices $\chi(A_1)$ and $\chi(A_2)$, and the convexity of the latter is well known.

As far as we are aware, the joint numerical range of more than 2 octonion hermitian matrices in $n \geq 3$ dimensions is not well understood. On the other hand, the convexity properties of joint numerical ranges of 2×2 hermitian matrices can be completely sorted out:

Theorem 3.3. *Let $A_1, \dots, A_9 \in \mathbb{O}^{2\times 2}$ hermitian matrices. Then $WJ(A_1, \dots, A_9)$ is convex if and only if the 10-tuple of hermitian matrices $\{A_1, \dots, A_9, I\}$ is linearly dependent (over the reals). In particular, if $p \leq 8$, then $WJ(A_1, \dots, A_p)$ is convex for every p-tuple of 2×2 hermitian octonion matrices $A_1, \dots, A_p$.*

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Note that the statement of Theorem 3.3 is analogous to the well known results of convexity of joint numerical ranges of complex and quaternion 2×2 hermitian matrices. Namely, Theorem 3.3 holds true for $\mathbb{C}$ and $\mathbb{H}$, with numbers $10, 9, 8$ replaced by $4, 3, 2$ and $6, 5, 4$, respectively. Observe that in all three cases $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$, the numbers 10, 6, and 4, respectively, are the real dimensions of the set of 2×2 hermitian matrices.

Note also that by Lemma 2.7, the interior of $\mathsf{K}_*^{2\times 2}$ consists of exactly the 2×2 positive definite matrices.

For the proof of Theorem 3.3 we need the following, perhaps independently interesting result.

Proposition 3.4. *Let V be a real subspace of real dimension k in $\mathsf{W}^{2\times 2}$. Assume that V has the following property: $x^*Ax = 0$, where $x \in \mathbb{O}^{2\times 1}$, for every $A \in V$ implies that $x = 0$. If $k \leq 8$, then V contains a matrix in the interior of $\mathsf{K}_*^{2\times 2}$.*

Proof. By the duality theory for pointed convex cones (see [4] for example) we need to show that

$$V^{\perp} \cap \mathsf{K}^{n\times n} = \{0\}. \tag{3.1}$$

The hypotheses of the theorem imply that the only positive semidefinite matrix A in $V^{\perp}$ of the form xx^*, $x \in \mathbb{O}^{2\times 1}$, is the zero matrix. Arguing by contradiction, assume (3.1) does not hold. Then $V^{\perp}$ contains a nonzero positive semidefinite matrix Q, which cannot be a real multiple of xx^*, for any $x \in \mathbb{O}^{2\times 1}$. The spectral decomposition for Q (see [7], also (2.11)) now implies that Q is positive definite. Since the real dimension of $V^{\perp}$ is at least 2, there exists $X \in \mathsf{W}^{2\times 2}$ such that Q and X are linearly independent (over $\mathbb{R}$). We claim that there is a linear combination $aQ + bX$, where $a, b \in \mathbb{R}$, of the form xx^*, $x \neq 0$, thereby obtaining a contradiction to the property of V hypothesized in Theorem 3.4.

To this end observe that the entries of Q and X are contained in the subalgebra of $\mathbb{O}$ generated by the two entries of Q and X in the $(1, 2)$-position. We may therefore assume that Q and X are matrices over $\mathbb{H}$. Applying a simultaneous congruence to Q and X, we may further assume that $Q = I$. Now clearly $\lambda I - X$, where λ is one of the (real) eigenvalues of X has the desired form (note that X cannot be a real multiple of I because of the assumption that Q and X are linearly independent). □

Proof of Theorem 3.3. The "only if" part.

Let $A_1, \ldots, A_9 \in \mathsf{W}^{2\times 2}$ be such that the 10-tuple $A_1, \ldots, A_9, I_2$ is linearly independent (over $\mathbb{R}$). We are going to prove $WJ(A_1, \ldots, A_9)$ is not convex.

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Consider a particular situation:

$$A_{0,j} = \begin{bmatrix} 0 & c_j \\ -c_j & 0 \end{bmatrix}, \quad j = 1, 2, \ldots, 7, \quad A_{0,8} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_{0,9} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then clearly

$$(\pm 1, 0, 0, 0, 0, 0, 0, 0, 0) \in WJ(A_{0,1}, \ldots, A_{0,9}).$$

We prove the non-convexity of $WJ(A_{0,1}, \ldots, A_{0,9})$ by showing that 0 does not belong to $WJ(A_{0,1}, \ldots, A_{0,9})$.

Indeed, suppose

$$\mathfrak{R}\left([x^* \quad y^*]\left(A_{0,j}\begin{bmatrix} x \\ y \end{bmatrix}\right)\right) = 0, \quad j = 1, 2, \ldots, 9, \tag{3.2}$$

for some $x, y \in \mathbb{O}$. We are going to prove that $x = y = 0$. The equality (3.2) amounts to the following system of equations:

$$\mathfrak{R}(x^* c_j y) = 0, \quad \text{for } j = 1, \ldots, 7, \quad xx^* = yy^*, \quad \mathfrak{R}(x^* y) = 0. \tag{3.3}$$

Write

$$x = \sum_{k=0}^{7} x_k c_k, \qquad y = \sum_{k=0}^{7} y_k c_k, \qquad x_k, y_k \in \mathbb{R}.$$

Then equations (3.3) take the form

$$\mathfrak{R}((x_0 c_0 + \sum_{k=1}^{7} x_k c_k) c_j (\sum_{\ell=0}^{7} y_\ell c_\ell)) = 0 \quad j = 1, 2, \ldots, 7, \tag{3.4}$$

$$x_0 y_0 + x_1 y_1 + \cdots + x_7 y_7 = 0, \quad \|x\| = \|y\|. \tag{3.5}$$

Equations (3.4) in turn can be rewritten as

$$-x_0 y_j + \sum_{k=1,\ k\neq j}^{7} x_k y_{\ell(j,k)} (-1)^{p(j,k)} + x_j y_0 = 0, \quad j = 1, \ldots, 7, \tag{3.6}$$

where $p(j,k) \in \{0, 1\}$ and $\ell(j,k)$ are found from the equation $c_j c_{\ell(j,k)} = (-1)^{p(j,k)} c_k$. Using the multiplication table, (3.6) and the first equation in (3.5) boil down to

$$\begin{bmatrix} y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\ -y_1 & y_0 & -y_4 & -y_7 & y_2 & -y_6 & y_5 & y_3 \\ -y_2 & y_4 & y_0 & -y_5 & -y_1 & y_3 & -y_7 & y_6 \\ -y_3 & y_7 & y_5 & y_0 & -y_6 & -y_2 & y_4 & -y_1 \\ -y_4 & -y_2 & y_1 & y_6 & y_0 & -y_7 & -y_3 & y_5 \\ -y_5 & y_6 & -y_3 & y_2 & y_7 & y_0 & -y_1 & -y_4 \\ -y_6 & -y_5 & y_7 & -y_4 & y_3 & y_1 & y_0 & -y_2 \\ -y_7 & -y_3 & -y_6 & y_1 & -y_5 & y_4 & y_2 & y_0 \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = 0. \tag{3.7}$$

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Observe that (3.7) has the form

$$(y_0 I + S(y)) \cdot [x_0 \quad x_1 \quad \dots \quad x_7]^T = 0, \tag{3.8}$$

where $S(y)$ is an 8×8 skewsymmetric real matrix such that $S(y)^2 = (-\sum_{j=1}^{7} y_j^2)I$. It is easy to see that $y_0 I + S(y)$ is invertible unless $y = 0$. (Indeed, the complex eigenvalues of $y_0 I + S(y)$ are $y_0 \pm \mathrm{i}\sqrt{y_1^2 + \cdots + y_7^2}$.) Now equation (3.8) shows that at least one of x or y must be zero. But then the second equation in (3.5) yields that both x and y are zeros.

It follows from the non-convexity of $WJ(A_{0,1}, \dots, A_{0,9})$ that $WJ(A_{0,1}, \dots, A_{0,9}, I)$ is also non-convex. Since $\mathsf{W}^{2\times 2}$ is 10-dimensional, we see that

$$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_9 \\ I \end{bmatrix} = T \cdot \begin{bmatrix} A_{0,1} \\ A_{0,2} \\ \vdots \\ A_{0,9} \\ I \end{bmatrix}$$

for some invertible real 10×10 matrix T. Now clearly

$$WJ(A_1, \dots, A_9, I) = T \cdot WJ(A_{0,1}, \dots, A_{0,9}, I).$$

Therefore, $WJ(A_1, \dots, A_9, I)$ is non-convex, and the non-convexity of $WJ(A_1, \dots, A_9, I)$ follows.

The "if" part. Let $A_1, \dots, A_9 \in \mathsf{W}^{2\times 2}$ be such $\{A_1, \dots, A_9, I\}$ are linearly dependent. Arguing by contradiction, suppose that $WJ(A_1, \dots, A_9)$ is not convex, so there exist vectors $q, p \in \mathbb{R}^{9\times 1}$ such that

$$q \notin WJ(A_1, \dots, A_9) \qquad \text{but} \qquad q \pm p \in WJ(A_1, \dots, A_9).$$

(The vector p is necessarily nonzero.) Replacing $A_1, \dots, A_9$ with $A_1 + \alpha_1 I, \dots, A_9 + \alpha_9 I$ for suitable real numbers $\alpha_1, \dots, \alpha_9$, we may assume that $q = 0$. Two cases may occur: (1) $A_1, \dots, A_9$ are linearly dependent. Then by Proposition 3.4, for some real numbers $\beta_1, \dots, \beta_9$ the matrix $Q := \beta_1 A_1 + \cdots + \beta_9 A_9$ is positive definite. Letting $x, y \in \mathbb{O}^{2\times 2}$, $\|x\| = \|y\| = 1$, be such that

$$(x^* A_1 x, \dots, x^* A_9 x) = p, \quad (y^* A_1 y, \dots, y^* A_9 y) = -p,$$

we obtain

$$x^* Q x = [\beta_1 \quad \beta_2 \quad \cdots \quad \beta_9]\, p, \quad y^* Q y = -[\beta_1 \quad \beta_2 \quad \cdots \quad \beta_9]\, p,$$

a contradiction to positive definiteness of Q. The second case: (2) $A_1, \dots, A_9$ are linearly independent. Then $I = \gamma_1 A_1 + \cdots + \gamma_9 A_9$ for some real numbers $\gamma_1, \dots, \gamma_9$.

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Not all the γ_j's are zero; say $\gamma_1 \neq 0$. Consider $W(I, A_2, \ldots, A_9)$. All three numerical ranges $WJ(A_1, \ldots, A_9)$, $W(I, A_2, \ldots, A_9)$, and $W(A_2, \ldots, A_9)$ are convex only simultaneously, so by Proposition 3.1 it suffices to prove the convexity of $W(A_2, \ldots, A_9)$. Now we repeat the proof of the "if" part of Theorem 3.3, replacing $A_1, \ldots, A_9$ with $A_2, \ldots, A_9$, and note that in this situation Theorem 3.4 is applicable regardless if $A_2, \ldots, A_9$ are linearly independent or not; thus, there is no need to consider case (2). □

4. Hermitian matrices with large eigenvalue multiplicity in a subspace. As another application of the properties of hermitian matrices in Section 2, we prove a result - Theorem 4.1 below - that asserts existence of a hermitian matrix with high eigenvalue multiplicity in a real subspace of hermitian matrices, provided the dimension of the subspace is sufficiently large. The proof of the theorem follows the approach of [9].

We say that a hermitian matrix $A \in \mathbb{O}^{n\times n}$ has the *greatest multiplicity* r ($r \geq 1$) if there exists (necessarily unique) $\alpha \in \mathbb{R}$ such that $A - \alpha I$ is negative semidefinite and $x^*(A - \alpha I)x = 0$ for all x in a real subspace of $\mathbb{O}^{n\times 1}$ of real dimension r. In this case, we say that A has the greatest multiplicity r at α. Note that the set $\{x \in \mathbb{O}^{n\times 1} : x^*Ax = 0\}$ is closed under multiplication on the right by reals, but not necessarily by octonions, therefore we cannot talk about octonion subspaces of such vectors x. Note also that every hermitian matrix $A \in \mathbb{O}^{n\times n}$ has the greatest multiplicity one. Indeed, take

$$\alpha = \min\{q \in \mathbb{R} : A - qI \text{ is negative semidefinite}\}.$$

Theorem 4.1. *Assume $n \geq 2$. Let V be a real subspace of real dimension k in $\mathsf{W}^{n\times n}$, and let $1 \leq r \leq n-1$. If $k \geq 1 + 8n(r-1) - \frac{(r-1)(r-2)}{2}$, then V contains a matrix with the greatest multiplicity r at $\alpha = 1$.*

Proof. The case $r = 1$ is trivial: Take any matrix $A \in V$ which is not negative semidefinite, and let

$$\mu := \max_{y\in\mathbb{O}^{n\times 1}\,:\,y^*y=1} y^*Ay > 0.$$

Then $B := \mu^{-1}A$ satisfies the required properties.

We proceed by induction on r. So assume there is $B \in V$ such that B has the greatest multiplicity p at $\alpha = 1$. Let $x_1, \ldots, x_p \in \mathbb{O}^{n\times 1}$ be a linearly independent (over the reals) set such that $B - I$ is negative semidefinite and $x^*(B - I)x = 0$ for every unit vector $x \in \mathbb{O}^{n\times 1}$ which is a real linear combination of $x_1, \ldots, x_p$. We will produce a matrix $A \in V$ that has the greatest multiplicity $p + 1$ at 1, assuming $k \geq 1 + 8np - p(p-1)/2$. Note that by Theorem 2.1 we have $Bx_j = x_j$, for $j = 1, 2, \ldots, p$.

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Suppose $A_1, \ldots, A_k$ form a basis in V. Consider the following system of linear equations:

$$\sum_{i=1}^{k} \alpha_i (A_i x_j) = 0, \quad j = 1, 2, \ldots, p, \tag{4.1}$$

where the α_i's are real variables. Applying the maps χ and ν we obtain an equivalent system

$$\sum_{i=1}^{k} \alpha_i \chi(A_i) \nu(x_j) = 0, \quad j = 1, 2, \ldots, p. \tag{4.2}$$

Passing to an orthonormal basis $\{b_1, \ldots, b_{8n}\}$ in $\mathbb{R}^{8n\times 1}$ of which $\{b_1, \ldots, b_p\}$ form a basis in Span $\{\nu(x_1), \ldots, \nu(x_p)\}$, the system (4.2) can in turn be rewritten in the form

$$\sum_{i=1}^{k} \alpha_i \chi(A_i) b_j = 0, \quad j = 1, 2, \ldots, p, \tag{4.3}$$

which amounts to having first p columns of $\sum_{i=1}^{k} \alpha_i \chi(A_i)$ equal to zero. Taking into account that each $\chi(A_i)$ are real symmetric, the system (4.3) boils down to at most

$$(8n - p)p + p(p + 1)/2 = 8np - \frac{p(p-1)}{2}$$

independent equations. Since

$$k \geq 1 + 8np - \frac{p(p-1)}{2},$$

the system (4.3), and therefore also (4.1), has a nontrivial solution $(\alpha_1^{(0)}, \ldots, \alpha_k^{(0)})$. Let

$$C = \sum_{i=1}^{k} \alpha_i^{(0)} A_i.$$

Then $C \neq 0$, and $Cx_j = 0$ for $j = 1, 2, \ldots, p$ (these equalities follows from (4.1) upon using the associative law which is legitimate since the $\alpha_i^{(0)}$'s are real). We assume C is not negative semidefinite (otherwise take $-C$ in place of C).

Consider the matrix $B + \alpha C$ where $\alpha \geq 0$. We claim that there is a value of α such that $B + \alpha C$ has the greatest multiplicity $p + 1$ at 1. Let

$$\alpha_0 := \arg\max_{\alpha\in\mathbb{R}} \{B + \alpha C - I \quad \text{is} \quad \text{negative} \quad \text{semidefinite}\}.$$

This is well defined, because, C being not negative semidefinite, for large $\alpha > 0$ the matrix $B + \alpha C - I$ is not negative semidefinite. Also, $\alpha_0 \geq 0$ because $B - I$ is negative

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semidefinite. Clearly $(B-\alpha C-I)x_j=0$ for $j=1,2,\dots,p$ and all $\alpha\geq 0$. Then we have

$$(\widetilde{B}+\alpha\widetilde{C}-I)\widetilde{x}_j=0,\quad j=1,2,\dots,p,$$

where $\widetilde{B}=\chi(B)$, $\widetilde{C}=\chi(C)$, and $\widetilde{x}_j=\nu(x_j)$. We will prove that

$$\dim\operatorname{Ker}(\widetilde{B}+\alpha_0\widetilde{C}-I)>p.$$

Suppose not. Then by well known property of semicontinuity of the dimension of the kernel of real matrices, there exists $\epsilon>0$ such that

$$\dim\operatorname{Ker}(\widetilde{B}+\alpha_0\widetilde{C}-I)\leq p,\quad \forall\ \ |\alpha-\alpha_0|<\epsilon.$$

It follows that

$$\operatorname{Ker}(\widetilde{B}+\alpha\widetilde{C}-I)=\operatorname{Span}_{\mathbb{R}}\{\widetilde{x}_1,\dots,\widetilde{x}_p\},\quad \forall\ \ |\alpha-\alpha_0|<\epsilon.$$

Thus, with respect to the orthogonal decomposition

$$\mathbb{R}^{8n\times 1}=\operatorname{Span}_{\mathbb{R}}\{\widetilde{x}_1,\dots,\widetilde{x}_p\}\oplus(\operatorname{Span}_{\mathbb{R}}\{\widetilde{x}_1,\dots,\widetilde{x}_p\})^{\perp},$$

we have

$$\widetilde{B}+\alpha\widetilde{C}-I=\begin{bmatrix}0&0\\0&Q(\alpha)\end{bmatrix},\quad \forall\ \ |\alpha-\alpha_0|<\epsilon,$$

where $Q(\alpha)$ is invertible for $|\alpha-\alpha_0|<\epsilon$ and negative semidefinite for $\alpha\leq\alpha_0$ by definition of α_0 and in view of Lemma 2.6. But then, since $Q(\alpha)$ is a continuous function of α, there exists $0<\epsilon'<\epsilon$ such that $Q(\alpha)$ is negative definite for all $|\alpha-\alpha_0|<\epsilon'$. In particular, $\widetilde{B}+\alpha\widetilde{C}-I$ is negative semidefinite for $\alpha_0<\alpha<\alpha_0+\epsilon'$, a contradiction to the definition of α_0. □

Note that the real dimension of the set of $n\times n$ hermitian octonion matrices is $4n^2-3n$. Therefore, the hypotheses of Theorem 4.1 cannot be satisfied if

$$8n(r-1)-\frac{(r-1)(r-2)}{2}\geq 4n^2-3n,$$

which implies

$$2r\geq 16n+3-\sqrt{224n^2+56n+1}.$$

Thus, the inequality $r\leq n-1$ in Theorem 4.1 can be replaced by

$$r<\frac{1}{2}\left(16n+3-\sqrt{224n^2+56n+1}\right).$$

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5. Concluding remark. In the case of hermitian matrices over the real, complexes, or quaternions, there are strong connections between convexity of joint numerical ranges, Bohnenblust's theorem [5] (which asserts existence of positive definite matrices in a real subspace of hermitian matrices under certain hypotheses), and existence of hermitian matrices with eigenvalues of high multiplicity in a given real subspace of hermitian matrices. In the real and complex cases these connections were explored and made precise in [2], [9], [12], and in [2], [12] for the quaternion case as well. However, much of this approach breaks down over octonions because of the general lack of spectral decomposition for hermitian matrices.

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REFERENCES

[1] Y.-H. Au-Yeung. On the convexity of the numerical range in quaternionic Hilbert space. *Linear and Multilinear Algebra*, 16:93–100, 1984.
[2] Y.-H. Au-Yeung and Y.T. Poon. A remark on the convexity and positive definiteness concerning Hermitian matrices. *Southeast Asian Bulletin of Mathematics*, 3:85–92, 1979.
[3] J.C. Baez. Octonions. *Bulletin of American Mathematical Society*, 39:145–205, 2002.
[4] A. Ben-Israel. Complex linear inequalities. *Inequalities, III (Proc. Third Sympos., Univ. California, Los Angeles, Calif., 1969)*, Academic Press, New York, 23–38, 1972.
[5] F. Bohnenblust. Joint positiveness of matrices. Unpublished manuscript, 1948.
[6] T. Dray, J. Janesky, and C.A. Manogue. Octonionic hermitian matrices with non-real eigenvalues. *Advances in Applied Clifford Algebras*, 10:193–216, 2000.
[7] T. Dray and C.A. Manogue. The octonionic eigenvalue problem. *Advances in Applied Clifford Algebras*, 8:341–364, 1998.
[8] T. Dray, C.A. Manogue, and S. Okubo. Orthonormal eigenbases over the octonions. *Algebras, Groups, and Geometry*, 19:163–180, 2002.
[9] S. Friedland and R. Loewy. Subspaces of symmetric matrices containing matrices with a multiple first eigenvalue. *Pacific Journal of Mathematics*, 62:389–399, 1976.
[10] E. Gutkin, E.A. Jonckheere, and M. Karow. Convexity of the joint numerical range: Topological and differential geometric viewpoints. *Linear Algebra and its Applications*, 376:143–171, 2004.
[11] S. Okubo. Eigenvalue problem for symmetric 3×3 octonionic matrix. *Advances in Applied Clifford Algebras*, 9:131–176, 1999.
[12] Y.T. Poon. Generalized numerical ranges, joint positive definiteness and multiple eigenvalues. *Proceedings of the American Mathematical Society*, 125:1625–1634, 1997.
[13] Y. Tian. Matrix representations of octonions and their applications. *Advances in Applied Clifford Algebras*, 10:61–90, 2000.