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MAXIMA OF THE Q-INDEX: GRAPHS WITHOUT LONG PATHS*

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Abstract. This paper gives tight upper bound on the largest eigenvalue $q(G)$ of the signless Laplacian of graphs with no paths of given order. Thus, let $S_{n,k}$ be the join of a complete graph of order k and an independent set of order $n-k$, and let $S_{n,k}^{+}$ be the graph obtained by adding an edge to $S_{n,k}$.

The main result of the paper is the following theorem:

Let $k \geq 1$, $n \geq 7k^2$, and let G be a graph of order n.

(i) If $q(G) \geq q(S_{n,k})$, then $P_{2k+2} \subset G$, unless $G = S_{n,k}$.

(ii) If $q(G) \geq q\left(S_{n,k}^{+}\right)$, then $P_{2k+3} \subset G$, unless $G = S_{n,k}^{+}$.

The main ingredient of our proof is a stability result of its own interest, about graphs with large minimum degree and with no long paths. This result extends previous work of Ali and Staton.

Key words. Signless Laplacian, Spectral radius, Forbidden paths, Stability theorem, Extremal problem.

AMS subject classifications. 05C50.

1. Introduction. Given a graph G, the Q-index of G is the largest eigenvalue $q(G)$ of its signless Laplacian $Q(G)$. In this paper we determine the maximum Q-index of graphs with no paths of given order. This extremal problem is related to other similar problems, so we shall start by an introductory discussion. Let $S_{n,k}$ be the join of a complete graph of order k and an independent set of order $n-k$ and let $S_{n,k}^{+}$ be the graph obtained by adding an edge to $S_{n,k}$. Write $\mathcal{G}(n)$ for the family of all graphs of order n, and P_l for the path of order l. Given graphs G and H, let $H \subset G$ indicate that H is a subgraph of G.

In the ground-breaking paper [7], Erdős and Gallai established many fundamental extremal relations about graphs with no path of given order, for example: *If G is a graph of order n with no P_{k+2}, then $e(G) \leq kn/2$.* The work of Erdős and Gallai

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caused a surge of later improvements and enhancements, not subsiding to the present day; below we mention some of these results and make a contribution of our own.

A nice and definite enhancement of the Erdős-Gallai result has been obtained by Balister, Gyori, Lehel and Schelp [2].

THEOREM 1.1. *Let $k \geq 1$, $n > (5k+4)/2$ and $G \in \mathcal{G}(n)$, and let G be connected.*

(i) *If $e(G) \geq e(S_{n,k})$, then $P_{2k+2} \subset G$, unless $G = S_{n,k}$.*

(ii) *If $e(G) \geq e\left(S_{n,k}^{+}\right)$, then $P_{2k+3} \subset G$, unless $G = S_{n,k}^{+}$.*

The main result of this paper is in the spirit of a recent trend in extremal graph theory involving spectral parameters of graphs; most often this is the largest eigenvalue $\mu(G)$ of the adjacency matrix of a graph G. The central question in this setup is the following one:

Problem A. *Given a graph F, what is the maximum $\mu(G)$ of a graph $G \in \mathcal{G}(n)$ with no subgraph isomorphic to F?*

Quite often, the results for $\mu(G)$ closely match the corresponding edge extremal results. For illustration, compare Theorem 1.1 with the following result, obtained in [10].

THEOREM 1.2. *Let $k \geq 1$, $n \geq 2^{4k+4}$ and $G \in \mathcal{G}(n)$.*

(i) *If $\mu(G) \geq \mu(S_{n,k})$, then $P_{2k+2} \subset G$, unless $G = S_{n,k}$.*

(ii) *If $\mu(G) \geq \mu\left(S_{n,k}^{+}\right)$, then $P_{2k+3} \subset G$, unless $G = S_{n,k}^{+}$.*

In fact, our paper contributes to an even newer trend in extremal graph theory, a variation of Problem A for the Q-index of graphs, where the central question is the following one:

Problem B. *Given a graph F, what is the maximum Q-index of a graph $G \in \mathcal{G}(n)$ with no subgraph isomorphic to F?*

This question has been resolved for various subgraphs, among which are the matchings. Thus, write M_k for a matching of k edges. In [11] Yu proved the following definite result about M_k.

THEOREM 1.3. *Let $k \geq 1$ and $G \in \mathcal{G}(n)$.*

(i) *If $2k+2 \leq n < (5k+3)/2$ and $q(G) \geq 4k$, then $M_{k+1} \subset G$, unless $G = K_{2k+1} \cup \overline{K}_{n-2k-1}$.*

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(ii) *If* $n = (5k+3)/2$ *and* $q(G) \geq 4k$*, then* $M_{k+1} \subset G$*, unless* $G = K_{2k+1} \cup \overline{K}_{n-2k-1}$ *or* $G = S_{n,k}$*.*

(iii) *If* $n > (5k+3)/2$ *and* $q(G) \geq q(S_{n,k})$*, then* $M_{k+1} \subset G$*, unless* $G = S_{n,k}$*.*

We are mostly interested in clause *(iii)* of this theorem. As it turns out, the focus on a subgraph as simple as M_k conceals a much stronger conclusion that can be drawn from the same premises. We arrive thus at the main result of the present paper.

Theorem 1.4. *Let* $k \geq 1$*,* $n \geq 7k^2$*, and* $G \in \mathcal{G}(n)$*.*

(i) *If* $q(G) \geq q(S_{n,k})$*, then* $P_{2k+2} \subset G$*, unless* $G = S_{n,k}$*.*

(ii) *If* $q(G) \geq q\left(S_{n,k}^{+}\right)$*, then* $P_{2k+3} \subset G$*, unless* $G = S_{n,k}^{+}$*.*

Our proof of Theorem 1.4 is quite complicated and builds upon several results, among which is a stability theorem enhancing previous results by Erdős and Gallai and Ali and Staton. We begin with a corollary of Theorems 1.9 and 1.12 of Erdős and Gallai [7].

Theorem 1.5. *Let* $k \geq 2$*,* G *be a* 2*-connected graph, and* u *be a vertex of* G*. If* $d(w) \geq k$ *for all vertices* $w \neq u$*, then* G *has a path of order* $\min\{\nu(G), 2k\}$*, with end vertex* u*.*

To state the next result set $L_{t,k} := K_1 \vee tK_k$, i.e., $L_{t,k}$ consists of t complete graphs of order $k+1$, all sharing a single common vertex; call the common vertex the *center* of $L_{t,k}$. In [1], Ali and Staton gave the following stability theorem.

Theorem 1.6. *Let* $k \geq 1$*,* $n \geq 2k+1$*,* $G \in \mathcal{G}(n)$*, and* $\delta(G) \geq k$*. If* G *is connected, then* $P_{2k+2} \subset G$*, unless* $G \subset S_{n,k}$*, or* $n = tk+1$ *and* $G = L_{t,k}$*.*

In the light of Theorem 1.1, the theorem of Ali and Staton suggests a possible continuation for P_{2k+3}, which however is somewhat more complicated to state and prove.

Theorem 1.7. *Let* $k \geq 2$*,* $n \geq 2k+3$*,* $G \in \mathcal{G}(n)$ *and* $\delta(G) \geq k$*. If* G *is connected, then* $P_{2k+3} \subset G$*, unless one of the following holds:*

(i) $G \subset S_{n,k}^{+}$*;*

(ii) $n = tk+1$ *and* $G = L_{t,k}$*;*

(iii) $n = tk+2$ *and* $G \subset K_1 \vee ((t-1)K_k \cup K_{k+1})$*;*

(iv) $n = (s+t)k+2$ *and* G *is obtained by joining the centers of two disjoint graphs* $L_{s,k}$ *and* $L_{t,k}$*.*

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In the remaining part of the paper, we give the proofs of Theorems 1.7 and 1.4.

2. Proofs. For graph notation and concepts undefined here, we refer the reader to [3]. For introductory material on the signless Laplacian, see the survey of Cvetković [4] and its references. In particular, let G be a graph, and X be a set of vertices of G. We write:

- $V(G)$ for the set of vertices of G, and $e(G), \nu(G)$ for the number of its edges and its vertices, respectively;

- $G[X]$ for the graph induced by X, and $E(X)$ for $E(G[X])$;

- $\Gamma(u)$ for the set of neighbors of a vertex u, and $d(u)$ for $|\Gamma(u)|$.

2.1. Proof of Theorem 1.7. Assume for a contradiction that $P_{2k+3} \nsubseteq G$. Let us first suppose that G is 2-connected and let $C = (v_1, \ldots, v_l)$ be a longest cycle in G. Set $V' := V(G) \backslash V(C)$. A theorem of Dirac [6] implies that $l \geq 2k$, and $P_{2k+3} \nsubseteq G$ implies that $l \leq 2k+1$. As C is maximal, no vertex in V' can be joined to consecutive vertices in C.

Suppose first that $l = 2k$. We shall show that the set V' is independent. Assume the opposite: Let $\{u, v\}$ be an edge in V', let $C(u) = \Gamma(u) \cap V(C)$ and $C(v) = \Gamma(v) \cap V(C)$. Since G is connected, $P_{2k+3} \nsubseteq G$ implies that $|C(u)| \geq k-1$ and $|C(v)| \geq k-1$.

If there is a vertex $w \in C(v) \backslash C(u)$, then the distance along C between w and any vertex in $C(u)$ is at least 3. Hence, $C(u)$ is contained in a segment of $2k-5$ consecutive vertices of C and so $C(u)$ itself contains consecutive vertices of C, a contradiction; hence, $C(v) \subset C(u)$ and, by symmetry, we conclude that $C(u) = C(v)$.

Finally, if $k \geq 4$, then $C(v)$ contains two vertices at distance 2 along C, and so C can be extended, a contradiction. The remaining simple cases $k = 2$ and 3 are left to the reader. Therefore, V' is independent.

Clearly, every vertex $u \in V'$ has exactly k neighbors in C and therefore, either $\Gamma(u) = \{v_1, v_3, \ldots, v_{2k-1}\}$ or $\Gamma(u) = \{v_2, v_4, \ldots, v_{2k}\}$. Let $u, w \in V'$, and assume that $\Gamma(u) = \{v_2, v_4, \ldots, v_{2k}\}$. If $\Gamma(w) = \{v_1, v_3, \ldots, v_{2k-1}\}$, then C can be extended; hence, $\Gamma(v) = \{v_2, v_4, \ldots, v_{2k}\}$ for every $v \in V'$.

To complete the case $l = 2k$ we shall show that $\{v_1, v_3, \ldots, v_{2k-1}\}$ is independent. Assume the opposite: Let $\{x, y\} \subset \{v_1, v_3, \ldots, v_{2k-1}\}$ and $\{x, y\} \in E(G)$. By symmetry we can assume that $x = v_1$ and $y = v_{2s+1}$. Taking $u \in V'$, we see that the sequence

$$u, v_2, v_3, \ldots, v_{2s+1}, v_1, v_{2k}, \ldots, v_{2s+2}, u$$

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is a cycle longer than C, a contradiction. Hence, the set $\{v_1, v_3, \ldots, v_{2k-1}\} \cup V'$ is independent and so $G \subset S_{n,k} \subset S_{n,k}^+$.

Suppose now that $l = 2k+1$. Clearly, $P_{2k+3} \nsubseteq G$ implies that V' is independent. If $u, v \in V'$ and $w \in \Gamma(v) \backslash \Gamma(u)$, the two neighbors of w along C do not belong to $\Gamma(u)$ because $P_{2k+3} \nsubseteq G$. Hence, $\Gamma(u)$ is a subset of $2k-2$ consecutive vertices of C and so u is joined to two consecutive vertices of C, a contradiction. Hence, all vertices of V' are joined to the same set of size k; by symmetry let this set be $\{v_2, v_4, \ldots, v_{2k}\}$.

We shall show that the set $\{v_1, v_3, \ldots, v_{2k-1}\}$ is independent. Indeed, assume that $\{v_{2s+1}, v_{2t+1}\} \in E(G)$ and $1 \leq 2s+1 < 2t+1 \leq 2k-1$. Taking $u, w \in V'$, we see that the sequence

$$u, v_{2s+2}, v_{2s+3}, \ldots, v_{2t+1}, v_{2s+1}, v_{2s}, \ldots, v_{2t+2}, w$$

is a path of order $2k+3$, contrary to our assumption. Hence, letting

$$V_2 := \{v_1, v_3, \ldots, v_{2k-1}, v_{2k+1}\} \cup V' \quad \text{and} \quad V_1 = V(G) \backslash V_2,$$

we find that $G \subset S_{n,k}^+$. This complete the proof for 2-connected graphs.

Finally suppose that G is not 2-connected. Let B be an end-block of G and u be its cut vertex. Clearly, $v(B) \geq k+1$; Theorem 1.5 implies that B contains a path of order $\min\{v(B), 2k\}$ with end vertex u. Since there are at least two end-blocks and $P_{2k+3} \nsubseteq G$, there is no end-block B with $v(B) > k+2$ and there is at most one end-block of order $k+2$. It is obvious that G contains at most two cut vertices, otherwise we have $P_{2k+3} \subset G$. If G contains one cut vertex, then each block of G is an end-block, and then *(ii)* or *(iii)* holds. If G contains two cut vertices, then *(iv)* holds, completing the proof. □

2.2. Proof of Theorem 1.4. Before going further, note that

$$q\left(S_{n,k}^+\right) > q(S_{n,k}) = \frac{n+2k-2+\sqrt{(n+2k-2)^2 - 8(k^2-k)}}{2}.$$

For $n \geq 7k^2$ and $k \geq 2$, we also find that

$$q\left(S_{n,k}^+\right) > q(S_{n,k}) > n+2k-2-\frac{2\left(k^2-k\right)}{n+2k-3} > n+2k-3. \tag{2.1}$$

If $q(G) \geq q(S_{n,k})$ and $k \geq 2$, the inequality of Das [5], implies that

$$\frac{2e(G)}{n-1} + n - 2 \geq q(G) \geq q(S_{n,k}) > n+2k-2-\frac{2\left(k^2-k\right)}{n+2k-3},$$

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and so,

$$e(G) > k\,(n-k)\,. \tag{2.2}$$

We shall also use the following bound on $q\,(G)$, which can be traced back to Merris [9],

$$q\,(G) \le \max_{u\in V(G)} \left\{ d\,(u) + \frac{1}{d\,(u)} \sum_{v\in\Gamma(u)} d\,(v) \right\}. \tag{2.3}$$

We first determine a crucial property used throughout the proof of Theorem 1.4.

PROPOSITION 2.1. *Let $k \ge 1$, $n \ge 7k^2$, and $G \in \mathcal{G}\,(n)$.*

(i) *If $q\,(G) \ge q\,(S_{n,k})$ and $P_{2k+2} \nsubseteq G$, then $\Delta\,(G) = n-1$;*

(ii) *If $q\,(G) \ge q\left(S^+_{n,k}\right)$ and $P_{2k+3} \nsubseteq G$, then $\Delta\,(G) = n-1$.*

Proof. We shall prove only *(ii)*, as *(i)* follows similarly. We claim that G is connected. Assume the opposite and let G_0 be a component of G, say of order $n_0 \le n-1$, such that $q\,(G_0) = q\,(G)$. Since $2n_0 - 2 \ge q\,(G_0) = q\,(G) > n$, we see that $n_0 > (5k+4)\,/2$ and Lemma 1.1 implies that $2e\,(G_0) \le e(S^+_{n_0,k}) = 2kn_0 - k^2 - k + 2$; hence, by the inequality of Das [5],

$$\begin{aligned} q\,(G) = q\,(G_0) &\le \frac{2e(G_0)}{n_0-1} + n_0 - 2 \le \frac{2kn - k^2 - 3k + 2}{n-2} + n - 3 \\ &= n + 2k - 3 - \frac{k^2-k-2}{n-2} \\ &< n + 2k - 2 - \frac{2\left(k^2-k\right)}{n+2k-3} \\ &\le q\,(S_{n,k})\,. \end{aligned}$$

This contradiction implies that G is connected.

Now, we shall prove that $\Delta\,(G) = n-1$. Assume for a contradiction that $\Delta\,(G) \le n-2$. Let u be a vertex for which the maximum in the right side of (2.3) is attained. Note that $d\,(u) \ge 2k$, for otherwise

$$q\,(G) \le d\,(u) + \frac{1}{d\,(u)} \sum_{v\in\Gamma(u)} d\,(v) \le d\,(u) + \Delta\,(G) \le n + 2k - 3 < q\left(S^+_{n,k}\right).$$

Furthermore, since G is connected, in view of Lemma 1.1,

$$\begin{aligned} \sum_{v\in\Gamma(u)} d\,(v) = 2e\,(G) - \sum_{v\in V(G)\setminus\Gamma(u)} d\,(v) &\le 2e\,(G) - d\,(u) - (n-1-d\,(u)) \\ &\le 2e\left(S^+_{n,k}\right) - n + 1 = (2k-1)\,n - k^2 - k + 3, \end{aligned}$$

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and so

$$q(G) \leq d(u) + \frac{(2k-1)n - k^2 - k + 3}{d(u)}.$$

The function $f(x) := x + \left((2k-1)n - k^2 - k + 3\right)/x$ is convex in x for $x > 0$; hence, its maximum is attained either for $x = 2k$ or for $x = n-2$. But we see that

$$q(G) \leq f(2k) = n + 2k - \frac{n + \left(k^2 + k\right) - 3}{2k} < n + 2k - 2 - \frac{2\left(k^2 - k\right)}{n + 2k - 3} \leq q(S_{n,k}),$$

and,

$$q(G) \leq f(n-2) = n + 2k - 3 - \frac{k^2 - 3k - 1}{n-2} < n + 2k - 2 - \frac{2\left(k^2 - k\right)}{n + 2k - 3} \leq q(S_{n,k}).$$

These contradictions show that $\Delta(G) = n - 1$. ▯

LEMMA 2.2. *Let $k \geq 2$, $n \geq 7k^2$, $G \in \mathcal{G}(n)$, $e(G) > k(n-k)$, and $\delta(G) \leq k-1$. Suppose also that G has a vertex u with $d(u) = n-1$. If $P_{2k+3} \nsubseteq G$, there exists an induced subgraph $H \subset G$, with $\nu(H) \geq n - k^2$, $\delta(H) \geq k$, and $u \in V(H)$.*

Proof. Define a sequence of graphs, $G_0 \supset G_1 \supset \cdots \supset G_r$ using the following procedure.

$G_0 := G$;

$i := 0$;

while $\delta(G_i) < k$ **do begin**

select a vertex $v \in V(G_i)$ with $d(v) = \delta(G_i)$;

$G_{i+1} := G_i - v$;

$i := i + 1$;

end.

Note that the while loop must exit before $i = k^2$. Indeed, by $P_{2k+3} \nsubseteq G_i$ Lemma 1.1 implies that

$$kn - ki - \left(k^2 + k\right)/2 + 1 \geq e(G_i) \geq e(G) - i(k-1) > k(n-k) - i(k-1);$$

hence, $i < k^2$. Letting $H = G_r$, where r is the last value of the variable i, the proof is completed. ▯

Proof of Theorem 1.4.

(i) Assume for a contradiction that $P_{2k+2} \nsubseteq G$. By Proposition 2.1, G has a vertex u with $d(u) = n-1$. If $k = 1$, then $P_4 \nsubseteq G$ and clearly $G = S_{n,1}$.

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Let $k \geq 2$. If $\delta(G) \geq k$, Theorem 1.6 implies that $G \subset S_{n,k}$ or $n = kt + 1$ and $G = L_{t,k}$. The latter case cannot hold because

$$q(L_{t,k}) \leq \max_{\{u,v\} \in E(L_{t,k})} \{d(u) + d(v)\} = n - 1 + k \leq n + 2k - 3 < q(S_{n,k}). \qquad (2.4)$$

In the first case, if $G \neq S_{n,k}$, then $q(G) < q(S_{n,k})$, completing the proof. Suppose now that $\delta(G) \leq k - 1$. By (2.2) we have $e(G) > k(n-k)$ and then Lemma 2.2 implies that there exists an induced subgraph H of order $n_1 \geq n - k^2$, with $\delta(H) \geq k$ and $u \in V(H)$. Let $H^{'} = G[V(G) \backslash V(H)]$. Theorem 1.6 implies that $H \subset S_{n_1,k}$, or $n_1 = tk + 1$ and $H = L_{t,k}$.

Assume first that $n_1 = tk + 1$ and $H = L_{t,k}$. Obviously, u is the center of H. Note that there is no edge between $V(H^{'})$ and $V(H) \setminus \{u\}$, for otherwise $P_{2k+2} \subset G$. Therefore,

$$e(H^{'}) = e(G) - e(H) - (n - n_1) > k(n-k) - \frac{(k+1)(n_1 - 1)}{2} - (n - n_1).$$

After some algebra, we find that $e(H^{'}) > \frac{1}{2}(k-1)(n-n_1)$; hence, $P_{k+1} \subset H^{'}$ (see [7]). Since u is a dominating vertex and $P_{k+1} \subset H$, we see that $P_{2k+2} \subset G$, a contradiction.

Assume now that $H \subset S_{n_1,k}$. Write I for the independent set of size $n_1 - k$ of H. Obviously, H contains a path P_{2k+1} with both ends in I. Thus, the set $V(H') \cup I$ is independent, for otherwise $P_{2k+2} \subset G$. Hence, $G \subset S_{n,k}$ and so $G = S_{n,k}$, completing the proof of *(i)*.

(ii) Assume for a contradiction that $P_{2k+3} \nsubseteq G$. By Proposition 2.1, G has a vertex u with $d(u) = n - 1$. Let $k = 1$. There is an edge in $G - u$, for otherwise $q(G) < q(S^{+}_{n,1})$. If there exist two edges in $G - u$, then $P_5 \subset G$. So $G - u$ induces exactly one edge, and $G = S^{+}_{n,1}$.

Let $k \geq 2$. If $\delta(G) \geq k$, in view of $\Delta(G) = n - 1$, Theorem 1.7 implies that either $G \subset S^{+}_{n,k}$ or $n = tk + 1$ and $G = L_{t,k}$, or $G \subset K_1 \vee ((t-1)K_k \cup K_{k+1})$. The inequality (2.4) shows that $G \neq L_{t,k}$, and $G \subset K_1 \vee ((t-1)K_k \cup K_{k+1})$ cannot hold because

$$\begin{aligned} q(K_1 \vee ((t-1)K_k \cup K_{k+1})) &\leq \max_{u \in V(K_1 \vee ((t-1)K_k \cup K_{k+1}))} \left\{ d(u) + \frac{1}{d(u)} \sum_{v \in \Gamma(u)} d(v) \right\} \\ &\leq n + k - 1 + \frac{k+1}{n-1} \\ &\leq n + 2k - 2 - \frac{2(k^2 - k)}{n + 2k - 3} < q\left(S^{+}_{n,k}\right). \end{aligned}$$

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In the first case, if $G \neq S^{+}_{n,k}$, then $q(G) < q\left(S^{+}_{n,k}\right)$, completing the proof. Suppose therefore that $\delta(G) \leq k-1$. By (2.2) we have $e(G) > k(n-k)$ and Lemma 2.2 implies that there exists an induced subgraph H of order $n_1 \geq n-k^2$, with $\delta(H) \geq k$ and $u \in V(H)$. Theorem 1.7 implies that H satisfies one of the conditions *(i)-(iv)*. Since u is a dominating vertex in H, condition *(iv)* is impossible.

Next, assume that H satisfies *(ii)* or *(iii)*. Clearly, $n_1 \geq n-k^2 \geq 3k+2$. Let t be the number of components of $H-u$; clearly $t \geq 3$. Suppose there are two components H_1 and H_2 of $H-u$, with edges between H_1 and H' and between H_2 and H'. Then either $P_{2k+3} \subset G$, or there is a cycle C_{2k+2} containing u; hence, $P_{2k+3} \subset G$ anyway. Thus, $H-u$ has $t-1$ components that are also components of $G-u$. Let H_0 be the remaining component of $H-u$; set $m = v(H_0)$ and note that $k \leq m \leq k+1$. Write H'' for the graph obtained by adding H_0 to H'. We shall show that $e(H'') > (k/2)\,v(H'')$. Indeed, otherwise we have

$$\begin{aligned}(k/2)(n-n_1+m) \geq e(H'') &= e(G) - e(H) + e(H_0) - (n-n_1)\\ &> k(n-k) - e(H) + e(H_0) - (n-n_1).\end{aligned}$$

Now, using the obvious inequalities

$$e(H) \leq n_1 - 1 + \frac{(k-1)(n_1-k-1)}{2} + \frac{(k+1)k}{2} \qquad \text{and} \qquad e(H_0) \geq (k-1)\,m/2,$$

together with $m \geq k$, $n_1 \geq n-k^2$ and $n \geq 7k^2$, we obtain a contradiction. Hence, $e(H'') > (k/2)\,v(H'')$ and so $P_{k+2} \subset H''$; since u is a dominating vertex and $P_{k+1} \subset H$, we get $P_{2k+3} \subset G$, which is a contradiction.

Finally, assume that $H \subset S^{+}_{n_1,k}$, that is to say, there exists $I \subset V(H)$ of size $n_1 - k$, such that I induces at most one edge on H. If I induces precisely one edge and there are edges between $V(H')$ and I, we see that $P_{2k+3} \subset G$, so $V(H') \cup I$ induces at most one edge. Hence, $G \subset S^{+}_{n,k}$ and $G = S^{+}_{n,k}$, completing the proof.

Assume now that I is independent and set $J = V(H) \backslash I$. Clearly, $\delta(H) \geq k$ implies that every vertex of I is joined to every vertex in J; hence, any vertex in I can be joined in H to the vertex u by a path of order $2k+1$. This implies that $V(H') \cup I$ contains no paths of order 3, otherwise $P_{2k+3} \subset G$; hence, the set $V(H') \cup I$ induces only isolated vertices and disjoint edges.

If $V(H') \cup I$ induces exactly one edge, we certainly have $G \subset S^{+}_{n,k}$. Assume now that $V(H') \cup I$ induces two or more edges. None of these edges has a vertex in I, as otherwise, using that u is dominating vertex, we can construct a P_{2k+3} in G. Likewise, we see that each of the ends of any edge in H' is joined only to u. We shall show that $q(G) < q(S_{n,k})$.

Let $(x_1, \ldots, x_n)$ be a positive unit eigenvector to $q(G)$. It is known, see, e.g., [4]

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that

$$q(G) = \sum_{\{i,j\} \in E(G)} (x_i + x_j)^2 .$$

Choose a vertex $v \in J \setminus \{u\}$ and let $\{i,j\}$ be an edge in H'. Letting $q = q(G)$, from the eigenequations for $Q(G)$, we have

$$(q-2)\, x_i = x_j + x_u \quad \text{and} \quad (q-2)\, x_j = x_i + x_u,$$

implying that $x_i = x_j = x_u / (q-3)$. On the other hand,

$$(q - d(v))\, x_v = \sum_{s \in \Gamma(v)} x_s > x_u,$$

implying that $x_v > x_i$ as $d(v) \geq |I| \geq n - k^2 - k > 3$.

For any $\{i,j\} \in E(H')$, remove the edge $\{i,j\}$ and join v to i and j. Write G' for the resulting graph. Obviously, $G' \subset S_{n,k}$. We see that

$$q(S_{n,k}) \geq q(G') \geq \sum_{\{i,j\} \in E(G')} (x_i + x_j)^2 > \sum_{\{i,j\} \in E(G)} (x_i + x_j)^2 = q(G),$$

a contradiction showing that $V(H') \cup I$ induces at most one edge and so $G \subset S_{n,k}^{+}$, completing the proof. $\square$

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REFERENCES

[1] A.A. Ali and W. Staton. On extremal graphs with no long paths. *Electron. J. Combin.*, 3:R20, 1996.
[2] P.N. Balister, E. Győri, J. Lehel, and R.H. Schelp. Connected graphs without long paths. *Discrete Math.*, 308:4487–4494, 2008.
[3] B. Bollobás. *Modern Graph Theory.* Graduate Texts in Mathematics, Vol. 184, Springer-Verlag, New York, 1998.
[4] D. Cvetković. Spectral theory of graphs based on the signless Laplacian. Research report, 2010, available at `http://www.mi.sanu.ac.rs/projects/signless_L_reportApr11.pdf`.
[5] K. Das. Maximizing the sum of the squares of the degrees of a graph. *Discrete Math.*, 285:57–66, 2004.
[6] G.A. Dirac. Some theorems on abstract graphs. *Proc. London Math Soc.*, 2:69–81, 1952.
[7] P. Erdős and T. Gallai. On maximal paths and circuits of graphs. *Acta Math. Acad. Sci. Hungar.*, 10:337–356, 1959.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 504-514, July 2014

[8] P. Erdős and A. Rényi. On a problem in the theory of graphs. *Publ. Math. Inst. Hungar. Acad. Sci.*, 7A:623–641, 1962.
[9] R. Merris. A note on Laplacian graph eigenvalues. *Linear Algebra Appl.*, 295:33–35, 1998.
[10] V. Nikiforov. The spectral radius of graphs without paths and cycles of specified length. *Linear Algebra Appl.*, 432:2243–2256, 2010.
[11] G. Yu. On the maximal signless Laplacian spectral radius of graphs with given matching number. *Proc. Japan Acad., Ser. A*, 84:163–166, 2008.