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TP_K COMPLETION OF PARTIAL MATRICES WITH ONE UNSPECIFIED ENTRY*

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Abstract. Every partial TP_2 (TP_1) matrix with one unspecified entry has a TP_2 (TP_1) completion. For a given m-by-n pattern with one unspecified entry, the minimum set of conditions characterizing TP_3 completability is given. These conditions are at most eight polynomial inequalities on the specified entries of the pattern. For $k \geq 3$, patterns with one unspecified entry that are TP_k completable are also characterized, and conditions are described for completability otherwise.

Key words. Totally positive-k matrix, Completion problem, Pattern, Polynomial conditions.

AMS subject classifications. 15A83, 15B48.

1. Introduction. An m-by-n matrix A is said to be TP_k (*totally positive-k*) if all its ℓ-by-ℓ minors are positive, for $\ell = 1, \ldots, k$. If $k = \min\{m, n\}$, so that all minors are positive, then A is called *totally positive*. A *partial* matrix is one in which some entries are specified, while the remaining, unspecified, entries are free to be chosen. A *partial TP_k* matrix is a partial matrix all of whose fully specified minors of order at most k are positive. A *completion* of a partial matrix is a particular choice of values for the unspecified entries, resulting in a conventional matrix. The *pattern* (which we may think of as an array) of a partial matrix is an inventory of which entries are specified (and which are unspecified). A pattern P of specified entries is called *TP_k completable* if every partial TP_k matrix with pattern P has a TP_k completion. A *matrix completion problem* asks which partial matrices have a completion enjoying an identified property. In studying the completion problem for TP matrices, understanding the completion problem for TP_k matrices for $k = 1, \ldots, n$ has proven to be helpful. It is obvious that a partial TP_1 matrix is always TP_1 completable. The problem of TP_2 completion has been completely solved in [6]. In particular, it is known that any pattern with only one unspecified entry is TP_2 completable. As a next step in this process, we consider the TP_k completion of patterns with one unspecified entry. It turns out that depending on the location of

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the unspecified entry and the value of k, the conditions for TP_k completability vary. Our objective is to describe such conditions.

In Section 2, some preliminary results are presented. Section 3 provides a family of pairs of minors that do not produce extra conditions for TP_k completability. In Section 4, an explicit description of the minimum set of conditions for TP_3 completability of patterns with one unspecified entry is given. In Section 5, a combinatorial characterization of TP_k completable patterns, $k \geq 4$, with one unspecified entry is given. Finally, in the Appendix, a list of conditions for TP_4 completability of a pattern is given.

2. Background. Using Tarski-Seidenberg principle [1] and the fact that the set of m-by-n TP_k matrices form a semialgebraic set, we know that, for any pattern of specified entries, there is a finite number of polynomial inequalities on the specified entries that characterize the TP_k completability of the pattern. However, for a given pattern, neither the polynomial inequalities nor the number of these can be obtained from Tarski-Seidenberg principle. Here, our purpose is to better understand efficient lists of such inequalities for TP_k completion of patterns with just one unspecified entry.

For $\gamma \subseteq \{1, \ldots, n\}$, the notation γ^c is used to denote the set $\{1, \ldots, n\} \backslash \gamma$. The set of real m-by-n matrices is denoted by $M_{m,n}$, when $m = n$, we use M_n instead of $M_{m,n}$. For $\alpha \subseteq \{1, \ldots, m\}$ and $\beta \subseteq \{1, \ldots, n\}$, the submatrix of A lying in the rows indexed by α (α^c) and the columns indexed by β (β^c) is denoted by $A[\alpha, \beta]$ ($A(\alpha, \beta)$). If α and β are sets of consecutive numbers, then $A[\alpha, \beta]$ is called a *contiguous* submatrix of A. A contiguous minor is defined similarly. A matrix A is called *TP_k-contiguous* if every contiguous minor of A of order $\ell = 1, \ldots, k$ is positive.

The following known result is a useful tool throughout this work, see [2, 3] for a proof.

LEMMA 2.1. *An m-by-n matrix A is TP_k if and only if it is TP_k-contiguous.*

Suppose A is an m-by-n partial TP_k matrix with one unspecified entry x in the position (i, j). Let C_x be the set of all contiguous square submatrices of A of order $1, \ldots, k$ that contain the unspecified entry x. The submatrix S_{ij} of A that contains all of the submatrices in C_x and has the minimum order is called the *kth surrounding submatrix* of A with respect to (i, j). Thus, the order of S_{ij} is at most $(2k-1)$-by-$(2k-1)$. Using Lemma 2.1, we have the following result that we use frequently.

LEMMA 2.2. *Let A be an m-by-n partial TP_k matrix with one unspecified entry x in the position (i, j). Then A is TP_k completable if and only if S_{ij} is TP_k completable.*

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Note that, S_{ij} is TP_k completable if and only if there is a value a_{ij} such that replacing x by a_{ij}, implies positivity of every minor in C_x. Consider $B = [b_{uv}] \in C_x$, and suppose x is in the (r, s) position of B. Then, by expanding the determinant of B along its rth row, we have

$$\det(B) > 0 \iff (-1)^{r+s} x \det(B_{rs}) + \sum_{\substack{p=1 \\ p \neq s}}^{\ell} (-1)^{r+p} b_{rp} \det(B_{rp}) > 0 \tag{2.1}$$

in which B_{rp} is the submatrix of B obtained by deleting the rth row and pth column. Since B is partial TP_k, $\det(B_{rs}) > 0$. Therefore, if $r + s$ is odd, then (2.1) gives an upper bound for x and if $r + s$ is even, then (2.1) gives a lower bound for x. Let L_{ij} be the set of all lower bounds for x obtained from the inequalities in (2.1) and U_{ij} be the set of all upper bounds for x obtained from the inequalities in (2.1), for each $B \in C_x$. In order to have a TP_k completion, for every $\ell \in L_{ij}$ and every $u \in U_{ij}$, the inequality $\ell < u$ should hold. This implies the following result.

Theorem 2.3. *Consider the partial TP_k matrix A with one unspecified entry in the position (i, j). Then, A is TP_k completable if and only if*

$$\max\{\ell\}_{\ell \in L_{ij}} < \min\{u\}_{u \in U_{ij}}.$$

We note that because TP_k contiguous is sufficient for TP_k, Lemma 2.2 and Theorem 2.3 can be applied to patterns with more than one unspecified entry as long as they are far enough apart. For instance, suppose the pattern P has only two unspecified entries (p, q) and (r, s), where S_{pq} and S_{rs} do not share any entry ($S_{pq} \cap S_{rs} = \emptyset$), then using Lemma 2.1, P is TP_k completable if and only if each of S_{pq} and S_{rs} is TP_k completable. This can be generalized as the following.

Lemma 2.4. *Suppose P is an m-by-n pattern with t unspecified entries in the positions (i_u, j_u), $u = 1, \ldots, t$. If for all $u, v \in \{1, \ldots, t\}$, with $u \neq v$, we have $S_{i_u j_u} \cap S_{i_v j_v} = \emptyset$, then P is TP_k completable if and only if each $S_{i_u j_u}, u \in \{1, \ldots, t\}$ is TP_k completable.*

Some of the inequalities in Theorem 2.3 are redundant, a family of such inequalities is described in the next section.

3. Unconditional pairs of minors. In the previous section, we presented the general conditions for TP_k completability of patterns, when the unspecified entries are far apart. However, computing all these conditions can be a tedious work. Providentially, in a partial TP_k matrix some of these lower bounds are always less than some of the upper bounds. A family of such pairs of bounds are described in this section.

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Consider a pattern P with exactly one unspecified entry x in the position (i,j). Let P_1 and P_2 be two subpatterns of P containing x where the sum of the indices of x in P_1 is even and the sum of the indices of x in P_2 is odd. Suppose A is a partial TP_k matrix with pattern P. Then, for the submatrix A_1 corresponding to P_1, the inequality $\det A_1 > 0$ implies a lower bound for x, say L_{A_1}, and for the submatrix A_2 corresponding to P_2, the inequality $\det A_2 > 0$ implies an upper bound for x, say U_{A_2}. If for any partial TP_k matrix A with pattern P, $L_{A_1} < U_{A_2}$, then the pair of subpatterns P_1 and P_2 is called an *unconditional pair of subpatterns*. Otherwise, such a pair is called a *conditional pair of subpatterns*. We use the term conditional (unconditional) minors (or submatrices) when it refers to minors (or submatrices). A trivial example for an unconditional pair of minors is a pair of 2-by-2 minors, since any partial TP_2 pattern with one unspecified entry is TP_2 completable. In this section, for a given pattern with one unspecified entry, a family of pairs of unconditional subpatterns is determined.

The following lemma is proved in [3].

Lemma 3.1. *Let $\tilde{T}_\ell = [t_{ij}] \in M_\ell$ be a permutation matrix with $t_{i,\ell-i+1} = 1$, for $i = 1,\dots,\ell$. If $A \in M_{m,n}$ is TP_k, then both A^T and $\tilde{T}_m A \tilde{T}_n$ are TP_k.*

If a pair of minors is conditional (unconditional), then depending on their situation and using Lemma 3.1, there can be three more pairs of conditional (unconditional) minors. This is useful in this work.

The following lemma can be obtained by the short-term Plüker identity; see [3]. We present a different proof using Sylvester's identity.

Lemma 3.2. *Let $A \in \mathcal{M}_{n,n+2}$, and $k,j \in \{2,\dots,n+1\}$. Suppose $\alpha = \{1,\dots,n\}$, $\beta = \{1,\dots,n+2\}\backslash\{j,k\}$. If $j < k$, then*

$$\begin{aligned}
&\det A[\alpha,\{1,k\}^c]\det A[\alpha,\{j,n+2\}^c] - \det A[\alpha,\{1,j\}^c]\det A[\alpha,\{k,n+2\}^c]\\
&\quad = \det A[\alpha,\{j,k\}^c]\det A[\alpha,\{1,n+2\}^c].
\end{aligned}$$

Proof. Let r_j and r_k be row vectors in $\mathbb{R}^{n+2}$ where $r_j[1] = (-1)^{1+j}$, $r_k[n+2] = (-1)^{n+k}$, and all other entries of both vectors are zero. Insert row r_j between rows $j-1$ and j of A and r_k between rows $k-2$ and $k-1$ of A to obtain the matrix A'. Rename the rows of A' from $1,\dots,j-1,r_j,j,j+1,\dots,k-2,r_k,k-1,\dots,n$ to $1,2,\dots,n+2$, with the same order. Then, A' is of the following form, where $A'[\beta,\{1,\dots,n+2\}] = A$.

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$$A' = \begin{array}{c|c|c|c|c|c|c|c|}
 & 1 & \dots & \mathrm{j} & \dots & \mathrm{k} & \dots & n+2 \\ \hline
1 & & & & & & & \\
\vdots & & & & & & & \\
j-1 & & & & & & & \\ \hline
r_j & (-1)^{1+j} & 0 & 0 & \dots & 0 & \dots & 0 \\ \hline
j & & & & & & & \\
\vdots & & & & & & & \\
k-2 & & & & & & & \\ \hline
r_k & 0 & \dots & 0 & \dots & 0 & 0 & (-1)^{n+k} \\ \hline
k-1 & & & & & & & \\
\vdots & & & & & & & \\
n & & & & & & & \\ \hline
\end{array}$$

Now, the matrix $B \in M_2$ with

$$B = \begin{bmatrix} \det A'[\beta \cup \{j\}, \beta \cup \{j\}] & \det A'[\beta \cup \{j\}, \beta \cup \{k\}] \\ \det A'[\beta \cup \{k\}, \beta \cup \{j\}] & \det A'[\beta \cup \{k\}, \beta \cup \{k\}] \end{bmatrix}$$

satisfies the Sylvester's identity (see [5]):

$$\det B = (\det A'[\beta])^{(n+2)-n-1} \det A' = \det A[\alpha, \{j,k\}^c] \det A[\alpha, \{1, n+2\}^c].$$

So, $\det A[\alpha, \{1,k\}^c] \det A[\alpha, \{j, n+2\}^c] - \det A[\alpha, \{1,j\}^c] \det A[\alpha, \{k, n+2\}^c]$
$= \det A[\alpha, \{j,k\}^c] \det A[\alpha, \{1, n+2\}^c]$. ▯

Using the above lemma, a family of unconditional pairs of minors is given as follows.

LEMMA 3.3. *Consider a partial TP$_k$ matrix A with only one unspecified entry x in the position (i,j). Suppose two ℓ-by-ℓ, $\ell \le k$, contiguous submatrices of A lie in rows $r_1, r_1+1, \dots, r_1+\ell-1$ and columns $c_1, c_1+1, \dots, c_1+\ell$. If $j \in \{c_1+1, \dots, c_1+\ell-1\}$, then the lower bound obtained from one of the minors is less than the upper bound obtained from the other minor.*

Proof. Without loss of generality suppose A is an ℓ-by-$(\ell+1)$ matrix, $\{r_1, r_1+1, \dots, r_1+\ell-1\} = \{1, 2, \dots, \ell\}$, $\{c_1, c_1+1, \dots, c_1+\ell\} = \{1, 2, \dots, \ell+1\}$, and let $i+j$ be even. The proof for $i+j$ odd is similar. By expanding the determinant of

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each of the submatrices along the ith row, we have

$$x \det A(\{i\},\{j,\ell+1\}) + \sum_{\substack{t=1\\t\neq j}}^{\ell} (-1)^{i+t} a_{it} \det A(\{i\},\{t,\ell+1\}) > 0, \text{ and}$$

$$-x \det A(\{i\},\{1,j\}) + \sum_{\substack{t=2\\t\neq j}}^{\ell+1} (-1)^{i+t-1} a_{it} \det A(\{(i\},\{1,t\}) > 0.$$

Therefore, there exists x such that

$$\frac{\displaystyle\sum_{\substack{t=2\\t\neq j}}^{\ell+1} (-1)^{i+t-1} a_{it} \det A(\{i\},\{1,t\})}{\det A(\{i\},\{1,j\})} > x > \frac{\displaystyle\sum_{\substack{t=1\\t\neq j}}^{\ell} (-1)^{i+t-1} a_{it} \det A(\{i\},\{t,\ell+1\})}{\det A(\{i\},\{j,\ell+1\})}$$

if and only if

$$\sum_{\substack{t=2\\t\neq j}}^{\ell} (-1)^{i+t-1} a_{it} [\det A(\{i\},\{1,t\}) \det A(\{i\},\{j,\ell+1\}) - \det A(\{i\},\{t,\ell+1\}) \det A(\{i\},\{1,j\})]$$
$$+ \Big[(-1)^{i+\ell} a_{i(\ell+1)} \det A(\{i\},\{j,\ell+1\}) + (-1)^{i+1} a_{i1} \det A(\{i\},\{1,j\})\Big] \det A(\{i\},\{1,\ell+1\}) > 0.$$

Using Lemma 3.2, this is equivalent to

$$\det A(\{i\},\{1,\ell+1\}) \left[\sum_{t=2}^{j-1} (-1)^{i+t-1} a_{it} (-\det A(\{i\},\{t,j\})) \sum_{t=j+1}^{\ell} (-1)^{i+t-1} a_{it} \det A(\{i\},\{t,j\})\right] +$$
$$+ \Big[(-1)^{i+\ell} a_{i(\ell+1)} \det A(\{i\},\{j,\ell+1\}) + (-1)^{i+1} a_{i1} \det A(\{i\},\{1,j\})\Big] \det A(\{i\},\{1,\ell+1\}) > 0$$
$$\Longleftrightarrow$$
$$\left[\sum_{t=2}^{j-1} (-1)^{i+t} a_{it} \det A(\{i\},\{t,j\}) + \sum_{t=j+1}^{\ell} (-1)^{i+t-1} a_{it} \det A(\{i\},\{t,j\})\right] \det A(\{i\},\{1,\ell+1\})$$
$$+ \Big[(-1)^{i+\ell} a_{i(\ell+1)} \det A(\{i\},\{j,\ell+1\}) + (-1)^{i+1} a_{i1} \det A(\{i\},\{1,j\})\Big] \det A(\{i\},\{1,\ell+1\}) > 0$$
$$\Longleftrightarrow$$
$$\left[\sum_{t=1}^{j-1} (-1)^{i+t} a_{it} \det A(\{i\},\{t,j\}) + \sum_{t=j+1}^{\ell+1} (-1)^{i+t-1} a_{it} \det A(\{i\},\{t,j\})\right] \det A(\{i\},\{1,\ell+1\}) > 0$$
$$\Longleftrightarrow \det A(\{i\},\{j\}) \det A(\{i\},\{1,\ell+1\}) > 0.$$

The last inequality holds since the matrix is partial TP_k. □

Therefore, in studying the TP_k completability of a given pattern, comparison of lower and upper bounds coming from subpatterns satisfying the assumption of Lemma

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3.3 need not be made. It is not known whether the remaining pairs of subpatterns are all conditional pairs. However, in the next section, we show that the inequalities obtained from some of the remaining pairs may be implied by the inequalities obtained from others (see the proof of Lemma 4.3 for example). Therefore, the set of inequalities obtained from all of the conditional pairs of subpatterns may not be the minimal set of polynomial inequalities on the specified entries (in addition to being partial TP_k) for a pattern to be TP_k completable. We call two conditional pairs of subpatterns *independent* if the conditions obtained from one pair cannot be implied from those of the other pair. The minimum set of independent pairs of conditional subpatterns, for $k = 3$, is described next.

4. TP_3 completion. In this section, necessary and sufficient conditions for TP_3 completability of a pattern with one unspecified entry, in the form of minimal polynomial inequalities in the specified entries, are given.

In studying the TP_3 completion problem, the smallest order to consider is a pattern of order 3-by-n, $n \geq 3$. Note that, a 3-by-n TP_3 matrix is also TP, so using Theorem 2.8 of [4], we have the following lemma. We also prove it directly by checking all possible conditional minors.

LEMMA 4.1. *Every partial TP_3 matrix of order 3-by-n, $n \geq 3$, with exactly one unspecified entry is TP_3 completable.*

Proof. Using Lemma 2.2, it is enough to consider a 3-by-5 partial TP_3 matrix A where the unspecified entry lies in the $(2,3)$ position. Using Lemma 3.3, the pairs of minors of order 3 are unconditional. Since A is TP_2 completable, the only minors that may produce conditions are $A[\{1,2,3\},\{1,2,3\}]$ and $A[\{2,3\},\{3,4\}]$. The upper bound/ lower bound condition formed by these minors is

$$\frac{a_{24}a_{33}}{a_{34}} < \frac{a_{13}(a_{21}a_{32} - a_{22}a_{31}) + a_{33}(a_{11}a_{22} - a_{12}a_{21})}{a_{11}a_{32} - a_{12}a_{31}}. \tag{4.1}$$

Since $a_{33} \det A[\{1,2,3\},\{1,2,4\}] > 0$, we have

$$a_{33}\left[a_{14}(a_{21}a_{32} - a_{22}a_{31}) + a_{34}(a_{11}a_{22} - a_{12}a_{21}) - a_{24}(a_{11}a_{32} - a_{12}a_{31})\right] > 0. \tag{4.2}$$

On the other hand, $a_{13}a_{34} > a_{14}a_{33}$ and $a_{21}a_{32} - a_{22}a_{31} > 0$, thus, using the inequality (4.2), we have

$$a_{13}a_{34}(a_{21}a_{32} - a_{22}a_{31}) + a_{34}a_{33}(a_{11}a_{22} - a_{12}a_{21}) - a_{24}a_{33}(a_{11}a_{32} - a_{12}a_{31}) > 0$$

which is exactly the inequality (4.1). $\square$

Let A be an m-by-n matrix, and suppose $\alpha \subseteq \{1,\ldots,m\}$ and $\beta \subseteq \{1,\ldots,n\}$ with $|\alpha| = |\beta|$. If x is an unspecified entry in the position $(i,j) \in \alpha \times \beta$, then the

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determinant of the submatrix $A[\alpha,\beta]$ is a function of x. The value of this function at $x=0$ is denoted by $\det A[\alpha,\beta](0)$.

Lemma 4.2. *Let P be a 4-by-4 partial TP_3 pattern with exactly one unspecified entry x lying in the position (i,j). Then P is TP_3 completable if and only if $(i,j) \neq (2,3),(3,2)$. Moreover, for a partial TP_3 matrix A with pattern P, if $(i,j)=(2,3)$, then A has a TP_3 completion if and only if*

$$\frac{-\det A[\{2,3,4\},\{1,2,3\}](0)}{\det A[\{3,4\},\{1,2\}]} < \frac{\det A[\{1,2\},\{3,4\}](0)}{\det A[\{1\},\{4\}]},$$

and if $(i,j)=(3,2)$, then A has a TP_3 completion if and only if

$$\frac{-\det A[\{1,2,3\},\{2,3,4\}](0)}{\det A[\{1,2\},\{3,4\}]} < \frac{\det A[\{3,4\},\{1,2\}](0)}{\det A[\{4\},\{1\}]}.$$

Proof. If x lies in any of the first or last columns (or rows), then using Lemma 2.2, and Lemma 4.1, the pattern is TP_3 completable. If x lies in the $(2,2)$ position, the only pairs of submatrices that contain x and do not lie in a 3-by-k submatrix, $k=3,4$, are the pairs of submatrices $A[\{1,2\},\{1,2\}], A[\{2,3,4\},\{2,3,4\}]$ and $A[\{1,2,3\},\{1,2,3\}], A[\{2,3,4\},\{2,3,4\}]$. Since the sums of the indices of x have the same parity in both submatrices in each pair, both of these pairs are unconditional. The case $(i,j)=(3,3)$ is similar.

If x lies in the position $(2,3)$, then the only pair of submatrices that contains x but do not lie in a 3-by-k submatrix, $k=3,4$, is $A[\{2,3,4\},\{1,2,3\}], A[\{1,2\},\{3,4\}]$. Therefore, the following inequality is a sufficient condition for a partial TP_3 matrix A to have a TP_3 completion

$$\frac{-\det A[\{2,3,4\},\{1,2,3\}](0)}{\det A[\{3,4\},\{1,2\}]} < \frac{\det A[\{1,2\},\{3,4\}](0)}{\det A[\{1\},\{4\}]}. \tag{4.3}$$

The following example shows that the above inequality is also a necessary condition for a TP_3 completion. Consider the partial TP_3 matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 4.9 & x & 8 \\ 1 & 6 & 10.9 & 16 \\ 1 & 8 & 15 & 31 \end{pmatrix},$$

where the inequality (4.3) does not hold. The inequality $\det A[\{2,3,4\},\{1,2,3\}] > 0$ implies $x > 8.645$, while $\det A[\{1,2\},\{3,4\}] > 0$ implies $x < 8$. Thus, there is no value for x that forms a TP_3 completion for A. The proof for the case of $(i,j)=(3,2)$ is similar. $\square$

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The following lemma explicitly describes the conditions for TP_3 completability of a 5-by-5 partial TP_3 matrix with one unspecified entry. If the sums of the indices of x in a submatrix B is odd (even), then the positivity of $\det B$ will result in an upper bound (lower bound) for x, we use the notation $\downarrow(\uparrow)B$ to emphasize this.

THEOREM 4.3. *Let A be a 5-by-5 partial TP_3 matrix with one unspecified entry in the position (i,j). Then*

i. if $i \in \{1,5\}$ or $j \in \{1,5\}$ or $(i,j) = (2,2)$ or $(i,j) = (4,4)$, then A has a TP_3 completion without any extra condition,

ii. if $(i,j) = (2,4)$, then A has a TP_3 completion if and only if

$$\frac{-\det A[\{2,3,4\},\{2,3,4\}](0)}{\det A[\{3,4\},\{2,3\}]} < \frac{\det A[\{1,2\},\{4,5\}](0)}{\det A[\{1\},\{5\}]},$$

if $(i,j) = (4,2)$, then A has a TP_3 completion if and only if

$$\frac{-\det A[\{2,3,4\},\{2,3,4\}](0)}{\det A[\{2,3\},\{3,4\}]} < \frac{\det A[\{4,5\},\{1,2\}](0)}{\det A[\{5\},\{1\}]},$$

iii. if $(i,j) = (2,3)$, then A has a TP_3 completion if and only if

$$\frac{-\det A[\{2,3,4\},\{3,4,5\}](0)}{\det A[\{3,4\},\{4,5\}]} < \frac{\det A[\{1,2,3\},\{1,2,3\}](0)}{\det A[\{1,3\},\{1,2\}]},$$

and

$$\frac{-\det A[\{2,3,4\},\{1,2,3\}](0)}{\det A[\{3,4\},\{1,2\}]} < \frac{\det A[\{1,2,3\},\{3,4,5\}](0)}{\det A[\{1,3\},\{4,5\}]},$$

the cases $(i,j) = (3,2), (3,4)$, or $(4,3)$ are similar.

iv. if $(i,j) = (3,3)$, then there are eight polynomial inequalities on the specified entries of A that need to hold in order for A to have a TP_3 completion.

Proof.

i. If $i \in \{1,5\}$ or $j \in \{1,5\}$, using Lemma 2.2, it is enough to consider a 3-by-5 (or 5-by-3) contiguous submatrix containing x, using Lemma 4.1, A has a TP_3 completion. If $(i,j) = (2,2)$ or $(4,4)$, it is enough to consider a 4-by-4 contiguous submatrix B of A such that x lies in the $(2,2)$ or $(3,3)$ position of B, respectively, using Lemma 4.2, A has a TP_3 completion.

ii. If $(i,j) = (2,4)$ or $(4,2)$, again it is enough to consider a 4-by-4 contiguous submatrix B of A containing x such that x lies in the $(2,3)$ or $(3,2)$ position of B, respectively. By Lemma 4.2, conditions for TP_3 completability of A are inequalities given in the statement.

iii. Suppose x is in the $(2,3)$ position. Using Lemma 2.2, the conditional pairs of submatrices lie in $A[\{1,2,3,4\},\{1,2,3,4,5\}]$. The only possible conditional cases

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that we need to consider are (a) both submatrices are of order 3; (b) one of the submatrices is of order 2 and the other is 3.

(a) In this case, there are nine such pairs that are listed below:

$$\begin{aligned}
&\downarrow A[\{1,2,3\},\{1,2,3\}], \uparrow A[\{1,2,3\},\{2,3,4\}],\\
&\downarrow A[\{1,2,3\},\{1,2,3\}], \uparrow A[\{2,3,4\},\{1,2,3\}],\\
&\downarrow A[\{1,2,3\},\{1,2,3\}], \uparrow A[\{2,3,4\},\{3,4,5\}], \qquad (4.4)\\
&\downarrow A[\{1,2,3\},\{3,4,5\}], \uparrow A[\{1,2,3\},\{2,3,4\}],\\
&\downarrow A[\{1,2,3\},\{3,4,5\}], \uparrow A[\{2,3,4\},\{1,2,3\}], \qquad (4.5)\\
&\downarrow A[\{1,2,3\},\{3,4,5\}], \uparrow A[\{2,3,4\},\{3,4,5\}],\\
&\downarrow A[\{2,3,4\},\{2,3,4\}], \uparrow A[\{1,2,3\},\{2,3,4\}],\\
&\downarrow A[\{2,3,4\},\{2,3,4\}], \uparrow A[\{2,3,4\},\{1,2,3\}],\\
&\downarrow A[\{2,3,4\},\{2,3,4\}], \uparrow A[\{2,3,4\},\{3,4,5\}].
\end{aligned}$$

Except the pairs of submatrices in (4.4) and (4.5), all other seven pairs lie in a 3-by-5 or 5-by-3 submatrix, so using Lemma 4.1, they are unconditional. Moreover, the matrices A_1 and A_2 below show that the two pairs in (4.4) and (4.5) are both conditional and independent. Therefore, the inequalities given in the statement must hold for a TP_3 completion.

$$A_1 = \begin{pmatrix} 10 & 60 & 6 & 5 & 2 \\ 1 & 8.99 & x & 8.61 & 6 \\ 1 & 9 & 8 & 10 & 8.35 \\ 1 & 10 & 11 & 300 & 500 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 2 & 2.95 & 5 & 7 \\ 1 & 3.2 & x & 16 & 26.25 \\ 1 & 4 & 12 & 31 & 54 \\ 1 & 5 & 18 & 51 & 97 \end{pmatrix}.$$

In the partial TP_3 matrix A_1, the inequalities $\det A_1[\{1,2,3\},\{1,2,3\}] > 0$ and $\det A_1[\{2,3,4\},\{3,4,5\}] > 0$ imply $7.9796 < x < 7.9753$, and in the partial TP_3 matrix A_2, the inequalities $\det A_2[\{1,2,3\},\{3,4,5\}] > 0$, and $\det A_2[\{2,3,4\},\{1,2,3\}] > 0$ imply $7.2 < x < 7.15542$. Therefore, there is no TP_3 completion for any of the matrices A_1 and A_2.

(b) In this case, the only pair with different parities for the sum of indices of x that are not contained in a 3-by-4 or 3-by-5 submatrix (or the transpose of those) is $\uparrow A[\{2,3,4\},\{1,2,3\}], \downarrow A[\{1,2\},\{3,4\}]$. This can be considered as an unspecified entry in the $(2,3)$ position of a 4-by-4 partial TP_3 matrix, using Lemma 4.2, the extra condition for TP_3 completion is

$$\frac{-\det A[\{2,3,4\},\{1,2,3\}](0)}{\det A[\{3,4\},\{1,2\}]} < \frac{\det A[\{1,2\},\{3,4\}](0)}{\det A[\{1\},\{4\}]} = \frac{a_{13}a_{24}}{a_{14}}. \qquad (4.6)$$

However, (4.6) is obtained by the inequality produced by matrices in (4.5) and the assumption of partial TP_3, as shown below, so it is not an independent condition for having a TP_3 completion.

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The inequality obtained from the pair in (4.5) is

$$\frac{-\det A[\{2,3,4\},\{1,2,3\}](0)}{\det A[\{3,4\},\{1,2\}]} < \frac{a_{13}\det A[\{2,3\},\{4,5\}]+a_{33}\det A[\{1,2\},\{4,5\}]}{\det A[\{1,3\},\{4,5\}]}.$$

So it is enough to show that

$$\frac{a_{13}\det A[\{2,3\},\{4,5\}]+a_{33}\det A[\{1,2\},\{4,5\}]}{\det A[\{1,3\},\{4,5\}]} < \frac{a_{13}a_{24}}{a_{14}}. \tag{4.7}$$

To show this, we have the following

$$\begin{aligned}
&a_{14}a_{25}(a_{33}a_{14}-a_{34}a_{13}) < a_{15}a_{24}(a_{33}a_{14}-a_{34}a_{13}) \iff\\
&a_{14}a_{25}a_{33}a_{14}-a_{14}a_{25}a_{34}a_{13}-a_{15}a_{24}a_{33}a_{14} < -a_{15}a_{24}a_{34}a_{13} \iff\\
&a_{13}a_{14}a_{24}a_{35}+a_{14}a_{25}a_{33}a_{14}-a_{14}a_{25}a_{34}a_{13}-a_{15}a_{24}a_{33}a_{14} <\\
&a_{13}a_{14}a_{24}a_{35}-a_{15}a_{24}a_{34}a_{13} \iff\\
&\frac{a_{13}(a_{24}a_{35}-a_{25}a_{34})+a_{33}(a_{14}a_{25}-a_{15}a_{24})}{a_{14}a_{35}-a_{15}a_{34}} < \frac{a_{13}a_{24}}{a_{14}},
\end{aligned}$$

this is exactly the inequality (4.7).
Therefore, there are two independent conditions necessary for the 5-by-5 pattern with $(2,3)$ unspecified. By Lemma 3.1, it can be shown that a similar result is true for the unspecified entry in positions $(3,2),(3,4)$, and $(4,3)$.

iv. For the case $(i,j)=(3,3)$, we need to consider the contiguous 5-by-5 matrix. In a similar way to the previous case, we consider two cases:
   (a) Suppose both pairs of submatrices are of order 3. There are nine 3-by-3 contiguous submatrices containing x:

$$\uparrow A[\{1,2,3\},\{1,2,3\}], \tag{4.8}$$
$$\uparrow A[\{1,2,3\},\{3,4,5\}], \tag{4.9}$$
$$\uparrow A[\{2,3,4\},\{2,3,4\}], \tag{4.10}$$
$$\uparrow A[\{3,4,5\},\{1,2,3\}], \tag{4.11}$$
$$\uparrow A[\{3,4,5\},\{3,4,5\}], \tag{4.12}$$
$$\downarrow A[\{1,2,3\},\{2,3,4\}], \tag{4.13}$$
$$\downarrow A[\{2,3,4\},\{1,2,3\}], \tag{4.14}$$
$$\downarrow A[\{2,3,4\},\{3,4,5\}], \tag{4.15}$$
$$\downarrow A[\{3,4,5\},\{2,3,4\}]. \tag{4.16}$$

These pairs create 20 intervals and the following 12 pairs of them satisfy conditions of Lemma 3.3, so they are unconditional:

$$\{(4.8),(4.13)\},\{(4.8),(4.14)\},\{(4.9),(4.13)\},\{(4.9),(4.15)\},$$

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$$\{(4.10),(4.13)\}, \{(4.10),(4.14)\}, \{(4.10),(4.15)\}, \{(4.10),(4.16)\},$$

$$\{(4.11),(4.14)\},\{(4.11),(4.16)\}, \{(4.12),(4.15)\}, \{(4.12),(4.16)\}.$$

The remaining eight pairs are

$$\{(4.8),(4.15)\},\{(4.8),(4.16)\},\{(4.12),(4.13)\},\{(4.12),(4.14)\}, \tag{4.17}$$

and

$$\{(4.9),(4.14)\},\{(4.9),(4.16)\},\{(4.11),(4.13)\},\{(4.11),(4.15)\}. \tag{4.18}$$

Using Lemma 3.1, it is enough to show that only one of the pairs in (4.17) and (4.18) are conditional. In the following matrices B_1 and B_2, the only conditional pairs of minors are $\{(4.8),(4.15)\}$ and $\{(4.9),(4.16)\}$, respectively, and the inequalities obtained from all other seven pairs hold. Thus, these eight pairs of minors are conditional and the conditions obtained from them are independent.

$$B_1 = \begin{pmatrix} 500 & 300 & 11 & 10 & 1 \\ 8.35 & 10 & 8 & 9 & 1 \\ 6 & 8.61 & x & 8.99 & 1 \\ 2 & 5 & 6 & 60 & 10 \\ 1 & 3 & 4 & 80 & 20 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3.2 & 4 & 5 \\ 1 & 2.95 & x & 12 & 18 \\ 1 & 5 & 16 & 31 & 51 \\ 1 & 7 & 26.25 & 54 & 97 \end{pmatrix}.$$

The polynomial inequalities obtained from conditional pairs of minors listed in (4.17) and (4.18) are listed below:

$$\begin{aligned}
\frac{-\det A[\{1,2,3\},\{1,2,3\}](0)}{\det A[\{1,2\},\{1,2\}]} &< \frac{\det A[\{2,3,4\},\{3,4,5\}](0)}{\det A[\{2,4\},\{4,5\}]}, \\
\frac{-\det A[\{1,2,3\},\{1,2,3\}](0)}{\det A[\{1,2\},\{1,2\}]} &< \frac{\det A[\{3,4,5\},\{2,3,4\}](0)}{\det A[\{4,5\},\{2,4\}]}, \\
\frac{-\det A[\{3,4,5\},\{3,4,5\}](0)}{\det A[\{4,5\},\{4,5\}]} &< \frac{\det A[\{1,2,3\},\{2,3,4\}](0)}{\det A[\{1,2\},\{2,4\}]}, \\
\frac{-\det A[\{3,4,5\},\{3,4,5\}](0)}{\det A[\{4,5\},\{4,5\}]} &< \frac{\det A[\{2,3,4\},\{1,2,3\}](0)}{\det A[\{2,4\},\{1,2\}]}, \\
\frac{-\det A[\{1,2,3\},\{3,4,5\}](0)}{\det A[\{1,2\},\{4,5\}]} &< \frac{\det A[\{2,3,4\},\{1,2,3\}](0)}{\det A[\{2,4\},\{1,2\}]}, \\
\frac{-\det A[\{1,2,3\},\{3,4,5\}](0)}{\det A[\{1,2\},\{4,5\}]} &< \frac{\det A[\{3,4,5\},\{2,3,4\}](0)}{\det A[\{4,5\},\{2,4\}]}, \\
\frac{-\det A[\{3,4,5\},\{1,2,3\}](0)}{\det A[\{4,5\},\{1,2\}]} &< \frac{\det A[\{1,2,3\},\{2,3,4\}](0)}{\det A[\{1,2\},\{2,4\}]}, \\
\frac{-\det A[\{3,4,5\},\{1,2,3\}](0)}{\det A[\{4,5\},\{1,2\}]} &< \frac{\det A[\{2,3,4\},\{3,4,5\}](0)}{\det A[\{2,4\},\{4,5\}]}.
\end{aligned}$$

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(b) Now, consider the pairs of submatrices containing x where one is of order 2 and the other one has order 3. The only cases for which they do not lie in a 3-by-5 or its transpose submatrix are

$$\uparrow A[\{1,2,3\},\{3,4,5\}], \downarrow A[\{3,4\},\{2,3\}], \tag{4.19}$$

$$\uparrow A[\{3,4,5\},\{1,2,3\}], \downarrow A[\{2,3\},\{3,4\}]. \tag{4.20}$$

Both of these pairs are contained in a 4-by-4 submatrix. The pair of submatrices in (4.19) creates the following condition:

$$\frac{-\det A[\{1,2,3\},\{3,4,5\}](0)}{\det A[\{1,2\},\{4,5\}]} < \frac{\det A[\{3,4\},\{2,3\}](0)}{\det A[\{4\},\{2\}]} = \frac{a_{32}a_{43}}{a_{42}}.$$

We show that this inequality is obtained from being partial TP_3 and the inequality produced by the pair of submatrices in (4.9) and (4.16), that is

$$\frac{-\det A[\{1,2,3\},\{3,4,5\}](0)}{\det A[\{1,2\},\{4,5\}]} < \frac{a_{32}\det A[\{4,5\},\{3,4\}] + a_{34}\det A[\{4,5\},\{2,3\}]}{\det A[\{4,5\},\{2,4\}]}.$$

It is enough to show that

$$\frac{a_{32}\det A[\{4,5\},\{3,4\}] + a_{34}\det A[\{4,5\},\{2,3\}]}{\det A[\{4,5\},\{2,4\}]} < \frac{a_{32}a_{43}}{a_{42}}. \tag{4.21}$$

The proof for the inequality (4.21) is similar to that of the inequality (4.7), and is omitted.

Similarly, we can show that the condition obtained from the submatrices in (4.11) and (4.15) and the assumption of partial TP_3 imply the condition obtained by (4.20). So the case of 2-by-2 versus 3-by-3 does not create extra independent conditions. Therefore, when $(i,j) = (3,3)$, there are eight independent conditions needed to ensure the TP_3 completability of a 5-by-5 pattern. $\square$

For an m-by-n, $m,n \geq 5$, pattern P with one unspecified entry, Lemma 2.2 can be used to reduce the checking for TP_3 completion of P to that of a subpattern of order at most 5-by-5. Theorem 4.3, then can be used in the corresponding S_{ij} to compute all the conditions for TP_3 completability.

5. TP_k completion of patterns with one unspecified entry. In this section, a characterization of TP_k completable patterns with one unspecified entry, $k \geq 4$, is given.

LEMMA 5.1. *Let A be a k-by-k partial TP matrix. For any $r \geq 0$, and $\ell > k$, matrix A can be contained contiguously in a partial TP_{k+r} matrix of order ℓ.*

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Proof. This can be shown by repeatedly using Lemma 2.4 in [4]. □

LEMMA 5.2. *Let P be an m-by-n pattern, $n \geq m \geq 4$, with one unspecified entry in the position (i,j). Suppose $i+j > 4$ and $i+j < m+n-2$. Then P is not TP_k completable, for $k \geq 4$.*

Proof. We show that there is a partial TP_k matrix with pattern P and with no TP_k completion. For this, start with a 4-by-4 partial TP matrix B with no TP completion. There is such a matrix by [4]. Considering the location of x in P, and using Lemma 5.1, extend B to a partial TP_k matrix A, such that A has pattern P. There is no TP_k completion for A, with $k \geq 4$, since otherwise there would be a TP_4 (and so TP) completion for B. This implies that the pattern P is not TP_k completable.

In order to find the submatrix B described above, we consider the following cases for (i,j)

1. If $i \leq m-3$ and $j \geq 4$, then x lies in the $(1,4)$ position of the submatrix $A[\{i,i+1,i+2,i+3\},\{j-3,j-2,j-1,j\}]$.
2. If $2 \leq i \leq m$ and $j = 3$, then x lies in the $(2,3)$ position of the submatrix $A[\{i-1,i,i+1,i+2\},\{1,2,3,4\}]$.
3. If $3 \leq i \leq m$ and $j = 2$, then x lies in the $(3,2)$ position of the submatrix $A[\{i-2,i-1,i,i+1\},\{1,2,3,4\}]$.
4. If $4 \leq i \leq m$ and $j = 1$, then x lies in the $(4,1)$ position of the submatrix $A[\{i-3,i-2,i-1,i\},\{1,2,3,4\}]$.
5. If $(i,j) = (m-2,n-1),(m-1,n-2),(m,n-3)$, then the submatrix $A[\{m-3,m-2,m-1,m\},\{n-3,n-2,n-1,n\}]$ has (i,j) in its off-diagonal entries.
6. If $(i,j) = (m-2,n-2),(m-1,n-3)$, since $i+j > 4$, then $m+n > 8$, so either $m > 4$ or $n > 4$, in the former case the submatrix $A[\{m-4,m-3,m-2,m-1\},\{n-3,n-2,n-1,n\}]$ has (i,j) in its off-diagonal entries, in the latter case the submatrix $A[\{m-3,m-2,m-1,m\},\{n-4,n-3,n-2,n-1\}]$ has (i,j) in its off-diagonal entries.
7. If $(i,j) = (m-2,n-3)$, then again using $i+j > 4$ the smallest possible values for i and j are $(i,j) = (4,6)$, $(i,j) = (5,5)$, and $(i,j) = (6,4)$. The submatrices $A[\{m-3,m-2,m-1,m\},\{n-5,n-4,n-3,n-2\}]$, $A[\{m-4,m-3,m-2,m-1\},\{n-4,n-3,n-2,n-1\}]$, and $A[\{m-5,m-4,m-3,m-2\},\{n-3,n-2,n-1,n\}]$ contain x on their anti-diagonal entries, respectively.

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In each of the above cases, the obtained 4-by-4 partial TP_4 pattern is not TP_4 completable; see [4]. This completes the proof. ∎

LEMMA 5.3. *Let P be an m-by-n pattern, $n \geq m \geq 4$, with one unspecified entry x in the position (i, j). Suppose $i + j \leq 4$ or $i + j \geq m + n - 2$. Then P is TP_k completable for $k \geq 4$.*

Proof. Let A be a partial TP_k matrix with pattern P. We consider the following cases:

1. If $i = 1$, then all of the contiguous minors of order at most k that contain x are contained in a k-by-$(k + 3)$ submatrix, say B. Since A is partial TP_k, and B has minors of order less than or equal to k, the submatrix B is in fact partial TP. The unspecified entry lies in the (i, j) position of B, with $i+j \leq 4$, using [4], B is TP (and therefore TP_k) completable. Using contiguity, there is a TP_k completion for A. By similar arguments, it can be shown that there is a TP_k completion for A if $j = 1$, $i = m$, and $j = n$.
2. If $i = j = 2$, then the contiguous minors of order at most k containing x are contained in a $(k+1)$-by-$(k+1)$ submatrix. However, the only minor of order $k + 1$ in this submatrix is the determinant of the entire submatrix, and that contains the unspecified entry, so there are no minors of order $k+1$ consisting of only specified entries. Therefore, the submatrix has only minors of order at most k, and because A is partial TP_k, the submatrix is partial TP. The unspecified entry lies in the $(2, 2)$ position in this partial TP submatrix, and as $i + j \leq 4$, the submatrix has a TP (and so TP_k) completion. Again using contiguity, there is a TP_k completion for A. By similar arguments, it follows that there is a TP_k completion for A if $i = m - 1$ and $j = n - 1$. ∎

Lemmas 5.2 and 5.3 imply the following results.

THEOREM 5.4. *For $m, n, k \geq 4$, an m-by-n pattern P with one unspecified entry in the (i, j) position is TP_k completable if and only if $i + j \leq 4$ or $i + j \geq m + n - 2$.*

THEOREM 5.5. *Let P be an m-by-n pattern with one unspecified entry. If P is TP_k completable, then P is TP_{k-1} completable.*

Proof. We know that every pattern with exactly one unspecified entry is both TP_1 and TP_2 completable. So the statement is true for $k < 4$. For $k \geq 4$, the result follows from Theorem 5.4. ∎

6. Appendix. Characterizing explicit conditions for TP_k completability, $k \geq 4$, using only upper bounds and lower bounds, requires tedious effort. Here we consider a 7-by-7 partial TP_4 matrix A where the $(4, 4)$ entry is unspecified. There are sixteen 4-by-4 contiguous minors containing the unspecified entry, eight of which produce

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lower bounds, and eight produce upper bounds. We use the following labelling for the submatrices of A of order 4 that produce upper bound (U_i) or lower bound (L_i) for x.

$$\begin{aligned}
&L_1 = A[\{1234\},\{1234\}],\ L_2 = A[\{1234\},\{3456\}],\ L_3 = A[\{2345\},\{2345\}],\\
&L_4 = A[\{2345\},\{4567\}],\ L_5 = A[\{3456\},\{1234\}],\ L_6 = A[\{3456\},\{3456\}],\\
&L_7 = A[\{4567\},\{1234\}],\ L_8 = A[\{4567\},\{4567\}],\ U_1 = A[\{1234\},\{2345\}],\\
&U_2 = A[\{1234\},\{4567\}],\ U_3 = A[\{2345\},\{1234\}],\ U_4 = A[\{2345\},\{3456\}],\\
&U_5 = A[\{3456\},\{2345\}],\ U_6 = A[\{3456\},\{4567\}],\ U_7 = A[\{4567\},\{1234\}],\\
&U_8 = A[\{4567\},\{3456\}].
\end{aligned}$$

So, there are 64 upper bound/lower bound pairs obtained from only 4-by-4 minors. From these minors 24 of them satisfy conditions of Lemma 3.3 and so they are unconditional pairs of minors. However, the following list of matrices together with Lemma 3.1 show that the remaining pairs of minors are conditional. In addition, there are eight conditions obtained from the TP_3 case. Moreover, there are more pairs of minors (of order 4-by-4 with either 2-by-2 or 3-by-3) that are not known to be conditional or unconditional.

6.1. A list of partial TP_4 matrices with no TP_4 completion.

$$L1 > U2 : \begin{pmatrix} 100 & 12 & 6 & 3 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 5 & 7 & 9 & 11 & 20 \\ 1 & 4 & 9 & x & 20 & 30 & 262.7 \end{pmatrix}.$$

$$L4 > U3 : \begin{pmatrix} 100 & 12 & 6 & 3 & 1 & 1 & 1 \\ 1 & 2 & 3 & 3 & 5 & 6 & 7 \\ 1 & 3 & 5 & x & 9 & 11 & 20 \\ 1 & 4 & 9 & 10 & 20 & 30 & 262.7 \end{pmatrix}.$$

$$L1 > U4 : \begin{pmatrix} 57 & 12.1 & 5.95 & 3.46 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6.1 \\ 1.15 & 3 & 4.998 & 7 & 9 & 11 \\ 1.09 & 4 & 9.2 & x & 20 & 30 \\ .999 & 5.4 & 15.49 & 26 & 40 & 200 \end{pmatrix}.$$

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$$L1 > U6 : \begin{pmatrix} 1 & 1.2 & 1 & 1.0099 & 1 & 1 & 1 \\ 1 & 4 & 5 & 6.989 & 8 & 9 & 10 \\ 1 & 6 & 9 & 16 & 21.0005 & 27 & 35.325 \\ 1.000101 & 7 & 12.0925 & x & 38 & 56.01 & 85 \\ 1 & 8 & 16 & 38.999 & 69.0001 & 116 & 206 \\ 1 & 9 & 22 & 64.55 & 136 & 268.325 & 577 \end{pmatrix}.$$

$$L2 > U5 : \begin{pmatrix} 1 & 1 & 1.108 & 1 & 1 & 1 \\ 1 & 1.55877 & 2.95047 & 4.41994 & 5.59978 & 5.70515 \\ 1 & 1.69127 & 4.15273 & 7.44402 & 10.4814 & 11.3799 \\ 1 & 3.20442 & x & 44.1068 & 71.1925 & 84.8527 \\ 1 & 3.66803 & 23.3806 & 59.95 & 100.867 & 127.887 \\ 1 & 4.16746 & 29.581 & 80.9181 & 144.658 & 206.988 \end{pmatrix}.$$

$$L2 > U3 : \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1.02546 & 1.7817 & 2.02443 & 2.28748 & 3.99454 \\ 1 & 1.27716 & 10.6119 & 13.7854 & 17.2935 & 40.8292 \\ 1 & 1.57735 & 21.3954 & x & 36.8739 & 103.157 \\ 1 & 2.13274 & 42.37 & 56.956 & 81.9643 & 347.708 \end{pmatrix}.$$

$$L2 > U6 : \begin{pmatrix} 1 & 1.173 & 1 & 1 & 1 \\ 1 & 1.66861 & 2.15911 & 2.54789 & 2.74514 \\ 1 & 2.43216 & 3.98167 & 5.85376 & 7.10803 \\ 1.0032 & x & 4.35477 & 7.75572 & 10.6744 \\ 1 & 2.744 & 4.92001 & 12.5234 & 20.7245 \\ 1 & 3.35 & 6.75196 & 77.476 & 176.401 \end{pmatrix}.$$

$$L2 > U7,\ L3 > U7 : \begin{pmatrix} 500 & 1 & 1 & 1.01 & 1 & 1 \\ 1 & 1.89299 & 2.69541 & 3.06855 & 3.39843 & 4.55957 \\ 1 & 2.98368 & 5.64721 & 7.41 & 9.22161 & 15.6336 \\ 1 & 3.7812 & 8.21308 & x & 16.775 & 33.0214 \\ 1 & 4.84931 & 11.7806 & 19.464 & 28.8714 & 65.2499 \\ 1 & 5.65706 & 14.8655 & 28.53 & 46.3549 & 129.497 \\ 1 & 7.44993 & 22.0492 & 51.7033 & 97.8835 & 10000000 \end{pmatrix}.$$

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$$L3 > U2: \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1.85217 & 2.2313 & 2.53312 & 2.95498 & 3.29232 \\ 1 & 2.46495 & 3.62929 & 6.17578 & 9.87553 & 13.0198 \\ 1 & 2.983 & x & 11.9684 & 23.14701 & 32.9 \\ 1 & 2.98379 & 7.015 & 20.1909 & 45.6336 & 81.9216 \end{pmatrix}.$$

$$L3 > U6: \begin{pmatrix} 1.0001 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1.2081 & 1.9291 & 2.75604 & 3.01261 & 3.61019 \\ 1 & 1.86343 & x & 11.8739 & 14.3995 & 21.3672 \\ 1.021 & 3.19352 & 14.3493 & 33.6922 & 43.3319 & 73.0577 \\ 1 & 4.05599 & 20.2848 & 49.0769 & 64.9434 & 116.177 \end{pmatrix}.$$

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