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TWO BY TWO UNITS*

R.E. HARTWIG† AND P. PATRíCIO‡

Abstract. In this paper, we will use outer inverses and the Brown-McCoy shift to characterize the existence of the inverse and group inverse of a 2×2 block matrix.

Key words. Outer inverses, Reflexive inverses, Generalized inverses, Pierce decomposition, Inverse, Group inverse, Block matrices.

AMS subject classifications. 15A09, 15B33, 16E50.

1. Introduction. A fundamental problem in matrix theory is the creation of a unit 2×2 matrix $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ over an associative ring R with unity 1, where we assume that either a or d is regular. Because $M = aE_{11} + bE_{21} + cE_{12} + dE_{22}$ with

$$E_{ij}E_{pq} = \delta_{jp}E_{iq}, \tag{1.1}$$

we see that this "vertical" (i.e., the block-matrix case) unit problem is related to the "horizontal" orthogonality problem:

$$\text{when is } xy = 0 = yx?$$

In general, the search for units is facilitated by existence of special elements. In particular, regular elements can be used, and the corresponding theory of generalized inverses may also be employed, leading to the search for group and Drazin inverses.

We say a is *2-regular* if $\hat{a}a\hat{a} = \hat{a}$ for some outer inverse (or 2-inverse) $\hat{a}$ of a.

An element a is *regular* if $aa^-a = a$ for some inner or 1-inverse a^- of a.

A *reflexive* or *1-2 inverse* a^+ satisfies

$$aa^+a = a, \quad a^+aa^+ = a^+.$$

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†Mathematics Department, North Carolina State University, Raleigh, NC 27695-8205, USA (hartwig@unity.ncsu.edu).

‡CMAT – Centro de Matemtica and Departamento de Matemática e Aplicações, Universidade do Minho, 4710-057 Braga, Portugal (pedro@math.uminho.pt). This author was financed by FEDER Funds through "Programa Operacional Factores de Competitividade – COMPETE" and by Portuguese Funds through FCT – "Fundação para a Ciência e a Tecnologia", within the project PEst-C/MAT/UI0013/2011.

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A reflexive inverse that commutes with a, if any, must be unique and is called the *group inverse* of a. It is usually denoted by $a^{\#}$.

In order to get a handle on invertibility, we shall use the Brown-McCoy (BM) shift ([1]) to derive a tractable expression for $I - MM^-$. On the other hand, to tackle the group inverse problem we shall employ a trick first used in [2].

Let us start by first making a short digression to address the story of outer or 2-inverses.

2. Outer inverses. Suppose that an element (or a matrix) m has an outer inverse $\hat{m}$ such that $\hat{m}m\hat{m} = \hat{m}$. Even though m need not be regular, there are several elements associated with m that are regular, such as $t = m\hat{m}m$. A more interesting element associated with m is the BM-shift,

$$\beta(m) = n = m - m\hat{m}m, \tag{2.1}$$

which depends on the choice of $\hat{m}$. We note that n may not be regular nor have an outer inverse.

Let us now investigate some of the relationships between m and its BM-shift n. In particular, we shall attempt to find other outer inverses of m. With no surprise when studying generalized inverses, this will become a study of idempotents. We begin by defining the idempotents

$$e = 1 - m\hat{m} \quad \text{and} \quad f = 1 - \hat{m}m. \tag{2.2}$$

It is clear that $\hat{m}n = 0 = n\hat{m}$ and $n = mf = em = nf = en = emf = enf$.

We now have

Lemma 2.1. *Let m be regular and let n be defined as in (2.1). With the notation (2.2), we have:*

- *(a) If n has a 2-inverse $\hat{n}$, then $\hat{n}e$, $f\hat{n}$, $f\hat{n}e$ and $\hat{m} + f\hat{n}e$ are also outer inverses of m.*
- *(b) n is regular iff m is regular. In which case $m^- = \hat{m} + fn^-e$ and both m^- and $(1 - \hat{m}m)m^-(1 - m\hat{m})$ are inner inverses of n.*
- *(c) If n^+ is given, then*

$$m^+ = \hat{m} + (1 - \hat{m}m)n^+(1 - m\hat{m}).$$

Proof. (a) $\hat{n}em\hat{n}e = \hat{n}n\hat{n}e = \hat{n}e$. Also $(\hat{m} + f\hat{n}e)m(\hat{m} + f\hat{n}e) = \hat{m}m(\hat{m} + f\hat{n}e) + f\hat{n}em(\hat{m} + f\hat{n}e) = \hat{m} + f\hat{n}e$, because $em = n = nf$.

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(b) Observe that $m(\hat{m} + fn^- e)m = m\hat{m}m + n = m$ [1, Lemma 1]. Next we have $nm^- n = emm^- mf = emf = n$, i.e., *any* inner inverse of m is also an inner inverse for n, for any choice of $\hat{m}$.

(c) This is left as an easy exercise. □

If n is 2-regular, we may repeat the shift process.

Indeed, set $r = \beta(n) = n - n\hat{n}n$, then $u = m - m\hat{m}m - n\hat{n}n$. Moreover, if we take, by Lemma 2.1, $\hat{\hat{m}} = f\hat{n}e$ then $r = m - m\hat{\hat{m}}m$, and we get a chain of outer inverses.

It follows at once that if m is regular, in which case n is also regular by Lemma 2.1 (c), then

$$mm^- = m\hat{m} + nn^-(1 - m\hat{m}) \text{ and } m^- m = \hat{m}m + (1 - \hat{m}m)n^- n$$

and hence that

$$1 - mm^- = (1 - nn^-)(1 - m\hat{m}) \text{ and } 1 - m^- m = (1 - \hat{m}m)(1 - n^- n).$$

We note that *each* term in brackets is an idempotent.

Moreover, we may further conclude that

$$m \text{ is a unit iff } (1 - nn^-)(1 - m\hat{m}) = 0 = (1 - \hat{m}m)(1 - n^- n).$$

If we repeat this shift process and set $r = n - n\hat{n}n$, then it follows that r is regular by Lemma 2.1 (c), and we obtain the following after suitable substitution:

1. $\hat{m}r = 0 = r\hat{m} = \hat{n}r = r\hat{n}$;
2. $1 - nn^- = (1 - rr^-)(1 - n\hat{n})$ and $1 - n^- n = (1 - \hat{n}n)(1 - r^- r)$;
3. $n^- = \hat{n} + (1 - \hat{n}n)r^-(1 - n\hat{n})$;
4. $1 - mm^- = (1 - rr^-)(1 - n\hat{n})(1 - m\hat{m})$ and $(1 - m^- m) = (1 - \hat{m}m)(1 - \hat{n}n)(1 - r^- r)$;
5. $m^- = [\hat{m} + (1 - \hat{m}m)\hat{n}(1 - m\hat{m})] + [(1 - \hat{m}m)(1 - \hat{n}n)r^-(1 - n\hat{n})(1 - m\hat{m})]$, in which the first bracket is another outer inverse of m.

In some cases, we require a third iteration of the BM-shift, say $s = r - r\hat{r}r$, for which

$$1 - rr^- = (1 - ss^-)(1 - r\hat{r}) \quad \text{and} \quad (1 - r^- r) = (1 - \hat{r}r)(1 - s^- s),$$

$$r^- = \hat{r} + (1 - \hat{r}r)s^-(1 - r\hat{r}).$$

If we substitute these into the expressions given by item 4, we arrive at:

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1. $1 - mm^- = (1 - ss^-)(1 - r\hat{r})(1 - n\hat{n})(1 - m\hat{m})$;
2. $1 - m^-m = (1 - \hat{m}m)(1 - \hat{n}n)(1 - \hat{r}r)(1 - s^-s)$.

It should be noted that these involve *five* idempotents.

We now obtain the following pyramid structure for m^-.

Theorem 2.2. *Given m, n, r and s as above with m regular, then $m^- = \lambda + \mu$ is a 1-inverse of m, where*

$$\lambda = \hat{m} + (1 - \hat{m}m)\hat{n}(1 - m\hat{m}) + (1 - \hat{m}m)(1 - \hat{n}n)\hat{r}(1 - n\hat{n})(1 - m\hat{m})$$

and

$$\mu = (1 - \hat{m}m)(1 - \hat{n}n)(1 - \hat{r}r)s^-(1 - r\hat{r})(1 - n\hat{n})(1 - m\hat{m}).$$

Using the facts that $0 = \hat{m}n = \hat{m}r = n\hat{m} = r\hat{m}$, it follows that $\lambda m\lambda = \lambda$, i.e., λ is another outer inverse of m.

The advantage of using the factored form with all these idempotents is that

(a) outer inverses are easy to calculate;
(b) finding an inner inverse of the final element s, is usually much easier than finding an inner inverse at an earlier stage.

To emphasize that the inner inverse has been obtained via the BM-shift we shall often refer to it as m^-_{BM}.

3. Using the Brown-McCoy shift. Let $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, and assume that d is regular. To compute a suitable inner inverse of M we shall use a three fold iteration of the Brown-McCoy shift by pivoting on the (2,2), (1,2) and (2,1) entries in M.

Step 1. Consider the outer inverse $\hat{M} = \begin{bmatrix} 0 & 0 \\ 0 & d^+ \end{bmatrix}$. Then

$$I - M\hat{M} = \begin{bmatrix} 1 & -cd^+ \\ 0 & 1 - dd^+ \end{bmatrix} \quad \text{and} \quad I - \hat{M}M = \begin{bmatrix} 1 & 0 \\ -d^+b & 1 - d^+d \end{bmatrix}. \tag{3.1}$$

Next, assuming eb and cf are regular, we define a sextet of idempotents:

$$\begin{array}{ll} e = 1 - dd^+, & f = 1 - d^+d, \\ g = 1 - (eb)(eb)^+, & h = 1 - (eb)^+(eb), \\ p = 1 - (cf)(cf)^+, & q = 1 - (cf)^+(cf). \end{array} \tag{3.2}$$

Note that $cfq = 0 = pcf = geb = ebh$.

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We now form the BM-shift matrix

$$N = \beta(M) = M - M\hat{M}M = \begin{bmatrix} \zeta & cf \\ eb & 0 \end{bmatrix},$$

where $\zeta = a - cd^+b$.

We could also use the matrix $T = \begin{bmatrix} cf & \zeta \\ 0 & eb \end{bmatrix}$ and observe that $NN^- = TT^-$, see [6].

Step 2. Take $\hat{N} = \begin{bmatrix} 0 & 0 \\ (cf)^+ & 0 \end{bmatrix}$. Then we compute

$$I - N\hat{N} = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad I - \hat{N}N = \begin{bmatrix} 1 & 0 \\ -(cf)^+\zeta & q \end{bmatrix}. \tag{3.3}$$

We then set

$$R = N(I - \hat{N}N) = \begin{bmatrix} \zeta & cf \\ eb & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(cf)^+\zeta & q \end{bmatrix} = \begin{bmatrix} p\zeta & 0 \\ eb & 0 \end{bmatrix}.$$

Step 3. Take $\hat{R} = \begin{bmatrix} 0 & (eb)^+ \\ 0 & 0 \end{bmatrix}$ and set $S = R - R\hat{R}R$. Then

$$R\hat{R} = \begin{bmatrix} p\zeta & 0 \\ eb & 0 \end{bmatrix} \begin{bmatrix} 0 & (eb)^+ \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & p\zeta(eb)^+ \\ 0 & (eb)(eb)^+ \end{bmatrix},$$

and likewise,

$$\hat{R}R = \begin{bmatrix} 0 & (eb)^+ \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p\zeta & 0 \\ eb & 0 \end{bmatrix} = \begin{bmatrix} (eb)^+(eb) & 0 \\ 0 & 0 \end{bmatrix}.$$

These give

$$I - R\hat{R} = \begin{bmatrix} 1 & -p\zeta(eb)^+ \\ 0 & g \end{bmatrix} \quad \text{and} \quad I - \hat{R}R = \begin{bmatrix} h & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.4}$$

Lastly, we compute

$$S = R(I - \hat{R}R) = \begin{bmatrix} \zeta & cf \\ eb & 0 \end{bmatrix} \begin{bmatrix} h & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} p\zeta h & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} w & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.5}$$

We recall by Lemma 2.1 that M is regular if and only if S is regular.

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Because of the sparse character of the matrix S, we may conclude that S is regular iff $w = p\zeta h$ is regular, in which case we can find an inner inverse $S^- = \begin{bmatrix} w^- & 0 \\ 0 & 0 \end{bmatrix}$.

We now combine (3.1), (3.3), (3.4) and (3.5), to compute

$$\begin{aligned} I - MM^- &= (1 - SS^-)(1 - R\hat{R})(1 - N\hat{N})(1 - M\hat{M}) \\ &= \begin{bmatrix} 1 - ww^- & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -p\zeta(eb)^+ \\ 0 & g \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -cd^+ \\ 0 & e \end{bmatrix}. \end{aligned}$$

If we now define $\alpha = (1 - ww^-)p$, then this takes the form

$$I - MM^- = \begin{bmatrix} \alpha & -\alpha[cd^+ + \zeta(eb)^+ e] \\ 0 & ge \end{bmatrix}. \tag{3.6}$$

Correspondingly, we also have

$$\begin{aligned} I - M^-M &= (1 - \hat{M}M)(I - \hat{N}N)(1 - \hat{R}R)(I - S^-S) \\ &= \begin{bmatrix} 1 & 0 \\ -d^+b & f \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(cf)^+\zeta & q \end{bmatrix} \begin{bmatrix} h & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 - w^-w & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

which becomes

$$\begin{aligned} I - M^-M &= \begin{bmatrix} h(1 - w^-w) & 0 \\ -d^+bh(1 - w^-w) - f(cf)^+\zeta h(1 - w^-w) & fq \end{bmatrix} \\ &= \begin{bmatrix} \beta & 0 \\ (-d^+b - f(cf)^+\zeta)\beta & fq \end{bmatrix}, \end{aligned}$$

where $\beta = h(1 - w^-w)$.

In conclusion, we can now give the parameters λ and μ, defined as in Theorem 2.2 , in product form

$$\begin{aligned} \lambda = &\begin{bmatrix} 0 & 0 \\ 0 & d^+ \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -d^+b & f \end{bmatrix} \begin{bmatrix} 0 & 0 \\ (cf)^+ & 0 \end{bmatrix} \begin{bmatrix} 1 & -cd^+ \\ 0 & e \end{bmatrix} \\ &+ \begin{bmatrix} 1 & 0 \\ -d^+b & f \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(cf)^+\zeta & q \end{bmatrix} \begin{bmatrix} 0 & (eb)^+ \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -cd^+ \\ 0 & e \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \mu = &\begin{bmatrix} 1 & 0 \\ -d^+b & f \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(cf)^+\zeta & q \end{bmatrix} \begin{bmatrix} h & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w^- & 0 \\ 0 & 0 \end{bmatrix} \\ &\times \begin{bmatrix} 1 & -p\zeta(eb)^+ \\ 0 & g \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -cd^+ \\ 0 & e \end{bmatrix}. \end{aligned}$$

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Thus,

$$M_{BM}^{-} = \lambda + \mu, \tag{3.7}$$

which comes from the repeated shift given by Theorem 2.2

$$M_{BM}^{-} = \hat{M} + \left(I - \hat{M}M\right) \times$$
$$\times \left[\hat{N} + \left(I - \hat{N}N\right)\left\{\hat{R} + \left(I - \hat{R}R\right)S^{-}\left(I - R\hat{R}\right)\right\}\left(I - N\hat{N}\right)\right]\left(I - M\hat{M}\right).$$

We may multiply out the products to obtain

$$M_{BM}^{-} = \begin{bmatrix} \sigma & \phi - \sigma t \\ \chi - s\sigma & d^{+} - s\phi - \chi t + \chi\zeta\phi + s\sigma t \end{bmatrix},$$

where

$$\phi = (eb)^{+}e,\ \chi = f(cf)^{+},\ t = cd^{+} + \zeta\phi,\ s = d^{+}b + \chi\zeta,\ w = p\zeta h,\ \text{and } \sigma = hw^{-}p.$$

In factored form, this becomes

$$M_{BM}^{-} = \begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix}\begin{bmatrix} \sigma & \phi \\ \chi & d^{+} + \chi\zeta\phi \end{bmatrix}\begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}.$$

Remark. The factored form of M_{BM}^{-} seems to suggest that we should start with the matrix $\begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} M \begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix}$ instead of M.

3.1. The triangular case. Consider the matrix $T = \begin{bmatrix} cf & \zeta \\ 0 & eb \end{bmatrix}$. Then

$$TT^{-} = \begin{bmatrix} (cf)(cf)^{+} + ww^{-}p & (1 - ww^{-})p\zeta(eb)^{+} \\ 0 & (eb)(eb)^{+} \end{bmatrix}, \tag{3.8}$$

and hence, $I - NN^{-} = I - TT^{-} = \begin{bmatrix} \alpha & -\alpha\zeta(eb)^{+} \\ 0 & g \end{bmatrix}$, where $\alpha = (1 - ww^{-})p$.

On the other hand, using he expression of $I - M\hat{M}$ given in equation (3.1), we arrive at

$$I - MM^{-} = (I - NN^{-})(I - M\hat{M}) = \begin{bmatrix} \alpha & \Gamma \\ 0 & ge \end{bmatrix},$$

where $\Gamma = -\alpha[cd^{+} + \zeta(eb)^{+}e] = \alpha\Gamma$ and $\Gamma(ge) = 0$. Clearly, this vanishes iff both α and ge vanish.

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We likewise obtain

$$(I-\hat{M}M)(I-N^-N)=\left[\begin{array}{cc}\beta & 0\\ (-d^+b-f(cf)^+\zeta)\beta & fq\end{array}\right], \qquad (3.9)$$

where $\beta=h(1-w^-w)$. We thus see that M has a left inverse iff $\beta=0$ and $fq=0$.

We may now use this to derive our main theorem.

4. Main theorem.

Theorem 4.1. *Let* $M=\left[\begin{array}{cc}a & c\\ b & d\end{array}\right]$ *with* d *regular. Suppose* $e=1-dd^+$ *and* $f=1-d^+d$*, and assume that* eb *and* cf *are regular. With the notation of (3.2) and* $\zeta=a-cd^+b$*, if* $w=p\zeta h$ *is regular, then the following hold:*

1. *M has a right inverse iff*
   (i) $eR=ebR$ *(or* $ge=0$*) and*
   (ii) $pR=(p\zeta h)R$ *(*$\alpha=0$ *or* $ww^-p=p$*).*
   In which case, M has a right inverse given by (3.8).
2. *M has a left inverse iff*
   (iii) $Rf=Rcf$ *(or* $fq=0$*) and*
   (iv) $Rh=R(p\zeta h)$ *(*$\beta=0$ *or* $hw^-w=h$*)*
   In which case, M has a left inverse given by (3.8).
3. *M is a unit iff all four conditions (i)–(iv) hold, in which case,*
   $$M^{-1}=\left[\begin{array}{cc}\sigma & \phi-\sigma t\\ \chi-s\sigma & d^+-s\phi-\chi t+\chi\zeta\phi+s\sigma t\end{array}\right],$$
   where $\phi=(eb)^+e$*,* $\chi=f(cf)^+$*,* $t=cd^++\zeta\phi$*,* $s=d^+b+\chi\zeta$*,* $w=p\zeta h$*, and* $\sigma=hw^-p$*.*

We note that a simplification only takes place after we multiply by M!

To simplify the conditions of Theorem 4.1, we shall need two standard results dating back to von Neumann [11].

Lemma 4.2. *Let* $a,b\in R$ *with* a *regular. Then (a)* $abR=aR$ *iff* $(1-a^-a)R+bR=R$*. (b)* $Rba=Ra$ *iff* $R(1-aa^-)+Rb=R$*.*

Lemma 4.3. *Let* d*,* cf *and* eb *be regular elements. With the notation of (3.2), we have*

(a) $dR+bR=\delta R$*, where* $\delta=dd^++eb(eb)^+e=\delta^2$*;*
(b) $Rc+Rd=R\varepsilon$ *where* $\varepsilon=d^+d+f(cf)^+cf=\varepsilon^2$*.*

Proof. (a) Let $k=dd^+r+bs\in dR+bR$, for some $r,s\in R$ and set $eb=m$. Since $k=dd^+k+(1-dd^+)k$, $dd^+k=dd^+r+dd^+bs$ and $(1-dd^+)k=(1-dd^+)bs=ms=$

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mm^+ms, it follows

$$\begin{aligned} k &= dd^+r + dd^+bs + mm^+(1-dd^+)ms \\ &= \left(dd^+ + mm^+(1-dd^+)\right)(dd^+r + dd^+bs + ms) \in \left(dd^+ + mm^+(1-dd^+)\right) R. \end{aligned}$$

The proof of part (b) is left to the reader. □

We now recall that an idempotent with a left (right) inverse must equal the unit element, i.e.,

$$\text{If } g^2 = g \text{ with } gR = R \text{ (or } Rg = R\text{), then } g = 1. \tag{4.1}$$

This leads us to the following essential result:

Corollary 4.4. *Let d, cf and eb be regular elements. Using the notation of (3.2), we have:*

(a) The following are equivalent:
(i) $dR + bR = R$; (ii) $\delta = dd^+ + eb(eb)^+e = 1$; (iii) $ebR = eR$.
(b) The following are equivalent:
(i) $Rc + Rd = R$; (ii) $\varepsilon = d^+d + f(cf)^+cf = 1$; (iii) $Rf = Rcf$.

The condition $dR + bR = R$ is a well known necessary condition for M to have a right inverse. Indeed, if $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ has a right inverse $X = \begin{bmatrix} x & r \\ y & z \end{bmatrix}$, then $br + dz = 1$ and $bR + dR = R$.

Likewise the condition $Rc + Rd = R$, which corresponds to $xc + yd = 1$, for some x and y, is necessary for M to have a left inverse.

Similarly, the condition $\alpha = 0$ of our main theorem is equivalent to any of the following

$$(i)\ (p\zeta h)R = pR,\ \ (ii)\ cfR + \zeta hR = R,\ \ (iii)\ \delta_2 = cf(cf)^+ + p\zeta h(p\zeta h)^+p = 1,$$

while the condition $\beta = 0$ can be replaced by any of the following

$$(i)\ Rp\zeta h = Rh,\ \ (ii)\ R = R(p\zeta) + Reb,\ \ (iii)\ \varepsilon_2 = (eb)^+eb + h(p\zeta h)^+p\zeta h = 1. \tag{4.2}$$

As such our main theorem can be written as

Corollary 4.5. *Let $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ with d, eb and cf regular. Under the same assumptions of Theorem (4.1), it follows that*

(A) M has a right inverse iff

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1. any of the following holds
(α) $eR = ebR$,
(β) $dR + bR = R$,
(γ) $\delta_1 = dd^+ + eb(eb)^+e = 1$;
2. plus any of the following holds
(α) $pR = wR$,
(β) $cfR + \zeta hR = R$,
(γ) $\delta_2 = cf(cf)^+ + ww^+p = 1$.

(B) M has a left inverse iff
1. any of the following holds
(α) $Rf = Rcf$,
(β) $Rc + Rd = R$,
(γ) $\varepsilon_1 = d^+d + f(cf)^+cf = 1$;
2. plus any of the following holds
(α) $Rw = Rh$,
(β) $R = R(p\zeta) + Reb$,
(γ) $\varepsilon_2 = (eb)^+eb + h(p\zeta h)^+p\zeta h = 1$.

We note in passing that

(i) if $\delta_1 = 1$, then $e = ebx$, where $x = (eb)^+e$;
(ii) if $\delta_2 = 1$, then $wy = p$, where $y = w^+$;
(iii) if $\varepsilon_1 = 1$, then $x'cf = f$ with $x' = f(cf)^+$;
(iv) if $\varepsilon_2 = 1$, then $y'w = h$ with $y' = hw^+$.

Conversely, if, for example, $ebx = e$ and $m = (1 - dd^+)b = eb$ then $dd^+ + mx$ is a right inverse of δ_1.

Remarks.

1. In the above procedure, we select first a reflexive inverse d^+ and then compute $(eb)^+, (cf)^+$ and the associate parameters p, q, g, h and ζ. After which we derive $w = p\zeta h$.
2. The conditions $ww^-p = p$ and $hw^-w = h$ are **exactly** the conditions needed for the invariance of $\sigma = hw^-p$ under $(.)^-$, see [5].
3. The existence of M^{-1} is guaranteed by the existence of four unit idempotents (equal to 1). This looks like some kind of determinantal condition.
4. No simplification of the form of M^{-1}, using the necessary condition of our main theorem, seems possible.
5. It comes as no surprise that when d is a *unit*, we obtain the known necessary and sufficient conditions on the Schur complement for M to be a unit. In this case, $e = f = 0$ and $p = q = g = h = 1$. Condition (1) of Theorem 4.1 just says that ζ has a right inverse, while conditions (2) tells us that ζ has a

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left inverse, so that $\zeta = a - cd^{-1}b$ is a unit.

6. When $d = 0$ then $e = f = 1$, $p = 1 - cc^+$, $q = 1 - c^+c$, $g = 1 - bb^+$, $h = 1 - b^+b$, $\phi = b^+$ and $\psi = c^+$. The condition $ge = 0$ becomes $bb^+ = 1$, while $fq = 0$ reduces to $c^+c = 1$. It is now clear that $w = (1-cc^+)a(1-b^+b)$ and $\sigma = (1 - b^+b)[(1 - cc^+)a(1 - b^+b)]^-(1 - cc^+)$. In this case
$$\begin{bmatrix} a & c \\ b & 0 \end{bmatrix}^{-1} = \begin{bmatrix} \sigma & b^+ - \sigma ab^+ \\ c^+ - c^+a\sigma & -c^+ab^+ + (c^+ab^+)\sigma(ab^+) \end{bmatrix}. \tag{4.3}$$
The right invertibility follows from the facts that $b\sigma = 0$ and $(1 - cc^+)a\sigma = 1 - cc^+$, while the left invertibility is a consequence of the conditions $\sigma c = 0$ and $\sigma a(1 - b^+b) = 1 - b^+b$.

5. The group inverse case. For M to have a group inverse, it suffices to show that $U = M + I - MM^-$ is invertible [10], in which case $M^\# = U^{-2}M$. Now, using (3.4), U takes the form of
$$U = \begin{bmatrix} a+\alpha & c+\Gamma \\ b & d+ge \end{bmatrix},$$
where $\alpha = (1 - ww^-)p$ and $\Gamma = -\alpha[cd^+ + \zeta(eb)^+e] = \alpha\Gamma$.

We note that the entries in the bottom row do *not* contain a or c, but only b and d. This suggests that it should be possible to use the second row to try and create an improved element.

In order to apply the conditions of Theorem 4.1, we must create a regular entry, preferably on the diagonal. In general, however, $ge + d$ need not be regular, and we shall have to make some extra assumptions. Rather than using U, we follow [2] in creating a simpler (2,2) element d'. Indeed, consider
$$U' = UT = \begin{bmatrix} a+\alpha & c+\Gamma \\ b & d+ge \end{bmatrix} \begin{bmatrix} 1 & (eb)^+e \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a+\alpha & x \\ b & \rho u \end{bmatrix} = \begin{bmatrix} a' & c' \\ b' & d' \end{bmatrix}, \tag{5.1}$$
where
$$a' = a + \alpha, \quad c' = c - \alpha cd^+ + [a + \alpha(1 - \zeta)](eb)^+e \ \text{ and } \ d' = \rho u, \tag{5.2}$$
together with
$$\alpha = (1 - ww^-)p, \quad \rho = 1 + dd^+b(eb)^+e \ \text{ and } \ u = d + 1 - dd^+. \tag{5.3}$$
We note in passing that ρ is a unit. We shall assume in addition to d, eb and cf being regular, that u **is regular**, which holds exactly when d^2 or d^2d^- is regular, see [6, Lemma 3.1].

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It is clear that $d' = \rho u$, will also be regular. We next introduce the "prime" idempotents

$$e' = 1 - d'd'^{+} = \rho(1 - uu^{+})\rho^{-1} \text{ and } f' = 1 - d'^{+}d' = 1 - u^{+}u. \tag{5.4}$$

and note that the regularity of u is equivalent to ef being regular, using [6, Lemma 3.3].

The expressions for u^{+}, uu^{+} and $u^{+}u$ are given in [6, Corollary 3.4]. Indeed,

(i) $u^{+} = d^{+} + f(ef)^{+}e(1 - d^{+})$,
(ii) $uu^{+} = dd^{+} + ed^{+} + ef(ef)^{+}e(1 - d^{+})$, and
(iii) $u^{+}u = d^{+}d + f(ef)^{+}ef$.

We further shall assume that $e'b$ and $c'f'$ are regular.

We may apply Theorem 4.1 to characterize the invertibility of $U' = \begin{bmatrix} a' & c' \\ b & d' \end{bmatrix}$, when we replace a by a', c by c' and d by d', as given in (5.2) and (5.3).

Also we observe that $e' = \rho(1 - uu^{+})\rho^{-1}$, $(e'b)^{+} = [(1 - uu^{+})\rho^{-1}b]^{+}\rho^{-1}$, $d'd'^{+} = \rho uu^{+}\rho^{-1}$, $d'^{+}d' = u^{+}u$, and $f' = 1 - u^{+}u$.

Using the form for M^{-}_{BM}, we may state $M^{\#} = [T(U')^{-1}]^{2}M$, where

$$T = \begin{bmatrix} 1 & \phi \\ 0 & 1 \end{bmatrix} \text{ and } U'^{-1} = \begin{bmatrix} 1 & 0 \\ -s' & 1 \end{bmatrix} \begin{bmatrix} \sigma' & \phi' \\ \chi' & d'^{+} + \chi'\zeta'\phi' \end{bmatrix} \begin{bmatrix} 1 & -t' \\ 0 & 1 \end{bmatrix}. \tag{5.5}$$

In this, we have:

(i) $\phi = (eb)^{+}e$, $\phi' = (e'b)^{+}e'$, and $\chi' = f'(c'f')^{+}$;
(ii) $\zeta' = a' - c'd'^{+}b = a + \alpha - c'u^{+}b$, where c' is given as in equation (5.2);
(iii) $p' = 1 - (c'f')(c'f')^{+}$ and $h' = 1 - (e'b)^{+}e'b$, in which $c'f' = c'(1 - u^{+}u)$ and $e'b = \rho(1 - uu^{+})\rho^{-1}$;
(iv) $t' = c'd' + \zeta'\phi'$ and $s' = d'^{+}b + \chi'\zeta'$;
(v) $w' = p'\zeta'h'$ and $\sigma' = h'(w')^{-}p'$.

We can now apply Corollary 4.5 to U', to give

THEOREM 5.1. *With the notation of (3.1), (5.2) and (5.4), and the above items (i) – (iv), if we assume that d, eb, cf, u, $e'b$, $c'f$ and u' are all regular, then $M^{\#}$ exists iff*

(i) $e'bR = e'R$, *i.e.,* $d'R + bR = R$, *or* $uR + \rho^{-1}bR = R$,
(ii) $Rf' = Rc'f'$, *i.e.,* $Ru + Rc' = R$,
(iii) $p'\zeta'h'R = p'R$, *i.e.,* $c'f'R + (\zeta'h')R = R$, *and*

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(iv) $Rh' = Rp'\zeta'h'$, *i.e.,* $R(1-uu^+) + Rp'\zeta'\rho^{-1} = R$.
In which case,

$$M^\# = \left(\begin{bmatrix} 1 & \phi \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -s' & 1 \end{bmatrix} \begin{bmatrix} \sigma' & \phi' \\ \chi' & d'^+ + \chi'\zeta'\phi' \end{bmatrix} \begin{bmatrix} 1 & -t' \\ 0 & 1 \end{bmatrix} \right)^2 \begin{bmatrix} a & c \\ b & d \end{bmatrix}. \quad (5.6)$$

Remarks. Note that $ef' = e(1-u^+u) = 0$, and thus, $c'f' = (c - \alpha cd^+)f'$. Yet, it is not clear how these equalities may be used to perform any simplification.

The four unit idempotents can be cleaned up somewhat. Indeed, we have:

1. $1 = d'd'^+ + e'b'(e'b')^+e$ reduces to
$$1 = uu^+ + (1-uu^+)\rho^{-1}b[(1-uu^+)\rho^{-1}b]^+(1-uu^+).$$
In other words, $(1-uu^+)\rho^{-1}bR = (1-uu^+)R$.
2. Recalling that $p' = 1 - c'(1-u^+u)[c'(1-u^+u)]^+$, $(e'b)^+ = [(1-uu^+)\rho^{-1}b]^+\rho^{-1}$, $h' = 1 - (e'b)^+(e'b) = 1 - [(1-uu^+)\rho^{-1}b]^+(1-uu^+)\rho^{-1}b$, and $\zeta' = a + \alpha - c'u^+\rho^{-1}b$, we see that condition (iii) of Theorem 5.1, reduces to
$$1 = c'(1-u^+u)[c'(1-u^+u)]^+ + p'\zeta'h'(p'\zeta'h')^+p'.$$
3. Part (ii) of Theorem 5.1 reduces to the condition that
$$1 = u^+u + (1-u^+u)[c'(1-u^+u)]^+c'(1-u^+u).$$
4. Lastly, we also see that
$$\begin{aligned} 1 &= (e'b)^+e'b + h'(p'\zeta'h')^+p'\zeta'h' \\ &= [(1-uu^+)\rho^{-1}b]^+(1-uu^+)\rho^{-1}b + h'(p'\zeta'h')^+p'\zeta'h'. \end{aligned}$$

5.1. Special cases. As special cases we consider the cases where $d = 0$, d is a unit and where $d^\#$ exists. The latter is equivalent to u being a unit.

Case: $d = 0$**.** If $d = 0$, then $d' = 1$, $u = \rho = 1$ and $e' = 0 = f'$. We then get $\phi' = 0, \phi = b^+$ and $\chi' = 0$. Moreover, $\zeta = a$, $p' = 1 = h' = 1$, $w = (1-cc^+)a(1-b^+b)$ and $\alpha = (1-ww^-)(1-cc^+)$.

The necessary and sufficient conditions of Theorem 5.1 reduce to ζ' being a unit. Now $c' = c + [a + \alpha(1-a)]b^+$, and hence, $\zeta' = a + \alpha - c'b$ reduces to

$$\zeta' = a(1-b^+b) - cb + \alpha[1 - (1-a)b^+b], \quad (5.7)$$

which was given in [8].

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Case: d is a unit. If d is a unit then $e = 0 = e' = f'$, and $u = d$ is a unit as well. Also $p = 1 = q$ and $h = 1 = g$. Hence, $w = p\zeta h = \zeta = a - cd^{-1}b$. Next we have $\alpha = 1 - \zeta\zeta^-$ and $c' = c - (1 - \zeta'\zeta'^-)cd^{-1}$. Now $\rho = 1$ and

$$\zeta' = a + \alpha - c'(d')^{-1}b = a + \alpha - [c - (1 - \zeta\zeta^-)cd^{-1}]d^{-1}b. \tag{5.8}$$

The conditions of Theorem 5.1 reduce to ζ' being a unit, where

$$\zeta' = (a - cd^{-1}b) + \alpha + (1 - \zeta\zeta^-)cd^{-2}b.$$

Case: $\mathbf{d}^\#$ exists. In this case, u is a unit and $d'd'^+ = d'^+d' = p' = h' = 1$ and $e' = f' = 0$. The conditions (i)–(iv) of Theorem 5.1 reduce to:

1. $R = R$,
2. $R = R$,
3. $\zeta'\zeta'^+$ has a right inverse,
4. $\zeta'^+\zeta'$ has a left inverse.

So, as a corollary, if $d^\#$ exists, as in [2], then M has a group inverse iff ζ' is a unit, with

$$\begin{aligned}
\zeta' &= a + \alpha - c'u^{-1}\rho^{-1}b,\\
c' &= c - \alpha cd^+ + [a + \alpha(1 - \zeta)](eb)^+e,\\
\rho &= 1 + dd^+b(eb)^+e,\\
\alpha &= (1 - ww^-)p.
\end{aligned}$$

6. Remarks and questions. We close with several remarks and open questions.

1. It is not clear why we get the related matrices $\begin{bmatrix} \zeta & cf \\ eb & 0 \end{bmatrix}$ and $\begin{bmatrix} \zeta & cf \\ eb & d \end{bmatrix}$ using two different methods.
2. Can one power M to give conditions for M^d to exist?
3. We assumed that ef is regular which is weaker than u being a unit, or equivalently, $d^\#$ existing.
4. Can we use the fact that $d - dd^dd$ is nilpotent?
5. When does $(p + q)^{-1}$ exist? When does $(p + q)^\#$ exist? When does $(p + q)^d$ exist? They are all related.
6. It seems that there is no nice simplification in general for the form of either M^{-1} or $M^\#$. However, when $d^\#$ exists there is a simplification as given in [2].

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