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INTERVAL SYSTEM OF MATRIX EQUATIONS
WITH TWO UNKNOWN MATRICES[*]

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Abstract. In this paper, we consider an interval system of matrix equations contained two equations with two unknown matrices as

$$\begin{cases} \mathbf{A_{11}}X + Y\mathbf{A_{12}} = \mathbf{C_1}, \\ \mathbf{A_{21}}X + Y\mathbf{A_{22}} = \mathbf{C_2}. \end{cases}$$

We define a solution set for this system and then we study some conditions that the solution set is bounded. Finally, we present a direct method and an iterative method for solving this interval system.

Key words. Interval matrix, Interval linear systems, System of matrix equations, Solution set.

AMS subject classifications. 65F30.

1. Introduction. Matrix equations and systems of matrix equations have many applications in sciences and engineering, such as electromagnetic scattering, structural mechanics and computation of the frequency response matrix in control theory. A sample of these systems of matrix equations is as the following:

$$\begin{cases} A_{11}X + YA_{12} = C_1, \\ A_{21}X + YA_{22} = C_2, \end{cases} \tag{1.1}$$

where A_{i1}, A_{i2} and C_i, for $i = 1, 2$, are the known real matrices of dimensions $m \times m$, $n \times n$ and $m \times n$, respectively, while the unknown matrices X and Y are $m \times n$ real matrices. The general state of this problem can be seen in [1].

The elements of A_{ij}, for $i, j = 1, 2$, occurring in practice are usually obtained from experiments, hence they may appear with uncertainties. We represent the uncertain elements in interval forms. Therefore, with the existence of uncertainties in data, the system of matrix equations (1.1) is transformed to the following interval system of

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matrix equations

$$\begin{cases} \mathbf{A_{11}}X + Y\mathbf{A_{12}} = \mathbf{C_1}, \\ \mathbf{A_{21}}X + Y\mathbf{A_{22}} = \mathbf{C_2}, \end{cases} \tag{1.2}$$

where $\mathbf{A_{ij}}$ and $\mathbf{C_i}$, for $i, j = 1, 2$, are interval matrices. Note that bold-face letters are used to show intervals. Some samples of interval matrix equations such as $\mathbf{A}X = \mathbf{B}$ and the interval Sylvester equation $\mathbf{A}X + X\mathbf{B} = \mathbf{C}$, have been considered previously; see [3, 16]

In this paper, we use notations $\mathbb{R}$ and $\mathbb{R}^{m \times n}$ as the field of real numbers and the vector space of $m \times n$ real matrices, respectively. We denote the set of all $m \times n$ interval matrices by $\mathbb{IR}^{m \times n}$.

For the interval matrix $\mathbf{A} = [\underline{A}, \overline{A}]$, the center matrix denoted by $\check{\mathbf{A}}$ and the radius matrix denoted by $\text{rad}(\mathbf{A})$ are respectively defined as

$$\check{\mathbf{A}} = \frac{1}{2}(\underline{A} + \overline{A}) \quad \text{and} \quad \text{rad}(\mathbf{A}) = \frac{1}{2}(\overline{A} - \underline{A}).$$

It is clear that $\mathbf{A} = [\check{\mathbf{A}} - \text{rad}(\mathbf{A}), \check{\mathbf{A}} + \text{rad}(\mathbf{A})]$.

We assume that the reader is familiar with a basic interval arithmetic and interval operators on the interval matrices. For more details, we refer to [7, 8]. An $n \times n$ interval matrix $\mathbf{A} = [\underline{A}, \overline{A}]$ is said to be regular if each $A \in \mathbf{A}$ is nonsingular. For two interval matrices $\mathbf{A} \in \mathbb{IR}^{m \times n}$ and $\mathbf{B} \in \mathbb{IR}^{k \times t}$, the Kronecker product denoted by $\otimes$ is defined by the $mk \times nt$ block interval matrix

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} \mathbf{a_{11}B} & \cdots & \mathbf{a_{1n}B} \\ \vdots & \ddots & \vdots \\ \mathbf{a_{m1}B} & \cdots & \mathbf{a_{mn}B} \end{bmatrix},$$

and $\text{vec}(\mathbf{A})$ is defined as an mn-interval vector and obtained by stacking the columns of $\mathbf{A}$, i.e.,

$$\text{vec}(\mathbf{A}) = (\mathbf{A}_{.1}, \mathbf{A}_{.2}, \ldots, \mathbf{A}_{.n})^T,$$

where $\mathbf{A}_{.j}$ is the jth column of $\mathbf{A}$. Note that these definitions are similar to those of real matrices.

THEOREM 1.1. *If $A, C \in \mathbb{R}^{m \times m}$, $B, D \in \mathbb{R}^{n \times n}$ and $X \in \mathbb{R}^{m \times n}$, then we have*

1. $(A + C) \otimes B = (A \otimes B) + (C \otimes B)$,
2. $B \otimes (A + C) = (B \otimes A) + (B \otimes C)$,
3. $(A \otimes B)(C \otimes D) = AC \otimes BD$,
4. $\text{vec}(AX) = (I_n \otimes A)\text{vec}(X)$,

5. $\mathrm{vec}(XB) = (B^T \otimes I_m)\mathrm{vec}(X),$
6. $\lambda(A \otimes B) = \lambda(A)\lambda(B),$
7. $\lambda((I_n \otimes A) + (B \otimes I_m)) = \lambda(A) + \lambda(B),$

where $\lambda(A)$ is an eigenvalue of A.

For discussion of convergence in the context of interval analysis, the distance between two intervals $\mathbf{x} = [\underline{x}, \overline{x}]$ and $\mathbf{y} = [\underline{y}, \overline{y}]$ is denoted by $\mathrm{d}(\mathbf{x}, \mathbf{y})$ and is defined as the following:

$$\mathrm{d}(\mathbf{x}, \mathbf{y}) = \max\{|\underline{x} - \underline{y}|, |\overline{x} - \overline{y}|\}.$$

For more information, see in [7]. Let $\{\mathbf{x_k}\}$ be a sequence of intervals. We say that $\{\mathbf{x_k}\}$ is convergent if there exists an interval $\mathbf{x}^*$ such that for every $\epsilon > 0$, there is a natural number $N = N(\epsilon)$ such that $\mathrm{d}(\mathbf{x_k}, \mathbf{x}^*) < \epsilon$ whenever $k > N$. A necessary and sufficient condition for convergence of sequence $\{\mathbf{x_k}\}$ is stated in the next theorem.

THEOREM 1.2. *The interval sequence* $\{\mathbf{x_k}\}$ *is convergent to* $\mathbf{x}^* = [\underline{x}^*, \overline{x}^*]$ *if and only if* $\underline{x}_k \to \underline{x}^*$ *and* $\overline{x}_k \to \overline{x}^*$ *in the sense of real sequences.*

2. Main results. Consider the interval system of matrix equations (1.2). We define the solution set for the system (1.2) by

$$\Sigma(X, Y) = \left\{(X, Y) : \begin{array}{l} X, Y \in \mathbb{R}^{m \times n}, A_{i1}X + YA_{i2} = C_i \\ \text{for some } A_{i1} \in \mathbf{A_{i1}}, A_{i2} \in \mathbf{A_{i2}}, C_i \in \mathbf{C_i} \end{array} ; \ i = 1, 2\right\}. \quad (2.1)$$

Similar to solving of interval linear systems [8, 13], the solution set of an interval system is generally of a complicated structure. But if $\Sigma(X, Y)$ is bounded, we look for an enclosure of this set, i.e., for a pair of interval matrices $(\mathbf{X}, \mathbf{Y})$ satisfying

$$\Sigma(X, Y) \subseteq (\mathbf{X}, \mathbf{Y}).$$

In the following two subsections, we will present a direct method to obtain an enclosure of $\Sigma(X, Y)$ and conditions under which $\Sigma(X, Y)$ is bounded. Also we will present an iterative algorithm to solve this important problem.

The description of a superset of $\Sigma(X, Y)$ given in the next theorem is similar to that which appeared in the pioneering works of Oettli and Prager [9] for interval linear systems.

THEOREM 2.1. *The solution set* $\Sigma(X, Y)$ *defined by (2.1) satisfies*

$$\Sigma(X, Y) \subseteq \left\{(X, Y) : \begin{array}{l} |\check{\mathbf{A}}_{\mathbf{i1}}X + Y\check{\mathbf{A}}_{\mathbf{i2}} - \check{\mathbf{C}}_{\mathbf{i}}| \leq \\ \mathrm{rad}(\mathbf{A_{i1}})|X| + |Y|\mathrm{rad}(\mathbf{A_{i2}}) + \mathrm{rad}(\mathbf{C_i}) \end{array} ; \ i = 1, 2\right\}. \quad (2.2)$$

Proof. Let $(X, Y) \in \Sigma(X, Y)$, then for some $A_{ij} \in \mathbf{A}_{ij}$ and $C_i \in \mathbf{C}_i$ for $i, j = 1, 2$, we have

$$A_{i1}X + YA_{i2} - C_i = 0.$$

Therefore, we have:

$$|\check{\mathbf{A}}_{\mathbf{i1}}X + Y\check{\mathbf{A}}_{\mathbf{i2}} - \check{\mathbf{C}}_{\mathbf{i}}| = |\check{\mathbf{A}}_{\mathbf{i1}} + Y\check{\mathbf{A}}_{\mathbf{i2}} - \check{\mathbf{C}}_{\mathbf{i}} - A_{i1}X - YA_{i2} + C_i|$$
$$\leq |\check{\mathbf{A}}_{\mathbf{i1}} - A_{i1}||X| + |Y||\check{\mathbf{A}}_{\mathbf{i2}} - A_{i2}| + |\check{\mathbf{C}}_{\mathbf{i}} - C_i|$$
$$\leq \operatorname{rad}(\mathbf{A}_{\mathbf{i1}})|X| + |Y|\operatorname{rad}(\mathbf{A}_{\mathbf{i2}}) + \operatorname{rad}(\mathbf{C}_{\mathbf{i}}).$$

So the proof is completed. ☐

2.1. Direct method. For obtaining an enclosure $(\mathbf{X}, \mathbf{Y})$ of $\Sigma(X, Y)$, suppose that $(X, Y) \in \Sigma(X, Y)$. Then by using Theorem 2.1, we have

$$\begin{cases} (\check{\mathbf{A}}_{\mathbf{11}})X - \operatorname{rad}(\mathbf{A}_{\mathbf{11}})|X| + Y(\check{\mathbf{A}}_{\mathbf{12}}) - |Y|\operatorname{rad}(\mathbf{A}_{\mathbf{12}}) \leq \overline{\mathbf{C}}_{\mathbf{1}}, \\ (\check{\mathbf{A}}_{\mathbf{11}})X + \operatorname{rad}(\mathbf{A}_{\mathbf{11}})|X| + Y(\check{\mathbf{A}}_{\mathbf{12}}) + |Y|\operatorname{rad}(\mathbf{A}_{\mathbf{12}}) \geq \underline{\mathbf{C}}_{\mathbf{1}}, \\ (\check{\mathbf{A}}_{\mathbf{21}})X - \operatorname{rad}(\mathbf{A}_{\mathbf{21}})|X| + Y(\check{\mathbf{A}}_{\mathbf{22}}) - |Y|\operatorname{rad}(\mathbf{A}_{\mathbf{22}}) \leq \overline{\mathbf{C}}_{\mathbf{2}}, \\ (\check{\mathbf{A}}_{\mathbf{21}})X + \operatorname{rad}(\mathbf{A}_{\mathbf{21}})|X| + Y(\check{\mathbf{A}}_{\mathbf{22}}) + |Y|\operatorname{rad}(\mathbf{A}_{\mathbf{22}}) \geq \underline{\mathbf{C}}_{\mathbf{2}}. \end{cases}$$

For given matrices X and Y, we write

$$|X| = T \circ X \quad \text{and} \quad |Y| = S \circ Y,$$

where $\circ$ denotes the so-called Hadamard product and the matrices T and S are sign matrices of X and Y, respectively. In order to find an element $\underline{x}_{ij}$ for fixed i and j of the interval matrix $\mathbf{X} = (\mathbf{x}_{\mathbf{ij}})$, we have to solve 2^{2mn} linear programming problems as the following: $\min x_{ij}$

subject to:

$$\begin{cases} (\check{\mathbf{A}}_{\mathbf{11}})X - \operatorname{rad}(\mathbf{A}_{\mathbf{11}})(T \circ X) + Y(\check{\mathbf{A}}_{\mathbf{12}}) - (S \circ Y)\operatorname{rad}(\mathbf{A}_{\mathbf{12}}) \leq \overline{\mathbf{C}}_{\mathbf{1}}, \\ (\check{\mathbf{A}}_{\mathbf{11}})X + \operatorname{rad}(\mathbf{A}_{\mathbf{11}})(T \circ X) + Y(\check{\mathbf{A}}_{\mathbf{12}}) + (S \circ Y)\operatorname{rad}(\mathbf{A}_{\mathbf{12}}) \geq \underline{\mathbf{C}}_{\mathbf{1}}, \\ (\check{\mathbf{A}}_{\mathbf{21}})X - \operatorname{rad}(\mathbf{A}_{\mathbf{21}})(T \circ X) + Y(\check{\mathbf{A}}_{\mathbf{22}}) - (S \circ Y)\operatorname{rad}(\mathbf{A}_{\mathbf{22}}) \leq \overline{\mathbf{C}}_{\mathbf{2}}, \\ (\check{\mathbf{A}}_{\mathbf{21}})X + \operatorname{rad}(\mathbf{A}_{\mathbf{21}})(T \circ X) + Y(\check{\mathbf{A}}_{\mathbf{22}}) + (S \circ Y)\operatorname{rad}(\mathbf{A}_{\mathbf{22}}) \geq \underline{\mathbf{C}}_{\mathbf{2}}, \end{cases}$$

and for all possible sign matrices T and matrices S. Hence, we need $4mn \times 2^{2mn}$ linear programming problems for finding the interval matrices X and Y, which makes the problem very troublesome even with small values of m and n.

Accordingly, we try to find an enclosure $(\mathbf{X}, \mathbf{Y})$ of $\Sigma(X, Y)$ by an easier technique. It can be shown that for any $A_{ij} \in \mathbf{A}_{\mathbf{ij}}$ and $C_i \in \mathbf{C}_{\mathbf{i}}$, the system of matrix equations

$$\begin{cases} A_{11}X + YA_{12} = C_1, \\ A_{21}X + YA_{22} = C_2 \end{cases}$$

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can be transformed to the following form

$$Gz = d,$$

where

$$G = \begin{bmatrix} I_n \otimes A_{11} & A_{12}^T \otimes I_m \\ I_n \otimes A_{21} & A_{22}^T \otimes I_m \end{bmatrix}, \quad d = \begin{bmatrix} \text{vec}(C_1) \\ \text{vec}(C_2) \end{bmatrix}, \quad z = \begin{bmatrix} \text{vec}(X) \\ \text{vec}(Y) \end{bmatrix}.$$

Now, we consider interval linear system of equations

$$\mathbf{G}z = \mathbf{d}, \tag{2.3}$$

where $\mathbf{G} \in \mathbb{IR}^{2mn \times 2mn}$ and $\mathbf{d} \in \mathbb{IR}^{2mn}$ are as the following:

$$\mathbf{G} = \begin{bmatrix} \mathbf{I_n} \otimes \mathbf{A_{11}} & \mathbf{A_{12}}^T \otimes \mathbf{I_m} \\ \mathbf{I_n} \otimes \mathbf{A_{21}} & \mathbf{A_{22}}^T \otimes \mathbf{I_m} \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} \text{vec}(\mathbf{C_1}) \\ \text{vec}(\mathbf{C_2}) \end{bmatrix}.$$

We assume that Γ is the solution set of (2.3). We define

$$\Theta = \left\{ \begin{bmatrix} \text{vec}(X) \\ \text{vec}(Y) \end{bmatrix} : (X, Y) \in \Sigma(X, Y) \right\}. \tag{2.4}$$

It is clear that $\Theta \subseteq \Gamma$.

Therefore, by solving interval linear system (2.3) and finding interval vector $\mathbf{z}$ as an enclosure of its solution set, we can specify the columns of the interval matrices $\mathbf{X}$ and $\mathbf{Y}$ which $(\mathbf{X}, \mathbf{Y})$ is an enclosure of $\Sigma(X, Y)$. To solve the interval linear system (2.3), see [2, 4, 7, 8, 11, 13, 14, 15].

As was mentioned before, the enclosure of $\Sigma(X, Y)$ is achievable if $\Sigma(X, Y)$ is a bounded set. The following theorem represent a condition in order that the solution set of the interval system (1.2) to be bounded.

THEOREM 2.2. *For all interval $m \times n$ matrices $\mathbf{C_1}$ and $\mathbf{C_2}$ the solution set defined by (2.1) is bounded if the system of inequalities*

$$|\check{\mathbf{A}}_{\mathbf{11}}X + Y\check{\mathbf{A}}_{\mathbf{12}}^T| \leq \text{rad}(\mathbf{A_{11}})|X| + |Y|\text{rad}(\mathbf{A_{12}}^T), \tag{2.5}$$

$$|\check{\mathbf{A}}_{\mathbf{21}}X + Y\check{\mathbf{A}}_{\mathbf{22}}^T| \leq \text{rad}(\mathbf{A_{21}})|X| + |Y|\text{rad}(\mathbf{A_{22}}^T), \tag{2.6}$$

have only the trivial solution $(X, Y) = (0, 0)$.

Proof. It is clear that $\Sigma(X, Y)$ is bounded if and only if the set Θ defined by (2.4) is bounded. Since we have $\Theta \subseteq \Gamma$, So it is sufficient to prove that Γ is bounded for

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each $2mn$-vector $\mathbf{d}$. The inequalities (2.5) and (2.6) are equivalent to the following inequalities

$$|\text{vec}(\check{\mathbf{A}}_{\mathbf{11}}X) + \text{vec}(Y\check{\mathbf{A}}_{\mathbf{12}}^T)| \le \text{vec}(\text{rad}(\mathbf{A}_{\mathbf{11}})|X|) + \text{vec}(|Y|\text{rad}(\mathbf{A}_{\mathbf{12}}^T)),$$

$$|\text{vec}(\check{\mathbf{A}}_{\mathbf{21}}X) + \text{vec}(Y\check{\mathbf{A}}_{\mathbf{22}}^T)| \le \text{vec}(\text{rad}(\mathbf{A}_{\mathbf{21}})|X|) + \text{vec}(|Y|\text{rad}(\mathbf{A}_{\mathbf{22}}^T)).$$

Based on Theorem 1.1, we can rewrite the above inequalities as

$$|(I_n \otimes \check{\mathbf{A}}_{\mathbf{11}})\text{vec}(X) + (\check{\mathbf{A}}_{\mathbf{12}}^T \otimes I_m)\text{vec}(Y)| \le (I_n \otimes \text{rad}(\mathbf{A}_{\mathbf{11}}))|\text{vec}(X)|$$
$$+ (\text{rad}(\mathbf{A}_{\mathbf{12}}^T) \otimes I_m)(|\text{vec}(Y)|), \tag{2.7}$$

$$|(I_n \otimes \check{\mathbf{A}}_{\mathbf{21}})\text{vec}(X) + (\check{\mathbf{A}}_{\mathbf{22}}^T \otimes I_m)\text{vec}(Y)| \le (I_n \otimes \text{rad}(\mathbf{A}_{\mathbf{21}}))|\text{vec}(X)|$$
$$+ (\text{rad}(\mathbf{A}_{\mathbf{22}}^T) \otimes I_m)(|\text{vec}(Y)|). \tag{2.8}$$

So, if the inequalities (2.5) and (2.6) have trivial solution then the inequalities (2.7) and (2.8) also have the trivial solution. As for the interval system (2.3), and the inequalities (2.7) and (2.8), the inequality

$$|\check{\mathbf{G}}z| \le \text{rad}(\mathbf{G})|z|$$

has only the trivial solution $z = 0$. Thus, from the result of [12], the solution set Γ is bounded for each $2mn$-vector $\mathbf{d}$. $\square$

In the next theorem, we give a necessary and sufficient condition for boundedness of $\Sigma(X, Y)$.

THEOREM 2.3. *Suppose that $\mathbf{A}_{\mathbf{11}}$ in the interval system of matrix equations is regular. Then for all interval matrices $\mathbf{C}_{\mathbf{1}}$ and $\mathbf{C}_{\mathbf{2}}$, the solution set of (1.2) is bounded if and only if $(A_{22}^T \otimes A_{11}) - (A_{12}^T \otimes A_{21})$ is nonsingular for each $A_{ij} \in \mathbf{A}_{\mathbf{ij}}$, $i,j = 1, 2$.*

Proof. The solution set of interval system of matrix equations (1.2) is bounded if and only if for all fixed $A_{ij} \in \mathbf{A}_{\mathbf{ij}}$, $i, j = 1, 2$, the matrix G defined by

$$G = \left[\begin{array}{cc} I_n \otimes A_{11} & A_{12}^T \otimes I_m \\ I_n \otimes A_{21} & A_{22}^T \otimes I_m \end{array} \right],$$

is nonsingular, or equivalently $\det(G) \ne 0$. Since A_{11} is a nonsingular matrix and A_{22} is a square matrix, from [6, p. 46], it follows that

$$\det(G) = \det(I_n \otimes A_{11}) \det\left((A_{22}^T \otimes I_m) - (I_n \otimes A_{21})(I_n \otimes A_{11}^{-1})(A_{12}^T \otimes I_m)\right)$$
$$= \det(I_n \otimes A_{11}) \det\left((A_{22}^T \otimes I_m) - (A_{12}^T \otimes A_{21}A_{11}^{-1})\right).$$

Hence, $\det(G) \ne 0$ if and only if

$$\det\left((A_{22}^T \otimes I_m) - (A_{12}^T \otimes A_{21}A_{11}^{-1})\right) \ne 0,$$

which is equivalent to

$$\det\left((A_{22}^T \otimes A_{11}) - (A_{12}^T \otimes A_{21})\right) \neq 0. \tag{2.9}$$

Therefore, the solution set of the interval system of matrix equations (1.2) is bounded if and only if for all fixed $A_{ij} \in \mathbf{A_{ij}}$, $i,j = 1,2$,

$$\det\left((A_{22}^T \otimes A_{11}) - (A_{12}^T \otimes A_{21})\right) \neq 0. \quad \square$$

The following corollary is an immediate consequence of the theorem.

COROLLARY 2.4. *Suppose that $\mathbf{A_{11}}$ in the interval system of matrix equations is regular. Then for all interval matrices $\mathbf{C_1}$ and $\mathbf{C_2}$, the solution set of (1.2) is bounded if $\left((\mathbf{A_{22}^T} \otimes \mathbf{A_{11}}) - (\mathbf{A_{12}^T} \otimes \mathbf{A_{21}})\right)$ is regular.*

For checking the regularity of interval matrices, see [5, 10]. If we choose $\mathbf{A_{11}} = \mathbf{A_{12}} = \mathbf{I}$, then we can state the following result.

COROLLARY 2.5. *In the interval system of matrix equations (1.2), assume that $\mathbf{A_{11}} = \mathbf{A_{12}} = \mathbf{I}$. Then the corresponding solution set is bounded if and only if*

$$\sigma(\mathbf{A_{21}}) \cap \sigma(\mathbf{A_{21}}) = \emptyset,$$

where

$$\sigma(\mathbf{A}) = \{\lambda \in \mathbb{C} : \exists x \neq 0 \ , \ Ax = \lambda x \ for \ some \ A \in \mathbf{A}\}.$$

Proof. Consider $\mathbf{A_{11}} = \mathbf{I_m}$ and $\mathbf{A_{12}} = \mathbf{I_n}$. Similar to the proof of previous theorem, the solution set is bounded if and only if

$$\det\left((A_{22}^T \otimes I_m) - (I_n \otimes A_{21})\right) \neq 0, \tag{2.10}$$

for each $A_{21} \in \mathbf{A_{21}}$ and $A_{22} \in \mathbf{A_{22}}$.

The relation (2.10) is equivalent to that $\lambda_i - \mu_j \neq 0$ for every eigenvalue λ_i, for $i = 1, \ldots, n$, of A_{22} and every eigenvalue μ_j, for $j = 1, \ldots, m$ of A_{21}. This implies that the solution set is bounded if and only if

$$\sigma(\mathbf{A_{21}}) \cap \sigma(\mathbf{A_{21}}) = \emptyset. \quad \square$$

EXAMPLE 2.6. Consider the interval system of matrix equations

$$\begin{cases} \mathbf{A}_{11}X + Y\mathbf{A_{12}} = \mathbf{C_1}, \\ \mathbf{A}_{21}X + Y\mathbf{A_{22}} = \mathbf{C_2}, \end{cases}$$

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in which

$$\mathbf{A_{11}} = \begin{bmatrix} [1,2] & [2,2.5] \\ [-2,-1] & [5,6] \end{bmatrix}, \quad \mathbf{A_{12}} = \begin{bmatrix} [0.1,0.3] & [2,2.5] \\ [-0.5,0.2] & [0.1,0.3] \end{bmatrix}, \quad \mathbf{C_1} = \begin{bmatrix} [3,4] & [-1,0] \\ [1,1] & [6,8] \end{bmatrix},$$

$$\mathbf{A_{21}} = \begin{bmatrix} [0.1,0.3] & [0,0.5] \\ [-0.4,-0.3] & [0.2,0.2] \end{bmatrix}, \quad \mathbf{A_{22}} = \begin{bmatrix} [3,4] & [-4,-3] \\ [1,2] & [6,7] \end{bmatrix}, \quad \mathbf{C_2} = \begin{bmatrix} [6,7] & [1,3] \\ [8,9] & [6,8] \end{bmatrix}.$$

Here, $\left((\mathbf{A_{22}^T} \otimes \mathbf{A_{11}}) - (\mathbf{A_{12}^T} \otimes \mathbf{A_{21}})\right)$ is regular. The proposed direct method shows that the enclosure of the solution set is a pair of interval matrices of $(\mathbf{X}_{Di}, \mathbf{Y}_{Di})$ where

$$\mathbf{X}_{Di} = \begin{bmatrix} [0.1211,4.1360] & [-7.9006,0.9944] \\ [0.0360,1.4322] & [-2.3282,2.1786] \end{bmatrix}, \quad \mathbf{Y}_{Di} = \begin{bmatrix} [0.4094,2.3682] & [0.2358,2.2193] \\ [0.8141,3.1215] & [0.9311,3.2972] \end{bmatrix}.$$

2.2. Iterative method. In this subsection, we present an iterative method for solving the interval system of matrix equations (1.2). Moreover, we discuss the condition of convergence for this method.

Let us rewrite (1.2) as the following:

$$\begin{cases} \mathbf{A_{11}}X = \mathbf{C_1} - Y\mathbf{A_{12}}, \\ Y\mathbf{A_{22}} = \mathbf{C_2} - \mathbf{A_{21}}X. \end{cases}$$

Assume that an initial guess $\mathbf{Y}_0$ is given. Now we define the following iteration equations:

$$\begin{cases} \mathbf{A_{11}}X_{k+1} = \mathbf{C_1} - Y_k\mathbf{A_{12}}, \\ Y_{k+1}\mathbf{A_{22}} = \mathbf{C_2} - \mathbf{A_{21}}X_{k+1}, \end{cases} \quad k \geq 0. \tag{2.11}$$

To find the interval matrices $\mathbf{X}_{k+1}$ and $\mathbf{Y}_{k+1}$ in (2.11), we need to solve an interval matrix equation such as $\mathbf{A}X = \mathbf{B}$. To this end, we consider an interval linear system of the form

$$\mathbf{A}X_{.j} = \mathbf{B}_{.j},$$

where $X_{.j}$ and $\mathbf{B}_{.j}$ are j^{th} columns of X and $\mathbf{B}$, respectively.

Having considered them, we introduce the following algorithm.

ALGORITHM 2.7.

Step 1. *Choose* $\mathbf{Y}_0$ *and set* $k = 0$.
Step 2. *Solve the interval matrix equations*

$$\mathbf{A_{11}}X_{k+1} = \mathbf{C_1} - \mathbf{Y_k}\mathbf{A_{12}}, \quad \mathbf{A_{22}^T}Y_{k+1}^T = \mathbf{C_2}^T - \mathbf{X_{k+1}^T}\mathbf{A_{21}^T}$$

and find the interval matrices $\mathbf{X}_{k+1}$ *and* $\mathbf{Y}_{k+1}$.

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Step 3. *If*

$$\max\{||\underline{X}_{k+1} - \underline{X}_k||, ||\overline{X}_{k+1} - \overline{X}_k||, ||\underline{Y}_{k+1} - \underline{Y}_k||, ||\overline{Y}_{k+1} - \overline{Y}_k||\} < \epsilon,$$

stop.
Otherwise, set

$$\mathbf{X}_k = \mathbf{X}_{k+1} \ , \ \mathbf{Y}_k = \mathbf{Y}_{k+1} \qquad\qquad if \ \ k = 0, \qquad (2.12)$$

$$\mathbf{X}_k = \mathbf{X}_{k+1} \cap \mathbf{X}_k \ , \ \mathbf{Y}_k = \mathbf{Y}_{k+1} \cap \mathbf{Y}_k \quad if \ \ k \geq 1. \qquad (2.13)$$

Set $k = k + 1$, and go to the Step 2.

As discussed in [7], The relation (2.13) implies that the algorithm converges in a finite number of iterations. By considering a particular condition, the next theorem expresses that the above algorithm converges in the absence of the relation (2.13).

THEOREM 2.8. *In the interval system (1.2), suppose* $\mathbf{A_{11}}$ *and* $\mathbf{A_{22}}$ *are regular matrices. If* $||A_{21}A_{11}^{-1}||||A_{12}A_{22}^{-1}|| < 1$ *for each* $A_{ij} \in \mathbf{A_{ij}}$, $i,j = 1,2$, *then the algorithm described above converges.*

Proof. From (2.11), we write

$$\begin{cases} A_{11}X_{k+1} = C_1 - Y_k A_{12}, \\ Y_{k+1}A_{22} = C_2 - A_{21}X_{k+1} \end{cases} \qquad (2.14)$$

for each $A_{ij} \in \mathbf{A_{ij}}$ and $C_i \in \mathbf{C_i}$, $i,j = 1,2$. Using the relations (2.14) together with the regularity of A_{11} and A_{22} yields

$$Y_{k+1} = C_2 A_{22}^{-1} - A_{21}A_{11}^{-1}C_1 A_{22}^{-1} + A_{21}A_{11}^{-1}Y_k A_{12}A_{22}^{-1}. \qquad (2.15)$$

Suppose (X^*, Y^*) is the exact solution of (2.14). So, Y^* satisfies

$$Y^* = C_2 A_{22}^{-1} - A_{21}A_{11}^{-1}C_1 A_{22}^{-1} + A_{21}A_{11}^{-1}Y^* A_{12}A_{22}^{-1}. \qquad (2.16)$$

If we apply (2.15) and (2.16), then we conclude that

$$\begin{aligned} Y_{k+1} - Y^* &= A_{21}A_{11}^{-1}(Y_k - Y^*)A_{12}A_{22}^{-1} \\ &= (A_{21}A_{11}^{-1})^2(Y_{k-1} - Y^*)(A_{12}A_{22}^{-1})^2 \\ &\quad \vdots \qquad\qquad \vdots \\ &= (A_{21}A_{11}^{-1})^{k+1}(Y_0 - Y^*)(A_{12}A_{22}^{-1})^{k+1}. \end{aligned}$$

Hence,

$$||Y_{k+1} - Y^*|| \leq ||A_{21}A_{11}^{-1}||^{k+1}||Y_0 - Y^*||||A_{12}A_{22}^{-1}||^{k+1}.$$

$$\boxed{\textbf{ELA}}$$

This shows that if $||A_{21}A_{11}^{-1}||\,||A_{12}A_{22}^{-1}|| < 1$, then each sequence of the real matrices $\{Y_k\}$ constructed by our algorithm converges, which implies the convergence of $\{\mathbf{Y}_k\}$. Again, from (2.14), we have

$$X_{k+1} = A_{11}^{-1}(C_1 - Y_K A_{12}).$$

It follows that

$$X_{k+1} - X^* = -A_{11}^{-1}(Y_k - Y^*)A_{12}.$$

So, convergence of $\{Y_k\}$ implies convergence of $\{X_k\}$ and we can conclude that $\{\mathbf{X}_k\}$ converges. $\square$

EXAMPLE 2.9.   Consider the interval system of matrix equations in Example (2.6). With $\mathbf{Y}_0 = \begin{bmatrix} [-1,1] & [-1,1] \\ [-1,1] & [-1,1] \end{bmatrix}$, by performing 4 iterations of the above algorithm we have:

$$\mathbf{X}_{It} = \begin{bmatrix} [0.4333, 3.5501] & [-6.1334, 0.0252] \\ [0.1793, 1.2667] & [-0.4889, 0.9067] \end{bmatrix}, \quad \mathbf{Y}_{It} = \begin{bmatrix} [0.4887, 2.0066] & [0.4429, 1.9582] \\ [0.9701, 2.8885] & [1.0909, 2.9937] \end{bmatrix}.$$

With ignoring (2.13) we obtain the following wider interval matrices

$$\tilde{\mathbf{X}}_{It} = \begin{bmatrix} [0.2549, 3.8652] & [-6.4952, 0.1732] \\ [0.0686, 1.3795] & [-1.7301, 0.9307] \end{bmatrix}, \quad \tilde{\mathbf{Y}}_{It} = \begin{bmatrix} [0.4149, 2.0142] & [0.4230, 2.0609] \\ [0.9366, 2.9380] & [1.0666, 3.0626] \end{bmatrix}.$$

From the result of examples (2.6) and (2.9), we conclude that:

$$(\mathbf{X}_{It}, \mathbf{Y}_{It}) \subseteq (\tilde{\mathbf{X}}_{It}, \tilde{\mathbf{Y}}_{It}) \subseteq (\mathbf{X}_{Di}.\mathbf{Y}_{Di}).$$

REMARK 2.10.   If the condition of the above theorem is not established, we may use the intersection of $(\mathbf{X}_{Di}, \mathbf{Y}_{Di})$ and $(\mathbf{X}_{It}, \mathbf{Y}_{It})$ for obtaining a sharper solution.

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