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SIMULTANEOUS DECOMPOSITION OF TWO EP MATRICES WITH APPLICATIONS*

JULIO BENÍTEZ† AND XIAOJI LIU‡

Abstract. We find a simple canonical form for EP complex matrices $\mathbf{A}$ and $\mathbf{B}$ under simultaneous unitary equivalence.

Key words. EP matrices, Unitarily equivalence, Generalized inverses.

AMS subject classifications. 15B99, 15A23, 15A09.

1. Introduction. For an $m \times n$ complex matrix $\mathbf{A}$, the symbols $\mathbf{A}^*$, $\mathcal{R}(\mathbf{A})$, $\mathcal{N}(\mathbf{A})$, and $\mathrm{rk}(\mathbf{A})$ will stand for the conjugate transpose, the column space, the null space, and the rank of $\mathbf{A}$, respectively. An $n \times n$ matrix $\mathbf{P}$ is an orthogonal projector if $\mathbf{P} = \mathbf{P}^2 = \mathbf{P}^*$. The symbol $\mathbf{I}_n$ will denote the identity matrix of order n. The zero matrix of order $n \times m$ will be denoted by $\mathbf{0}_{n,m}$, and $\mathbf{0}_n$ will be used instead of $\mathbf{0}_{n,n}$. When there is no danger of confusion with the size, a zero matrix will be denoted, simply, by $\mathbf{0}$. Furthermore, $\mathbf{A}^\dagger$ will stand for the Moore-Penrose inverse of $\mathbf{A} \in \mathbb{C}_{n,m}$ i.e., the unique matrix satisfying the four equations

$$\mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{A}, \qquad \mathbf{A}^\dagger\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger, \qquad (\mathbf{A}\mathbf{A}^\dagger)^* = \mathbf{A}\mathbf{A}^\dagger, \qquad (\mathbf{A}^\dagger\mathbf{A})^* = \mathbf{A}^\dagger\mathbf{A}.$$

It is known that any matrix $\mathbf{A} \in \mathbb{C}_{n,m}$ has a Moore-Penrose inverse (see e.g. [21]). Moreover, it can be easily proved that $\mathbf{A}\mathbf{A}^\dagger$ is the orthogonal projector onto $\mathcal{R}(\mathbf{A})$ and $\mathbf{A}^\dagger\mathbf{A}$ is the orthogonal projector onto $\mathcal{R}(\mathbf{A}^*)$.

The following lemma, which we will use below, establishes a canonical form for a pair of orthogonal projectors (see e.g., [8, 13, 16]).

Lemma 1.1 (CS decomposition). *Let $\mathbf{P}_1, \mathbf{P}_2 \in \mathbb{C}_{n,n}$ be two orthogonal projectors.*

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†Universidad Politécnica de Valencia, Instituto de Matemática Multidisciplinar, Valencia, 46022, Spain (jbenitez@mat.upv.es).

‡Faculty of Science, Guangxi University for Nationalities, Nanning 530006, China, and Guangxi Key Laboratory of Hybrid Computational and IC Design Analysis, Nanning 530006, China (xiaojiliu72@126.com).

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Then there exists a unitary matrix $\mathbf{U} \in \mathbb{C}_{n,n}$ *such that*

$$\mathbf{P}_1 = \mathbf{U} \begin{bmatrix} \mathbf{I} & & & & & \\ & \mathbf{0} & & & & \\ & & \mathbf{I} & & & \\ & & & \mathbf{I} & & \\ & & & & \mathbf{0} & \\ & & & & & \mathbf{0} \end{bmatrix} \mathbf{U}^*, \quad \mathbf{P}_2 = \mathbf{U} \begin{bmatrix} \widehat{\mathbf{C}}^2 & \widehat{\mathbf{C}}\widehat{\mathbf{S}} & & & & \\ \widehat{\mathbf{C}}\widehat{\mathbf{S}} & \widehat{\mathbf{S}}^2 & & & & \\ & & \mathbf{I} & & & \\ & & & \mathbf{0} & & \\ & & & & \mathbf{I} & \\ & & & & & \mathbf{0} \end{bmatrix} \mathbf{U}^*,$$

where $\widehat{\mathbf{C}}$ *and* $\widehat{\mathbf{S}}$ *are positive diagonal real matrices such that* $\widehat{\mathbf{C}}^2 + \widehat{\mathbf{S}}^2 = \mathbf{I}$*, the symbol* $\mathbf{I}$ *denotes identity matrices of various sizes, and the corresponding blocks in the two projection matrices are of the same size.*

A square matrix $\mathbf{A} \in \mathbb{C}_{n,n}$ is an EP matrix (short for equal projection matrix) provided $\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger\mathbf{A}$. If $\mathbf{A}$ is an EP matrix, we shall denote by $\mathbf{P}_\mathbf{A}$ the orthogonal projector onto $\mathcal{R}(\mathbf{A})$, i.e., $\mathbf{P}_\mathbf{A} = \mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger\mathbf{A}$.

The following definition is borrowed from [24].

Definition 1.2. Let $\mathcal{X}$ and $\mathcal{Y}$ be two nontrivial subspaces of $\mathbb{C}^n$ and $r = \min\{\dim \mathcal{X}, \dim \mathcal{Y}\}$. We define the *canonical angles* $\theta_1, \ldots, \theta_r \in [0, \pi/2]$ between $\mathcal{X}$ and $\mathcal{Y}$ by

$$\cos\theta_i = \sigma_i(\mathbf{P}_\mathcal{X}\mathbf{P}_\mathcal{Y}), \quad i = 1, \ldots, r,$$

where the real numbers $\sigma_1(\mathbf{P}_\mathcal{X}\mathbf{P}_\mathcal{Y}), \ldots, \sigma_r(\mathbf{P}_\mathcal{X}\mathbf{P}_\mathcal{Y}) \geq 0$ are the r greatest singular values of the matrix $\mathbf{P}_\mathcal{X}\mathbf{P}_\mathcal{Y}$, and $\mathbf{P}_\mathcal{S}$ stands for the orthogonal projector onto the subspace $\mathcal{S} \subset \mathbb{C}^n$.

2. Simultaneous decomposition of two EP matrices. The main result of this paper is a simultaneous decomposition of two EP matrices $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{n,n}$ up to a unitarily equivalence. Without loss of generality we assume that $\mathrm{rk}(\mathbf{B}) \leq \mathrm{rk}(\mathbf{A})$.

Theorem 2.1. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{n,n}$ *be two EP matrices such that* $\mathrm{rk}(\mathbf{B}) \leq \mathrm{rk}(\mathbf{A})$. *Let* p *be the multiplicity of the canonical angle* 0*,* s *the multiplicity of the canonical angle* $\pi/2$*, and* $\theta_1, \ldots, \theta_r \in]0, \pi/2[$ *the remaining canonical angles between* $\mathcal{R}(\mathbf{A})$ *and* $\mathcal{R}(\mathbf{B})$. *Let* $q = \mathrm{rk}(\mathbf{A}) - (r + p)$. *Then there exists a unitary matrix* $\mathbf{U} \in \mathbb{C}_{n,n}$ *such that* $\mathbf{A}$ *and* $\mathbf{B}$ *can be written as*

$$\mathbf{A} = \mathbf{U} \left[\begin{array}{c|ccc|c} \mathbf{A}_1 & \mathbf{0} & \mathbf{A}_2 & \mathbf{A}_3 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0}_{r+s} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_4 & \mathbf{0} & \mathbf{A}_5 & \mathbf{A}_6 & \mathbf{0} \\ \hline \mathbf{A}_7 & \mathbf{0} & \mathbf{A}_8 & \mathbf{A}_9 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] \mathbf{U}^* \tag{2.1}$$

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and

$$\mathbf{B} = \mathbf{U} \begin{bmatrix} \mathbf{C}\mathbf{B}_1\mathbf{C} & \mathbf{C}\mathbf{B}_1\mathbf{S} & \mathbf{C}\mathbf{B}_2 & \mathbf{0} \\ \mathbf{S}^*\mathbf{B}_1\mathbf{C} & \mathbf{S}^*\mathbf{B}_1\mathbf{S} & \mathbf{S}^*\mathbf{B}_2 & \mathbf{0} \\ \mathbf{B}_3\mathbf{C} & \mathbf{B}_3\mathbf{S} & \mathbf{B}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{U}^*, \tag{2.2}$$

where

(i) $\mathbf{A}_1 \in \mathbb{C}_{r+s,r+s}$, $\mathbf{A}_5 \in \mathbb{C}_{q-s,q-s}$, $\mathbf{A}_9 \in \mathbb{C}_{p,p}$, $\mathbf{B}_1 \in \mathbb{C}_{r+s,r+s}$, $\mathbf{B}_2 \in \mathbb{C}_{r+s,p}$, $\mathbf{B}_3 \in \mathbb{C}_{p,r+s}$, *and* $\mathbf{B}_4 \in \mathbb{C}_{p,p}$.

(ii) $\mathbf{C} = \operatorname{diag}(\cos\theta_1, \ldots, \cos\theta_r) \oplus \mathbf{0}_s$, $\mathbf{S} = \begin{bmatrix} \operatorname{diag}(\sin\theta_1,\ldots,\sin\theta_r) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_s & \mathbf{0}_{s,q-s} \end{bmatrix}$,

(iii) $\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 \\ \mathbf{A}_4 & \mathbf{A}_5 & \mathbf{A}_6 \\ \mathbf{A}_7 & \mathbf{A}_8 & \mathbf{A}_9 \end{bmatrix}$ *and* $\begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_3 & \mathbf{B}_4 \end{bmatrix}$ *are nonsingular.*

Proof. By Lemma 1.1 applied to the orthogonal projectors $\mathbf{P_A}$ and $\mathbf{P_B}$, there exist $\theta_1, \ldots, \theta_r \in]0, \pi/2[$, $p, q, r, s \in \{0, 1, \ldots, n\}$, and a unitary matrix $\mathbf{U} \in \mathbb{C}_{n,n}$ such that

$$\mathbf{P_A} = \mathbf{U} \left(\begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \oplus \mathbf{I}_p \oplus \mathbf{I}_q \oplus \mathbf{0} \oplus \mathbf{0} \right) \mathbf{U}^*$$

and

$$\mathbf{P_B} = \mathbf{U} \left(\begin{bmatrix} \widehat{\mathbf{C}}^2 & \widehat{\mathbf{C}}\widehat{\mathbf{S}} \\ \widehat{\mathbf{C}}\widehat{\mathbf{S}} & \widehat{\mathbf{S}}^2 \end{bmatrix} \oplus \mathbf{I}_p \oplus \mathbf{0} \oplus \mathbf{I}_s \oplus \mathbf{0} \right) \mathbf{U}^*,$$

where $\widehat{\mathbf{C}} = \operatorname{diag}(\cos\theta_1, \ldots, \cos\theta_r)$ and $\widehat{\mathbf{S}} = \operatorname{diag}(\sin\theta_1, \ldots, \sin\theta_r)$. Evidently, we have

$$\widehat{\mathbf{C}}\widehat{\mathbf{S}} = \widehat{\mathbf{S}}\widehat{\mathbf{C}} \qquad \text{and} \qquad \widehat{\mathbf{C}}^2 + \widehat{\mathbf{S}}^2 = \mathbf{I}_r. \tag{2.3}$$

It is clear that $\operatorname{rk}(\mathbf{A}) = \operatorname{rk}(\mathbf{P_A}) = r + p + q$. Also, it is easy to see that the matrix $\begin{bmatrix} \widehat{\mathbf{C}} & -\widehat{\mathbf{S}} \\ \widehat{\mathbf{S}} & \widehat{\mathbf{C}} \end{bmatrix}$ is unitary and in particular is nonsingular. By using

$$\begin{bmatrix} \widehat{\mathbf{C}} & -\widehat{\mathbf{S}} \\ \widehat{\mathbf{S}} & \widehat{\mathbf{C}} \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{C}} & \widehat{\mathbf{S}} \\ \mathbf{0} & \mathbf{0}_r \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{C}}^2 & \widehat{\mathbf{C}}\widehat{\mathbf{S}} \\ \widehat{\mathbf{C}}\widehat{\mathbf{S}} & \widehat{\mathbf{S}}^2 \end{bmatrix}$$

and $\operatorname{rk}\left(\begin{bmatrix} \widehat{\mathbf{C}} & \widehat{\mathbf{S}} \\ \mathbf{0} & \mathbf{0}_r \end{bmatrix} \right) = r$, we get $\operatorname{rk}(\mathbf{B}) = \operatorname{rk}(\mathbf{P_B}) = r + p + s$. By the rank hypothesis, we obtain $s \leq q$. We shall denote $\mathbf{N} = [\mathbf{I}_s \mid \mathbf{0}_{s,q-s}] \in \mathbb{C}_{s,q}$,

$$\mathbf{C} = \begin{bmatrix} \widehat{\mathbf{C}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_s \end{bmatrix} \in \mathbb{C}_{r+s,r+s} \qquad \text{and} \qquad \mathbf{S} = \begin{bmatrix} \widehat{\mathbf{S}} & \mathbf{0} \\ \mathbf{0} & \mathbf{N} \end{bmatrix} \in \mathbb{C}_{r+s,r+q}.$$

Evidently,

$$\begin{bmatrix} \mathbf{C}^2 & \mathbf{C}\mathbf{S} \\ \mathbf{S}^*\mathbf{C} & \mathbf{S}^*\mathbf{S} \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{C}}^2 & \mathbf{0} & \widehat{\mathbf{C}}\widehat{\mathbf{S}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_s & \mathbf{0} & \mathbf{0}_{s,q} \\ \widehat{\mathbf{C}}\widehat{\mathbf{S}} & \mathbf{0} & \widehat{\mathbf{S}}^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{q,s} & \mathbf{0} & \mathbf{N}^*\mathbf{N} \end{bmatrix}, \qquad \mathbf{N}^*\mathbf{N} = \begin{bmatrix} \mathbf{I}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{q-s} \end{bmatrix}.$$

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By means of a suitable permutation, we can assume that

$$\mathbf{P_A} = \mathbf{U}\left(\begin{bmatrix} \mathbf{I}_{r+s} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{r+s} \oplus \mathbf{I}_{q-s} \end{bmatrix} \oplus \mathbf{I}_p \oplus \mathbf{0}\right)\mathbf{U}^* \tag{2.4}$$

and

$$\mathbf{P_B} = \mathbf{U}\left(\begin{bmatrix} \mathbf{C}^2 & \mathbf{CS} \\ \mathbf{S}^*\mathbf{C} & \mathbf{S}^*\mathbf{S} \end{bmatrix} \oplus \mathbf{I}_p \oplus \mathbf{0}\right)\mathbf{U}^*. \tag{2.5}$$

Let $m = 2r + s + q$. Observe that the first summands in (2.4) and (2.5) are $m \times m$ matrices. We partition $\mathbf{A}$ as follows:

$$\mathbf{A} = \mathbf{U}\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 \\ \mathbf{A}_4 & \mathbf{A}_5 & \mathbf{A}_6 \\ \mathbf{A}_7 & \mathbf{A}_8 & \mathbf{A}_9 \end{bmatrix}\mathbf{U}^*, \qquad \mathbf{A}_1 \in \mathbb{C}_{m,m},\ \mathbf{A}_5 \in \mathbb{C}_{p,p}. \tag{2.6}$$

Since $\mathbf{A}$ is EP, it follows $\mathbf{A} = \mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{P_A}\mathbf{A}$ and $\mathbf{A} = \mathbf{A}\mathbf{P_A}$. From (2.4) and (2.6) we obtain the blocks $\mathbf{A}_3, \mathbf{A}_6, \mathbf{A}_7, \mathbf{A}_8, \mathbf{A}_9$ are zero, and if we denote

$$\mathbf{P} = \begin{bmatrix} \mathbf{I}_{r+s} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{r+s} \oplus \mathbf{I}_{q-s} \end{bmatrix} \in \mathbb{C}_{m,m}, \tag{2.7}$$

then we get

$$\mathbf{A}_1 = \mathbf{P}\mathbf{A}_1 = \mathbf{A}_1\mathbf{P}, \qquad \mathbf{A}_2 = \mathbf{P}\mathbf{A}_2, \qquad \mathbf{A}_4 = \mathbf{A}_4\mathbf{P}.$$

By employing these equalities, $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_4$ can be written as

$$\mathbf{A}_1 = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{0} & \mathbf{A}_{22} \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} \mathbf{A}_{13} \\ \mathbf{0} \\ \mathbf{A}_{23} \end{bmatrix}, \quad \mathbf{A}_4 = \begin{bmatrix} \mathbf{A}_{31} & \mathbf{0} & \mathbf{A}_{32} \end{bmatrix},$$

where $\mathbf{A}_{11} \in \mathbb{C}_{r+s,r+s}, \mathbf{A}_{22} \in \mathbb{C}_{q-s,q-s}, \mathbf{A}_{13} \in \mathbb{C}_{r+s,p}, \mathbf{A}_{23} \in \mathbb{C}_{q-s,p}, \mathbf{A}_{31} \in \mathbb{C}_{p,r+s}$, and $\mathbf{A}_{32} \in \mathbb{C}_{p,q-s}$. Thus, (by a renaming of the subindexes) $\mathbf{A}$ can be written as in (2.1). Furthermore, the size of $\mathbf{\Delta} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_5 \end{bmatrix}$ is $r + q + p$, which is equal to the rank of $\mathbf{A}$. Also it is evident that $\mathrm{rk}(\mathbf{A}) = \mathrm{rk}(\mathbf{\Delta})$, which implies that $\mathbf{\Delta}$ is nonsingular.

We partition $\mathbf{B}$ as follows:

$$\mathbf{B} = \mathbf{U}\begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 \\ \mathbf{B}_4 & \mathbf{B}_5 & \mathbf{B}_6 \\ \mathbf{B}_7 & \mathbf{B}_8 & \mathbf{B}_9 \end{bmatrix}\mathbf{U}^*, \qquad \mathbf{B}_1 \in \mathbb{C}_{m,m},\ \mathbf{B}_5 \in \mathbb{C}_{p,p}.$$

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As for $\mathbf{A}$, the blocks $\mathbf{B}_3, \mathbf{B}_6, \mathbf{B}_7, \mathbf{B}_8$, and $\mathbf{B}_9$ are zero, and if we denote

$$\mathbf{Q} = \begin{bmatrix} \mathbf{C}^2 & \mathbf{CS} \\ \mathbf{S}^*\mathbf{C} & \mathbf{S}^*\mathbf{S} \end{bmatrix}, \tag{2.8}$$

then

$$\mathbf{B}_1 = \mathbf{QB}_1, \qquad \mathbf{B}_1 = \mathbf{B}_1\mathbf{Q}, \qquad \mathbf{B}_2 = \mathbf{QB}_2, \qquad \mathbf{B}_4 = \mathbf{B}_4\mathbf{Q}. \tag{2.9}$$

Let us partition

$$\mathbf{B}_1 = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}, \qquad \mathbf{B}_{11} \in \mathbb{C}_{r+s,r+s}.$$

From the first equality of (2.9), we obtain

$$\mathbf{B}_{11} = \mathbf{C}(\mathbf{CB}_{11} + \mathbf{SB}_{21}), \quad \mathbf{B}_{12} = \mathbf{C}(\mathbf{CB}_{12} + \mathbf{SB}_{22}), \quad \mathbf{B}_{21} = \mathbf{S}^*(\mathbf{CB}_{11} + \mathbf{SB}_{21}),$$

and

$$\mathbf{B}_{22} = \mathbf{S}^*(\mathbf{CB}_{12} + \mathbf{SB}_{22}).$$

If we denote $\mathbf{H}_1 = \mathbf{CB}_{11} + \mathbf{SB}_{21}$ and $\mathbf{H}_2 = \mathbf{CB}_{12} + \mathbf{SB}_{22}$, then $\mathbf{B}_1 = \begin{bmatrix} \mathbf{CH}_1 & \mathbf{CH}_2 \\ \mathbf{S}^*\mathbf{H}_1 & \mathbf{S}^*\mathbf{H}_2 \end{bmatrix}$. From the second equality of (2.9) we get $\mathbf{CH}_1 = \mathbf{C}(\mathbf{H}_1\mathbf{C}+\mathbf{H}_2\mathbf{S}^*)\mathbf{C}$, $\mathbf{CH}_2 = \mathbf{C}(\mathbf{H}_1\mathbf{C}+\mathbf{H}_2\mathbf{S}^*)\mathbf{S}$, $\mathbf{S}^*\mathbf{H}_1 = \mathbf{S}^*(\mathbf{H}_1\mathbf{C} + \mathbf{H}_2\mathbf{S}^*)\mathbf{C}$, and $\mathbf{S}^*\mathbf{H}_2 = \mathbf{S}^*(\mathbf{H}_1\mathbf{C} + \mathbf{H}_2\mathbf{S}^*)\mathbf{S}$. If we define $\mathbf{H} = \mathbf{H}_1\mathbf{C} + \mathbf{H}_2\mathbf{S}^*$, then $\mathbf{B}_1$ can be written as

$$\mathbf{B}_1 = \begin{bmatrix} \mathbf{CHC} & \mathbf{CHS} \\ \mathbf{S}^*\mathbf{HC} & \mathbf{S}^*\mathbf{HS} \end{bmatrix}, \qquad \mathbf{H} \in \mathbb{C}_{r+s,r+s}.$$

In the same way, by using the third and fourth equalities of (2.9), there exist $\mathbf{E} \in \mathbb{C}_{r+s,p}$ and $\mathbf{F} \in \mathbb{C}_{p,r+s}$ such that

$$\mathbf{B}_2 = \begin{bmatrix} \mathbf{CE} \\ \mathbf{S}^*\mathbf{E} \end{bmatrix}, \qquad \mathbf{B}_4 = \begin{bmatrix} \mathbf{FC} & \mathbf{FS} \end{bmatrix}.$$

Thus, by renaming the blocks, $\mathbf{B}$ can be written as in (2.2).

To find the canonical angles between $\mathcal{R}(\mathbf{A})$ and $\mathcal{R}(\mathbf{B})$, we appeal to Definition 1.2. From (2.4), (2.5), (2.7) and (2.8), obviously,

$$\mathbf{P_A P_B} = \mathbf{U}(\mathbf{PQ} \oplus \mathbf{I}_p \oplus \mathbf{0})\mathbf{U}^*, \tag{2.10}$$

and if we denote $\mathbf{X} = \mathbf{0}_{r+s} \oplus \mathbf{I}_{q-s}$, then

$$\mathbf{PQ} = \begin{bmatrix} \mathbf{I}_{r+s} & \mathbf{0} \\ \mathbf{0} & \mathbf{X} \end{bmatrix} \begin{bmatrix} \mathbf{C}^2 & \mathbf{CS} \\ \mathbf{S}^*\mathbf{C} & \mathbf{S}^*\mathbf{S} \end{bmatrix} = \begin{bmatrix} \mathbf{C}^2 & \mathbf{CS} \\ \mathbf{XS}^*\mathbf{C} & \mathbf{XS}^*\mathbf{S} \end{bmatrix}.$$

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But we have

$$\mathbf{XS}^* = \begin{bmatrix} \mathbf{0}_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{q-s} \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{S}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_s \\ \mathbf{0} & \mathbf{0}_{q-s,s} \end{bmatrix} = \mathbf{0}.$$

Furthermore, from the definitions of $\mathbf{C}$ and $\mathbf{S}$, and (2.3), we have

$$\mathbf{C}^2 + \mathbf{SS}^* = \begin{bmatrix} \widehat{\mathbf{C}}^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_s \end{bmatrix} + \begin{bmatrix} \widehat{\mathbf{S}}^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{NN}^* \end{bmatrix} = \mathbf{I}_r \oplus \mathbf{I}_s = \mathbf{I}_{r+s},$$

$$\begin{aligned} \mathbf{S}(\mathbf{C} \oplus \mathbf{I}_{q-s}) &= \begin{bmatrix} \widehat{\mathbf{S}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_s & \mathbf{0}_{s,q-s} \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{C}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{q-s} \end{bmatrix} \\ &= \begin{bmatrix} \widehat{\mathbf{C}}\widehat{\mathbf{S}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_s & \mathbf{0}_{s,q-s} \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{C}}\widehat{\mathbf{S}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{s,q} \end{bmatrix} = \mathbf{CS}, \end{aligned}$$

and

$$\mathbf{S}^*\mathbf{S} + \left(\mathbf{C}^2 \oplus \mathbf{I}_{q-s}\right) = \begin{bmatrix} \widehat{\mathbf{S}}^2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{q-s} \end{bmatrix} + \begin{bmatrix} \widehat{\mathbf{C}}^2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{q-s} \end{bmatrix} = \mathbf{I}_{r+q},$$

which imply that $\left[\begin{smallmatrix} \mathbf{C} & \mathbf{S} \\ -\mathbf{S}^* & \mathbf{C} \oplus \mathbf{I}_{q-s} \end{smallmatrix}\right]$ is unitary. From now on, we let

$$\mathbf{W} = \begin{bmatrix} \mathbf{C} & \mathbf{S} \\ -\mathbf{S}^* & \mathbf{C} \oplus \mathbf{I}_{q-s} \end{bmatrix}.$$

Hence, the fact that $\mathbf{C}$ is diagonal and

$$\mathbf{PQ} = \begin{bmatrix} \mathbf{C}^2 & \mathbf{CS} \\ \mathbf{0} & \mathbf{0}_{r+q} \end{bmatrix} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{r+q} \end{bmatrix} \begin{bmatrix} \mathbf{C} & \mathbf{S} \\ -\mathbf{S}^* & \mathbf{C} \oplus \mathbf{I}_{q-s} \end{bmatrix}$$

yield that $\cos\theta_1, \dots, \cos\theta_r$, and 0 (repeated $s + r + q$ times) are the singular values of $\mathbf{PQ}$. From (2.10), we obtain that $\cos\theta_1, \dots, \cos\theta_r$; 0 (repeated $s + r + q$ times); and 1 (repeated p times) are singular values of $\mathbf{P_A P_B}$. Since there must be $\min\{\mathrm{rk}(\mathbf{A}), \mathrm{rk}(\mathbf{B})\} = \mathrm{rk}(\mathbf{B}) = r + p + s$ canonical angles between $\mathcal{R}(\mathbf{A})$ and $\mathcal{R}(\mathbf{B})$, these canonical angles are 0 (repeated p times), $\theta_1, \dots, \theta_r$, and $\pi/2$ (repeated s times).

It remains to prove that (maintaining the notation of the proof) that $\mathbf{\Lambda} = \left[\begin{smallmatrix} \mathbf{H} & \mathbf{E} \\ \mathbf{F} & \mathbf{B}_5 \end{smallmatrix}\right]$ is nonsingular. To this end, since $\mathbf{\Lambda}$ is $(r+s+p) \times (r+s+p)$, it is sufficient to prove

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$\mathrm{rk}(\mathbf{\Lambda}) = r + s + p$. Observe that

$$\mathbf{U}^*\mathbf{B}\mathbf{U} = \begin{bmatrix} \mathbf{CHC} & \mathbf{CHS} & \mathbf{CE} & \mathbf{0} \\ \mathbf{S}^*\mathbf{HC} & \mathbf{S}^*\mathbf{HS} & \mathbf{S}^*\mathbf{E} & \mathbf{0} \\ \mathbf{FC} & \mathbf{FS} & \mathbf{B}_5 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{C} & -\mathbf{S} & \mathbf{0} & \mathbf{0} \\ \mathbf{S}^* & \mathbf{C}\oplus\mathbf{I}_{q-s} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{H} & \mathbf{0} & \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{r+q} & \mathbf{0} & \mathbf{0} \\ \mathbf{F} & \mathbf{0} & \mathbf{B}_5 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{C} & \mathbf{S} & \mathbf{0} & \mathbf{0} \\ -\mathbf{S}^* & \mathbf{C}\oplus\mathbf{I}_{q-s} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix},$$

where $\mathbf{I}$ denotes an identity matrix of suitable size. In addition, since $\mathbf{W}$ is unitary and in particular is nonsingular, we get $r + p + s = \mathrm{rk}(\mathbf{B}) = \mathrm{rk}\left(\left[\begin{smallmatrix} \mathbf{H} & \mathbf{E} \\ \mathbf{F} & \mathbf{B}_5 \end{smallmatrix}\right]\right)$. ∎

Two particular cases are described in next two results.

COROLLARY 2.2. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{n,n}$ be two EP matrices. The following statements are equivalent:*

(i) $\mathbf{P_A}\mathbf{P_B} = \mathbf{P_B}\mathbf{P_A}$.
(ii) *The matrices $\mathbf{A}$ and $\mathbf{B}$ can be written as*

$$\mathbf{A} = \mathbf{U}\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_3 & \mathbf{A}_4 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{U}^*, \qquad \mathbf{B} = \mathbf{U}\begin{bmatrix} \mathbf{0}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_4 & \mathbf{B}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 & \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{U}^*, \tag{2.11}$$

where $\mathbf{U} \in \mathbb{C}_{n,n}$ is unitary, $\mathbf{A}_1 \in \mathbb{C}_{q,q}$, $\mathbf{A}_4, \mathbf{B}_4 \in \mathbb{C}_{p,p}$, $\mathbf{B}_1 \in \mathbb{C}_{s,s}$, $\left[\begin{smallmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \mathbf{B}_4 & \mathbf{B}_3 \\ \mathbf{B}_2 & \mathbf{B}_1 \end{smallmatrix}\right]$ are nonsingular.

Proof. The proof of (ii) ⇒ (i) is trivial. Let us prove the converse. We can clearly assume that $\mathrm{rk}(\mathbf{B}) \geq \mathrm{rk}(\mathbf{A})$. Since $\mathbf{P_A}\mathbf{P_B}$ is an orthogonal projector, its singular values are 0 or 1. Hence, there is no canonical angle between $\mathcal{R}(\mathbf{A})$ and $\mathcal{R}(\mathbf{B})$ in $]0, \pi/2[$. By Theorem 2.1, we can write matrices $\mathbf{A}$ and $\mathbf{B}$ as in (2.1) and (2.2) with $r = 0$, and therefore, $\mathbf{C} = \mathbf{0}_s$ and $\mathbf{S} = [\,\mathbf{I}_s \ \mathbf{0}_{s,q-s}\,]$. Evidently, we get

$$\mathbf{S}^*\mathbf{B}_1\mathbf{S} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{q-s} \end{bmatrix}, \qquad \mathbf{B}_3\mathbf{S} = \begin{bmatrix} \mathbf{B}_3 & \mathbf{0}_{p,q-s} \end{bmatrix}, \qquad \mathbf{S}^*\mathbf{B}_2 = \begin{bmatrix} \mathbf{B}_2 \\ \mathbf{0}_{q-s,p} \end{bmatrix}.$$

By a simultaneous permutation of the rows and the columns of $\mathbf{A}$ and $\mathbf{B}$, we get

$$\mathbf{A} = \mathbf{U}\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_4 & \mathbf{A}_5 & \mathbf{A}_6 & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_7 & \mathbf{A}_8 & \mathbf{A}_9 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{U}^*, \quad \mathbf{B} = \mathbf{U}\begin{bmatrix} \mathbf{0}_s & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{q-s} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_4 & \mathbf{B}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_2 & \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{U}^*.$$

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By joining some blocks of the matrices $\mathbf{A}$ and $\mathbf{B}$, and renaming the blocks of these matrices, we obtain the theorem. □

Based on Theorem 2.1, we now give a simpler proof of Corollary 2.3 than that given in [4, Corollary 3.9]

COROLLARY 2.3. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{n,n}$ be two EP matrices. The following statements are equivalent:*

(i) $\mathbf{AB} = \mathbf{BA}$.
(ii) *The matrices $\mathbf{A}$ and $\mathbf{B}$ can be written as*

$$\mathbf{A} = \mathbf{U}(\mathbf{A}_1 \oplus \mathbf{A}_2 \oplus \mathbf{0} \oplus \mathbf{0})\mathbf{U}^*, \qquad \mathbf{B} = \mathbf{U}(\mathbf{0} \oplus \mathbf{B}_1 \oplus \mathbf{B}_2 \oplus \mathbf{0})\mathbf{U}^*,$$

where $\mathbf{U} \in \mathbb{C}_{n,n}$ is unitary, $\mathbf{A}_1 \in \mathbb{C}_{q,q}$, $\mathbf{A}_2, \mathbf{B}_1 \in \mathbb{C}_{p,p}$, and $\mathbf{B}_2 \in \mathbb{C}_{s,s}$ are nonsingular matrices such that $\mathbf{A}_1\mathbf{B}_1 = \mathbf{B}_1\mathbf{A}_1$.

Proof. We only prove (i) ⇒ (ii) because the other implication is trivial. Since $\mathbf{A}$ is EP, there exists a unitary matrix $\mathbf{V}$ such that $\mathbf{A} = \mathbf{V}(\mathbf{K} \oplus \mathbf{0})\mathbf{V}^*$, where $\mathbf{K} \in \mathbb{C}_{k,k}$ is nonsingular (see e.g. [9, Section 4.3]). From $\mathbf{AB} = \mathbf{BA}$ we get that $\mathbf{B}$ can be written as $\mathbf{B} = \mathbf{V}(\mathbf{X} \oplus \mathbf{Y})\mathbf{V}^*$ with $\mathbf{X} \in \mathbb{C}_{k,k}$. Since $\mathbf{P_A} = \mathbf{V}(\mathbf{I}_k \oplus \mathbf{0})\mathbf{V}^*$ and $\mathbf{P_B} = \mathbf{V}(\mathbf{XX}^\dagger \oplus \mathbf{YY}^\dagger)\mathbf{V}^*$ we obtain $\mathbf{P_A P_B} = \mathbf{P_B P_A}$. Corollary 2.2 yields that matrices $\mathbf{A}$ and $\mathbf{B}$ can be written as in (2.11). From $\mathbf{AB} = \mathbf{BA}$ we get $\mathbf{A}_2\mathbf{B}_4 = \mathbf{0}$, $\mathbf{A}_2\mathbf{B}_3 = \mathbf{0}$, $\mathbf{A}_4\mathbf{B}_3 = \mathbf{0}$, $\mathbf{B}_4\mathbf{A}_3 = \mathbf{0}$, $\mathbf{B}_2\mathbf{A}_3 = \mathbf{0}$, and $\mathbf{B}_2\mathbf{A}_4 = \mathbf{0}$. From these, we get

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{B}_4 & \mathbf{B}_3 \\ \mathbf{B}_2 & \mathbf{B}_1 \end{bmatrix} \begin{bmatrix} \mathbf{A}_3 \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

and

$$\begin{bmatrix} \mathbf{0} & \mathbf{B}_2 \end{bmatrix} \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_2 & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{B}_4 & \mathbf{B}_3 \\ \mathbf{B}_2 & \mathbf{B}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

The nonsingularity of $\left[\begin{smallmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \mathbf{B}_4 & \mathbf{B}_3 \\ \mathbf{B}_2 & \mathbf{B}_1 \end{smallmatrix}\right]$ imply that $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{B}_2$, and $\mathbf{B}_3$ are null. By a renaming blocks, the proof is concluded. □

3. Some applications. In this section, we obtain several results based on Theorem 2.1.

The following lemma concerns representations based on Theorem 2.1 of two EP matrices and their Moore-Penrose inverses under certain conditions and will be used extensively in the computations below.

LEMMA 3.1. *Let $\mathbf{A}$ and $\mathbf{B} \in \mathbb{C}_{n,n}$ be two EP matrices such that* $\mathrm{rk}(\mathbf{A}) = \mathrm{rk}(\mathbf{B})$ *and $\mathbf{P_A} - \mathbf{P_B}$ is nonsingular. Then $\mathbf{C} = \mathrm{diag}(\cos\theta_1, \ldots, \cos\theta_r) \oplus \mathbf{0}_s$, $\mathbf{S} =$*

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$\mathrm{diag}(\sin\theta_1, \ldots, \sin\theta_r) \oplus \mathbf{I}_s$,

$$\mathbf{A} = \mathbf{U} \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{r+s} \end{bmatrix} \mathbf{U}^*, \qquad \mathbf{B} = \mathbf{U} \begin{bmatrix} \mathbf{C}\mathbf{B}_1\mathbf{C} & \mathbf{C}\mathbf{B}_1\mathbf{S} \\ \mathbf{S}^*\mathbf{B}_1\mathbf{C} & \mathbf{S}^*\mathbf{B}_1\mathbf{S} \end{bmatrix} \mathbf{U}^*, \tag{3.1}$$

and

$$\mathbf{A}^\dagger = \mathbf{U} \begin{bmatrix} \mathbf{A}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{r+s} \end{bmatrix} \mathbf{U}^*, \qquad \mathbf{B}^\dagger = \mathbf{U} \begin{bmatrix} \mathbf{C}\mathbf{B}_1^{-1}\mathbf{C} & \mathbf{C}\mathbf{B}_1^{-1}\mathbf{S} \\ \mathbf{S}^*\mathbf{B}_1^{-1}\mathbf{C} & \mathbf{S}^*\mathbf{B}_1^{-1}\mathbf{S} \end{bmatrix} \mathbf{U}^*. \tag{3.2}$$

Proof. We write $\mathbf{A}$ and $\mathbf{B}$ as in Theorem 2.1. Since $\mathbf{P_A} - \mathbf{P_B}$ is nonsingular, $2r + q + s = n$ and $p = 0$. Since $\mathrm{rk}(\mathbf{A}) = \mathrm{rk}(\mathbf{B})$, then $q = s$. Therefore, $\mathbf{C}$, $\mathbf{S}$, $\mathbf{A}$, and $\mathbf{B}$ can be expressed as in the statement of the theorem. The representation of $\mathbf{A}^\dagger$ is evident. Since $\left[\begin{smallmatrix} \mathbf{C} & \mathbf{S} \\ -\mathbf{S}^* & \mathbf{C} \end{smallmatrix}\right]$ is unitary,

$$\begin{aligned} \mathbf{B}^\dagger &= \left(\mathbf{U} \begin{bmatrix} \mathbf{C} & -\mathbf{S} \\ \mathbf{S}^* & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{C} & \mathbf{S} \\ -\mathbf{S}^* & \mathbf{C} \end{bmatrix} \mathbf{U}^* \right)^\dagger \\ &= \mathbf{U} \begin{bmatrix} \mathbf{C} & -\mathbf{S} \\ \mathbf{S}^* & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}^\dagger \begin{bmatrix} \mathbf{C} & \mathbf{S} \\ -\mathbf{S}^* & \mathbf{C} \end{bmatrix} \mathbf{U}^* \\ &= \mathbf{U} \begin{bmatrix} \mathbf{C} & -\mathbf{S} \\ \mathbf{S}^* & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{B}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{C} & \mathbf{S} \\ -\mathbf{S}^* & \mathbf{C} \end{bmatrix} \mathbf{U}^* \\ &= \mathbf{U} \begin{bmatrix} \mathbf{C}\mathbf{B}_1^{-1}\mathbf{C} & \mathbf{C}\mathbf{B}_1^{-1}\mathbf{S} \\ \mathbf{S}^*\mathbf{B}_1^{-1}\mathbf{C} & \mathbf{S}^*\mathbf{B}_1^{-1}\mathbf{S} \end{bmatrix} \mathbf{U}^*. \end{aligned} \tag{3.3}$$

The proof is finished. ◻

Let $\mathbf{A}$ and $\mathbf{B}$ be two EP matrices, and let p, r, s and q have the same meaning as in Theorem 2.1. If $\mathrm{rk}(\mathbf{A}) \leq \mathrm{rk}(\mathbf{B})$, then

$$\mathrm{rk}(\mathbf{P_A} - \mathbf{P_B}) = \mathrm{rk}(\mathbf{A}) + \mathrm{rk}(\mathbf{B}) - 2p = 2r + q + s \tag{3.4}$$

and

$$\mathrm{rk}(\mathbf{P_A}\mathbf{P_B} - \mathbf{P_B}\mathbf{P_A}) = 2r. \tag{3.5}$$

The reader is referred to [11] for a deeper insight of a pair of orthogonal projectors, concretely, the equalities (3.4) and (3.5) appeared as part of Theorem 26 and Corollary 55 in [11]. In particular, we have that the nonsingularity of $\mathbf{P_A} - \mathbf{P_B}$ implies $p = 0$; and the nonsingularity of $\mathbf{P_A}\mathbf{P_B} - \mathbf{P_B}\mathbf{P_A}$ implies $q = s = p = 0$, $\mathrm{rk}(\mathbf{A}) = \mathrm{rk}(\mathbf{B}) = r$. Hence, the nonsingularity of $\mathbf{P_A}\mathbf{P_B} - \mathbf{P_A}\mathbf{P_B}$ yields the nonsingularity of $\mathbf{P_A} - \mathbf{P_B}$ (observe that we can drop $\mathrm{rk}(\mathbf{A}) \leq \mathrm{rk}(\mathbf{B})$). In next Corollary 3.2 we will give a kind of converse. To establish this converse, we need the concept of co-EP matrices, which

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will be immediately defined. Following [6], we say that $\mathbf{X} \in \mathbb{C}_{n,n}$ is *co-EP* when $\mathbf{X}\mathbf{X}^\dagger - \mathbf{X}^\dagger\mathbf{X}$ is nonsingular.

Corollary 3.2. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{n,n}$ *be two EP matrices such that* $\mathrm{rk}(\mathbf{A}) = \mathrm{rk}(\mathbf{B})$ *and* $\mathbf{P_A} - \mathbf{P_B}$ *is nonsingular. Then the following statements are equivalent:*

(i) $\mathbf{P_A}\mathbf{P_B} - \mathbf{P_B}\mathbf{P_A}$ *is nonsingular.*
(ii) $\mathbf{AB} - \mathbf{BA}$ *is nonsingular.*
(iii) $\mathbf{AB}$ *is co-EP.*

Under this equivalence one has $\mathbf{AB}(\mathbf{AB})^\dagger = \mathbf{P_A}$ *and* $(\mathbf{AB})^\dagger\mathbf{AB} = \mathbf{P_B}$.

Proof. Maintaining the notation of Theorem 2.1, the hypotheses of the corollary yield $q = s$ and $2r + q + s = n$.

(i) ⇔ (ii): By (3.1), we have

$$\mathbf{AB} - \mathbf{BA} = \mathbf{U}\begin{bmatrix} \mathbf{A}_1\mathbf{C}\mathbf{B}_1\mathbf{C} - \mathbf{C}\mathbf{B}_1\mathbf{C}\mathbf{A}_1 & \mathbf{A}_1\mathbf{C}\mathbf{B}_1\mathbf{S} \\ -\mathbf{S}^*\mathbf{B}_1\mathbf{C}\mathbf{A}_1 & \mathbf{0} \end{bmatrix}\mathbf{U}^*.$$

Hence, $\det(\mathbf{AB}-\mathbf{BA}) = \det(\mathbf{A}_1)^2 \det(\mathbf{B}_1)^2 \det(\mathbf{C})^2 \det(\mathbf{S})^2$. Recall that Theorem 2.1 implies that $\mathbf{A}_1$ and $\mathbf{B}_1$ are both nonsingular. Lemma 3.1 yields that $\mathbf{S}$ is nonsingular. Hence, $\mathbf{AB} - \mathbf{BA}$ is nonsingular ⇔ $\mathbf{C}$ is nonsingular ⇔ $s = 0$ ⇔ $2r = n$ ⇔ $\mathbf{P_A}\mathbf{P_B} - \mathbf{P_B}\mathbf{P_A}$ is nonsingular.

(i) ⇒ (iii): If we set $\mathbf{X} = \mathbf{A}_1\mathbf{C}\mathbf{B}_1$, then (3.1) yields

$$\mathbf{AB} = \mathbf{U}\begin{bmatrix} \mathbf{XC} & \mathbf{XS} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{U}^* = \mathbf{U}\begin{bmatrix} \mathbf{X} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{C} & \mathbf{S} \\ -\mathbf{S}^* & \mathbf{C} \end{bmatrix}\mathbf{U}^*. \tag{3.6}$$

Since $\left[\begin{smallmatrix} \mathbf{C} & \mathbf{S} \\ -\mathbf{S}^* & \mathbf{C} \end{smallmatrix}\right]$ is unitary, it follows

$$(\mathbf{AB})^\dagger = \mathbf{U}\begin{bmatrix} \mathbf{C} & -\mathbf{S} \\ \mathbf{S}^* & \mathbf{C} \end{bmatrix}\begin{bmatrix} \mathbf{X}^\dagger & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{U}^* = \mathbf{U}\begin{bmatrix} \mathbf{C}\mathbf{X}^\dagger & \mathbf{0} \\ \mathbf{S}^*\mathbf{X}^\dagger & \mathbf{0} \end{bmatrix}\mathbf{U}^*. \tag{3.7}$$

Since $\mathbf{P_A}\mathbf{P_B} - \mathbf{P_B}\mathbf{P_A}$ is nonsingular, $s = 0$. Hence, $\mathbf{C}$ is nonsingular, and from the definition of matrix $\mathbf{X}$, we can see that $\mathbf{X}$ is nonsingular. Equalities (2.4) and (2.5), and the above representations of $\mathbf{AB}$ and $(\mathbf{AB})^\dagger$ lead to

$$\mathbf{AB}(\mathbf{AB})^\dagger = \mathbf{U}\begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_r \end{bmatrix}\mathbf{U}^* = \mathbf{P_A}, \quad (\mathbf{AB})^\dagger\mathbf{AB} = \mathbf{U}\begin{bmatrix} \mathbf{C}^2 & \mathbf{CS} \\ \mathbf{S}^*\mathbf{C} & \mathbf{S}^*\mathbf{S} \end{bmatrix}\mathbf{U}^* = \mathbf{P_B}.$$

Thus, $\mathbf{AB}(\mathbf{AB})^\dagger - (\mathbf{AB})^\dagger\mathbf{AB} = \mathbf{P_A} - \mathbf{P_B}$ is nonsingular, i.e., $\mathbf{AB}$ is co-EP.

(iii) ⇒ (i): Assume that $\mathbf{AB}$ is co-EP. Theorem 2.3 of [6] implies that $(\mathbf{AB}) + (\mathbf{AB})^*$ is nonsingular. From (3.1) we have $(\mathbf{AB}) + (\mathbf{AB})^* = \mathbf{U}\left[\begin{smallmatrix} \mathbf{XC}+\mathbf{CX}^* & \mathbf{XS} \\ \mathbf{S}^*\mathbf{X}^* & \mathbf{0} \end{smallmatrix}\right]\mathbf{U}^*$, where $\mathbf{X} = \mathbf{A}_1\mathbf{C}\mathbf{B}_1$. The nonsingularity of $(\mathbf{AB}) + (\mathbf{AB})^*$ yields the nonsingularity

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of $\mathbf{XS}$, which implies that $\mathbf{X}$ is nonsingular. Hence, $\mathbf{C}$ is nonsingular, which implies that $\mathbf{P_A P_B} - \mathbf{P_B P_A}$ is nonsingular. ☐

A square matrix $\mathbf{A}$ is said to be *group invertible* if there exists a matrix $\mathbf{X}$ such that

$$\mathbf{AXA} = \mathbf{A}, \qquad \mathbf{XAX} = \mathbf{X}, \qquad \mathbf{XA} = \mathbf{AX}. \tag{3.8}$$

It can be proved (see e.g. [3, Chapter 4]) that for a square matrix $\mathbf{A}$ there is at most one matrix $\mathbf{X}$ satisfying (3.8). Such matrix will be denoted by $\mathbf{A}^{\#}$. We will use the following result due to Cline (see [10] or [3, Section 4.4]).

THEOREM 3.3. *Let a square matrix* $\mathbf{A}$ *have the full-rank factorization* $\mathbf{A} = \mathbf{FG}$. *Then* $\mathbf{A}$ *has a group inverse if and only if* $\mathbf{GF}$ *is nonsingular, in which case* $\mathbf{A}^{\#} = \mathbf{F}(\mathbf{GF})^{-2}\mathbf{G}$.

COROLLARY 3.4. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{n,n}$ *be EP matrices such that* $\mathbf{P_A P_B} - \mathbf{P_B P_A}$ *is nonsingular. Then* $\mathbf{AB}$ *is group invertible.*

Proof. By representing $\mathbf{A}$ and $\mathbf{B}$ as in (2.1), (2.2) and recalling that the nonsingularity of $\mathbf{P_A P_B} - \mathbf{P_B P_A}$ implies $q = s = p = 0$, in particular, the nonsingularity of $\mathbf{P_A} - \mathbf{P_B}$ and $\mathrm{rk}(\mathbf{A}) = \mathrm{rk}(\mathbf{B})$, we obtain from Lemma 3.1 that

$$\mathbf{AB} = \mathbf{U} \begin{bmatrix} \mathbf{XC} & \mathbf{XS} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{U}^* = \mathbf{U} \begin{bmatrix} \mathbf{X} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{C} & \mathbf{S} \end{bmatrix} \mathbf{U}^*, \tag{3.9}$$

where $\mathbf{X} = \mathbf{A}_1\mathbf{C}\mathbf{B}_1 \in \mathbb{C}_{r,r}$. We have $\mathrm{rk}(\mathbf{AB}) = \mathrm{rk}(\mathbf{XC}) = r$ because $\mathbf{X}$ and $\mathbf{C}$ are nonsingular. We let $\mathbf{F} = \mathbf{U}\left[\begin{smallmatrix} \mathbf{X} \\ \mathbf{0} \end{smallmatrix}\right] \in \mathbb{C}_{n,r}$ and $\mathbf{G} = \begin{bmatrix} \mathbf{C} & \mathbf{S} \end{bmatrix} \mathbf{U}^* \in \mathbb{C}_{r,n}$. Evidently $\mathrm{rk}(\mathbf{F}) = \mathrm{rk}(\mathbf{X}) = r$ because $\mathbf{X}$ is nonsingular. Furthermore, $r = \mathrm{rk}(\mathbf{I}_r) = \mathrm{rk}(\mathbf{GG}^*) = \mathrm{rk}(\mathbf{G})$. Hence, (3.9) is a full-rank factorization of $\mathbf{AB}$. From the nonsingularity of $\mathbf{CX}$ and Theorem 3.3, we get that $\mathbf{AB}$ is group invertible. ☐

Furthermore, Theorem 3.3 allows us to give a representation of $(\mathbf{AB})^{\#}$ under the hypotheses of Corollary 3.4.

$$(\mathbf{AB})^{\#} = \mathbf{U} \begin{bmatrix} \mathbf{X} \\ \mathbf{0} \end{bmatrix} (\mathbf{CX})^{-2} \begin{bmatrix} \mathbf{C} & \mathbf{S} \end{bmatrix} \mathbf{U}^* = \mathbf{U} \begin{bmatrix} \mathbf{C}^{-1}\mathbf{X}^{-1} & \mathbf{C}^{-1}\mathbf{X}^{-1}\mathbf{C}^{-1}\mathbf{S} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{U}^*, \tag{3.10}$$

where $\mathbf{X} = \mathbf{A}_1\mathbf{C}\mathbf{B}_1$.

EXAMPLE 1. The converse of Corollary 3.4 does not hold in general, as the following example shows. Let $\mathbf{A} = \mathrm{diag}(1, 0)$ and $\mathbf{B} = \mathrm{diag}(0, 1)$. Evidently, $\mathbf{A}$ and $\mathbf{B}$ are two orthogonal projectors and thus, $\mathbf{A} = \mathbf{A}^\dagger = \mathbf{P_A}$ and $\mathbf{B} = \mathbf{B}^\dagger = \mathbf{P_B}$. However, $\mathbf{AB} = \mathbf{0}$ is group invertible and $\mathbf{P_A P_B} - \mathbf{P_B P_A}$ is singular.

Evidently, we have $\|\mathbf{AB}\|_2 \leq \|\mathbf{A}\|_2 \|\mathbf{B}\|_2$ for any pair of conformable matrices $\mathbf{A}$ and $\mathbf{B}$. We have a sharper bound provided $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{n,n}$ are EP, $\mathrm{rk}(\mathbf{A}) = \mathrm{rk}(\mathbf{B})$, and

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$\mathbf{P_A} - \mathbf{P_B}$ is nonsingular. Furthermore, we find some bounds for the norm of several expressions.

Corollary 3.5. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{n,n}$ be EP matrices and $\theta_1 < \cdots < \theta_r$ be the canonical angles between $\mathcal{R}(\mathbf{A})$ and $\mathcal{R}(\mathbf{B})$.*

(i) *If* $\mathrm{rk}(\mathbf{A}) = \mathrm{rk}(\mathbf{B})$ *and* $\mathbf{P_A} - \mathbf{P_B}$ *is nonsingular, then*

$$\|\mathbf{AB}\|_2 \leq \|\mathbf{A}\|_2 \|\mathbf{B}\|_2 \cos\theta_1.$$

(ii) *If* $\mathbf{P_A P_B} - \mathbf{P_B P_A}$ *is nonsingular, then*
(a) $\|(\mathbf{AB})^\dagger\|_2 \leq \|\mathbf{A}^\dagger\|_2 \|\mathbf{B}^\dagger\|_2 / \cos\theta_r$.
(b) $\frac{\sin^2\theta_1}{\cos\theta_1} \frac{\|\mathbf{B}^\dagger\|_2}{\|\mathbf{A}\|_2} \leq \|(\mathbf{AB})^\dagger - \mathbf{B}^\dagger \mathbf{A}^\dagger\|_2 \leq \|\mathbf{A}^\dagger\|_2 \|\mathbf{B}^\dagger\|_2 \frac{\sin^2\theta_r}{\cos\theta_r}$.
(c) $\|(\mathbf{AB})^\#\|_2 \leq \|\mathbf{A}^\dagger\|_2 \|\mathbf{B}^\dagger\|_2 / \cos^3\theta_r$.

Proof. Let us represent $\mathbf{A}$ and $\mathbf{B}$ as in (3.1) and let us define $\mathbf{X} = \mathbf{A}_1 \mathbf{C} \mathbf{B}_1$.

(i): By (3.1), (3.3) and (3.6), we have

$$\|\mathbf{AB}\|_2 = \|\mathbf{X}\|_2 = \|\mathbf{A}_1 \mathbf{C} \mathbf{B}_1\|_2 \leq \|\mathbf{A}_1\|_2 \|\mathbf{C}\|_2 \|\mathbf{B}_1\|_2 = \|\mathbf{A}\|_2 \|\mathbf{B}\|_2 \|\mathbf{C}\|_2.$$

Observe that $\|\mathbf{C}\|_2 = \max\{\cos\theta_i : 1 \leq i \leq r\}$.

(ii): Since $\mathbf{P_A P_B} - \mathbf{P_B P_A}$ is nonsingular, $\mathbf{C}$ is nonsingular. From (3.2), (3.3) and (3.7), we have

$$\begin{aligned}\|(\mathbf{AB})^\dagger\|_2 &= \|\mathbf{X}^{-1}\|_2 = \|\mathbf{B}_1^{-1} \mathbf{C}^{-1} \mathbf{A}_1^{-1}\|_2 \\ &\leq \|\mathbf{B}_1^{-1}\|_2 \|\mathbf{C}^{-1}\|_2 \|\mathbf{A}_1^{-1}\|_2 = \|\mathbf{A}^\dagger\|_2 \|\mathbf{B}^\dagger\|_2 \|\mathbf{C}^{-1}\|_2.\end{aligned}$$

To prove (a), observe that $\|\mathbf{C}^{-1}\|_2 = \max\{(\cos\theta_i)^{-1} : 1 \leq i \leq r\}$.

In order to prove (b), define $\mathbf{Y} = \mathbf{B}_1^{-1} \mathbf{C} \mathbf{A}_1^{-1}$. From (3.2) and (3.7) we have

$$\begin{aligned}(\mathbf{AB})^\dagger - \mathbf{B}^\dagger \mathbf{A}^\dagger &= \mathbf{U} \begin{bmatrix} \mathbf{C}(\mathbf{X}^{-1} - \mathbf{Y}) & \mathbf{0} \\ \mathbf{S}^*(\mathbf{X}^{-1} - \mathbf{Y}) & \mathbf{0} \end{bmatrix} \mathbf{U}^* \\ &= \mathbf{U} \begin{bmatrix} \mathbf{C} & -\mathbf{S} \\ \mathbf{S}^* & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{X}^{-1} - \mathbf{Y} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{U}^*.\end{aligned}$$

Thus, $\|(\mathbf{AB})^\dagger - \mathbf{B}^\dagger \mathbf{A}^\dagger\|_2 = \|\mathbf{X}^{-1} - \mathbf{Y}\|_2$. But, $\mathbf{X}^{-1} - \mathbf{Y} = \mathbf{B}_1^{-1}(\mathbf{C}^{-1} - \mathbf{C})\mathbf{A}_1^{-1}$. Therefore,

$$\begin{aligned}\|(\mathbf{AB})^\dagger - \mathbf{B}^\dagger \mathbf{A}^\dagger\|_2 &= \|\mathbf{B}_1^{-1}(\mathbf{C}^{-1} - \mathbf{C})\mathbf{A}_1^{-1}\|_2 \\ &\leq \|\mathbf{B}_1^{-1}\|_2 \|\mathbf{C}^{-1} - \mathbf{C}\|_2 \|\mathbf{A}_1^{-1}\|_2 = \|\mathbf{A}^\dagger\|_2 \|\mathbf{B}^\dagger\|_2 \|\mathbf{C}^{-1} - \mathbf{C}\|_2.\end{aligned}$$

By considering the function $f :]0, \pi/2[\to \mathbb{R}$ given by $f(\theta) = \frac{1}{\cos\theta} - \cos\theta$, we have $\|\mathbf{C}^{-1} - \mathbf{C}\|_2 = \frac{1}{\cos\theta_r} - \cos\theta_r$. In addition, observe that $\mathbf{C}^{-1} - \mathbf{C} = \mathbf{C}^{-1}(\mathbf{I}_r - \mathbf{C}^2) =$

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$\mathbf{C}^{-1}\mathbf{S}^2$. Thus,

$$\begin{aligned}\|\mathbf{B}^\dagger\|_2 = \|\mathbf{B}_1^{-1}\|_2 &= \|\mathbf{B}_1^{-1}(\mathbf{C}^{-1}-\mathbf{C})\mathbf{A}_1^{-1}\mathbf{A}_1\mathbf{C}\mathbf{S}^{-2}\|_2\\ &\le \|(\mathbf{AB})^\dagger - \mathbf{B}^\dagger\mathbf{A}^\dagger\|_2\|\mathbf{A}\|_2\|\mathbf{S}^{-2}\mathbf{C}\|_2.\end{aligned}$$

By studying the function $g(\theta) = \cos\theta/\sin^2\theta$, we have $\|\mathbf{S}^{-2}\mathbf{C}\|_2 = \cos\theta_1/\sin^2\theta_1$. Thus, (b) is proved.

Recall that by Corollary 3.4, the matrix $\mathbf{AB}$ is group invertible. By (3.10) and if we denote $\mathbf{Z} = \mathbf{C}^{-1}\mathbf{B}_1^{-1}\mathbf{C}^{-1}\mathbf{A}_1^{-1}\mathbf{C}^{-1}$ we have

$$(\mathbf{AB})^\# = \mathbf{U}\begin{bmatrix}\mathbf{ZC} & \mathbf{ZS}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{U}^* = \mathbf{U}\begin{bmatrix}\mathbf{Z} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{C} & \mathbf{S}\\ -\mathbf{S}^* & \mathbf{C}\end{bmatrix}\mathbf{U}^*.$$

Hence, $\|(\mathbf{AB})^\#\|_2 = \|\mathbf{Z}\|_2 \le \|\mathbf{C}^{-1}\|_2^3\|\mathbf{B}^\dagger\|_2\|\mathbf{A}^\dagger\|_2$. Thus, (c) is proved. □

Example 2. This example shows that the bounds established in Corollary 3.5 cannot be improved. Let

$$\mathbf{A} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \quad \text{and} \quad \mathbf{B} = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}.$$

Since $\mathrm{rk}(\mathbf{A}) = \mathrm{rk}(\mathbf{B}) = 1$, according to Definition 1.2 there is only one canonical angle between $\mathcal{R}(\mathbf{A})$ and $\mathcal{R}(\mathbf{B})$. Since $\mathbf{A}$ and $\mathbf{B}$ are orthogonal projectors, $\mathbf{A}^\dagger = \mathbf{A}$, $\mathbf{B}^\dagger = \mathbf{B}$, and $\|\mathbf{A}\|_2 = \|\mathbf{B}\|_2 = 1$. In addition, $\mathbf{A}$ and $\mathbf{B}$ are EP matrices, $\mathbf{P}_\mathbf{A} = \mathbf{A}$, and $\mathbf{P}_\mathbf{B} = \mathbf{B}$. Now,

$$\mathbf{AB} = \mathbf{P}_\mathbf{A}\mathbf{P}_\mathbf{B} = \frac{1}{2}\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} \qquad \text{and} \qquad \mathbf{BA} = \mathbf{P}_\mathbf{B}\mathbf{P}_\mathbf{A} = \frac{1}{2}\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}.$$

Let us note that $\mathbf{A}$ and $\mathbf{B}$ satisfy any condition of Corollary 3.5.

The singular values of $\mathbf{P}_\mathbf{A}\mathbf{P}_\mathbf{B}$ are 0 and $\sqrt{2}/2$ (because the singular values of $\mathbf{P}_\mathbf{A}\mathbf{P}_\mathbf{B}$ are the square root of the eigenvalues of $(\mathbf{P}_\mathbf{A}\mathbf{P}_\mathbf{B})(\mathbf{P}_\mathbf{A}\mathbf{P}_\mathbf{B})^*$). Hence, the unique canonical angle between $\mathcal{R}(\mathbf{A})$ and $\mathcal{R}(\mathbf{B})$ is $\pi/4$. Also,

$$\|\mathbf{AB}\|_2 = \max\{\sigma : \sigma \text{ is a singular value of } \mathbf{AB}\} = \sqrt{2}/2.$$

Hence, the bound of Corollary 3.5 (i) cannot be improved.

Now, $(\mathbf{AB})^\dagger = (\mathbf{AB})^*\left[(\mathbf{AB})(\mathbf{AB})^*\right]^\dagger = \begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$. Hence, $\|(\mathbf{AB})^\dagger\|_2 = \sqrt{2}$. Thus, the bound of Corollary 3.5 (ii.a) cannot be improved.

Since $(\mathbf{AB})^\dagger - \mathbf{B}^\dagger\mathbf{A}^\dagger = \frac{1}{2}\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$, we have $\|(\mathbf{AB})^\dagger - \mathbf{B}^\dagger\mathbf{A}^\dagger\|_2 = \sqrt{2}/2$. The bounds of Corollary 3.5 (ii.b) cannot be improved.

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Since $2\mathbf{AB}$ is an idempotent, $(2\mathbf{AB})^{\#} = 2\mathbf{AB}$. Hence, $(\mathbf{AB})^{\#} = 4\mathbf{AB}$, and therefore, $\|(\mathbf{AB})^{\#}\|_2 = 4\|\mathbf{AB}\|_2 = 2\sqrt{2}$, which shows that the bound of Corollary 3.5 (ii.c) cannot be improved.

Example 3. By means of numerical experiments, we show how good is the bound of Corollary 3.5 (i). In the following m-file (that can be executed in Matlab, Octave, or Freemat), we compute $\|\mathbf{AB}\|_2/(\|\mathbf{A}\|_2\|\mathbf{B}\|_2 \cos\theta_1)$ for 29 pairs of 4×4 matrices $\mathbf{A}$ and $\mathbf{B}$ randomly chosen.

```
function example
x = zeros(1,29);
for i=1:29
    % find two EP 4x4 random matrices according Lemma 3.1
    aux1 = rand(2,2); aux2 = rand(2,2);
    A1 = aux1*aux1'+eye(2); B1 = aux2*aux2'+eye(2);
    % XX^* + I is always nonsingular
    A = [A1 zeros(2,2); zeros(2,2) zeros(2,2)];
    th = pi/2*rand(2,1); %two random angles in [0,pi/2]
    C = diag(cos(th)); S = diag(sin(th));
    B = [C*B1*C C*B1*S; S'*B1*C S'*B1*S];
    PA = A*pinv(A); PB = B*pinv(B); sigma = svd(PA*PB);
    % There are two canonical angles
    s1 = sigma(1); s2 = sigma(2);
    % cos(theta1) = s1; cos(theta2) = s2
    x(i) = norm(A*B)/(norm(A)*norm(B)*s1);
end
bar(1:29,x,0.2)
```

One execution of this file produces Figure 3.1. We can see that in general $\|\mathbf{AB}\|_2 \neq \|\mathbf{A}\|_2\|\mathbf{B}\|_2 \cos\theta_1$.

We can perform analogous numerical experiments to test the remaining bounds. We can insert the following lines in the above code.

```
pab = pinv(A*B); pa = pinv(A); pb = pinv(b);
(norm(pab)*s2)/(norm(pa)*norm(pb))
(norm(pab-pb*pa)*s2)/(norm(pa)*norm(pb)*(1-s2^2))
(norm(pab-pb*pa)*s1*norm(A))/(norm(pb)*(1-s1^2))
giAB = pinv(pab*(A*B)^3*pab);
(norm(giAB)*s2^3)/(norm(pa)*norm(pb))
```

Observe that we have used $(\mathbf{AB})^{\#} = \big((\mathbf{AB})^{\dagger}(\mathbf{AB})^3(\mathbf{AB})^{\dagger}\big)^{\dagger}$ (see [7, Remark 1]).

From Definition 1.2, it is evident that for a pair of orthogonal projectors $\mathbf{P}_1$ and

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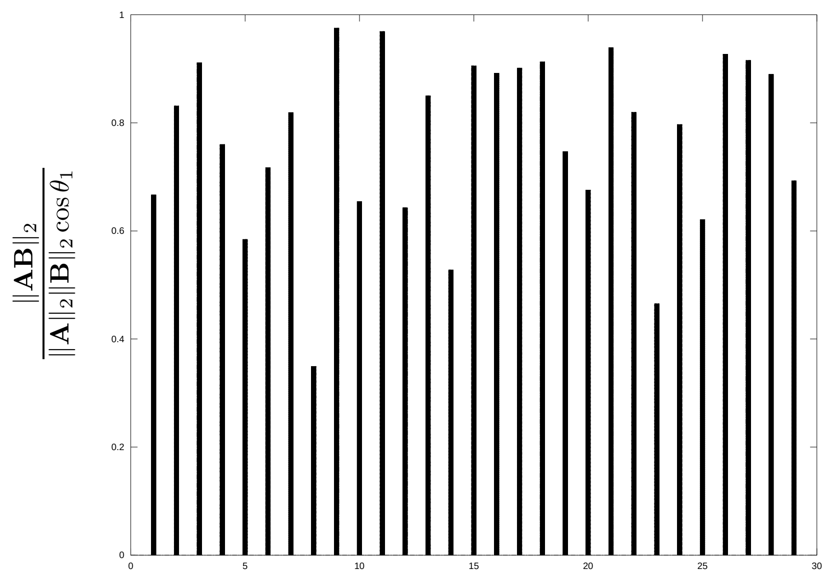

FIG. 3.1.

$\mathbf{P}_2$, one has $\|\mathbf{P}_1\mathbf{P}_2\|_2 = \cos\theta_1$, where θ_1 is the least canonical angle between $\mathcal{R}(\mathbf{P}_1)$ and $\mathcal{R}(\mathbf{P}_2)$, which shows again that the bound in Corollary 3.5 (i) cannot be improved. Another related known equality is the following: $\|(\mathbf{P}_1\mathbf{P}_2)^\dagger\| = 1/\cos\theta_r$, where θ_r is the greatest canonical angle between $\mathcal{R}(\mathbf{P}_1)$ and $\mathcal{R}(\mathbf{P}_2)$ (see [11, Lemma 36]).

The right inequality of Corollary 3.5 (ii.b) is related with the reverse order law for the Moore-Penrose inverse. There are known characterizations of the reverse order law (see [1] and [14]). We get that under the hypotheses of Theorem 3.5, if $\theta_r = 0$, then $(\mathbf{AB})^\dagger = \mathbf{B}^\dagger\mathbf{A}^\dagger$.

The nonsingularity of linear combinations of two matrices has been a widely studied topic, see e.g. [2, 5, 12, 15, 18, 19, 20, 22, 23]. Based on Theorem 2.1, we deduce a result concerning this topic.

COROLLARY 3.6. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{n,n}$ *be EP matrices. If* 0 *is not a canonical angle between* $\mathcal{R}(\mathbf{A})$ *and* $\mathcal{R}(\mathbf{B})$*; and* $\text{rk}(\mathbf{A}) = \text{rk}(\mathbf{B})$*, then* $\mathcal{N}(a\mathbf{A} + b\mathbf{B}) = \mathcal{N}(\mathbf{A}) \cap \mathcal{N}(\mathbf{B})$ *for any* a, b *nonzero complex numbers.*

Proof. Let us represent $\mathbf{A}$ and $\mathbf{B}$ as in Theorem 2.1. Since $\text{rk}(\mathbf{A}) = \text{rk}(\mathbf{B})$ and 0

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is not a canonical angle between $\mathcal{R}(\mathbf{A})$ and $\mathcal{R}(\mathbf{B})$, then $s = q$ and $p = 0$. We have

$$a\mathbf{A} + b\mathbf{B} = \mathbf{U} \begin{bmatrix} a\mathbf{A}_1 + b\mathbf{C}\mathbf{B}_1\mathbf{C} & b\mathbf{C}\mathbf{B}_1\mathbf{S} & \mathbf{0} \\ b\mathbf{S}^*\mathbf{B}_1\mathbf{C} & b\mathbf{S}^*\mathbf{B}_1\mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{U}^*.$$

Pick $\mathbf{x} \in \mathcal{N}(a\mathbf{A} + b\mathbf{B})$. By writing $\mathbf{x} = \mathbf{U} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{bmatrix}$, where $\mathbf{u}, \mathbf{v} \in \mathbb{C}_{r+s,1}$, we get

$$(a\mathbf{A}_1 + b\mathbf{C}\mathbf{B}_1\mathbf{C})\mathbf{u} + b\mathbf{C}\mathbf{B}_1\mathbf{S}\mathbf{v} = \mathbf{0} \qquad \text{and} \qquad b\mathbf{S}^*\mathbf{B}_1\mathbf{C}\mathbf{u} + b\mathbf{S}^*\mathbf{B}_1\mathbf{S}\mathbf{v} = \mathbf{0}. \tag{3.11}$$

Since $b \neq 0$ and $\mathbf{S}$ is nonsingular, the last of the equalities of (3.11) yields

$$\mathbf{B}_1\mathbf{C}\mathbf{u} + \mathbf{B}_1\mathbf{S}\mathbf{v} = \mathbf{0}. \tag{3.12}$$

By substitution of (3.12) into the first of the equalities of (3.11) we get $a\mathbf{A}_1\mathbf{u} = \mathbf{0}$, hence $\mathbf{u} = \mathbf{0}$ in view of $a \neq 0$ and the nonsingularity of $\mathbf{A}_1$. Substituting $\mathbf{u} = \mathbf{0}$ into (3.12), we have $\mathbf{v} = \mathbf{0}$ (recall that $\mathbf{B}_1$ and $\mathbf{S}$ are nonsingular). From $\mathbf{u} = \mathbf{0}$ and $\mathbf{v} = \mathbf{0}$ it is trivial to obtain $\mathbf{A}\mathbf{x} = \mathbf{B}\mathbf{x} = \mathbf{0}$, and thus, $\mathcal{N}(a\mathbf{A} + b\mathbf{B}) \subset \mathcal{N}(\mathbf{A}) \cap \mathcal{N}(\mathbf{B})$. The opposite inclusion is trivial. ∎

Corollary 3.7. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{n,n}$ be two EP matrices such that* $\mathrm{rk}(\mathbf{A}) = \mathrm{rk}(\mathbf{B})$ *and $\mathbf{P_A} - \mathbf{P_B}$ is nonsingular. Then $a\mathbf{A} + b\mathbf{B}$ is nonsingular for all $a, b \in \mathbb{C} \setminus \{0\}$.*

Proof. Since $\mathbf{P_A} - \mathbf{P_B}$ is nonsingular, then $\mathcal{R}(\mathbf{P_A}) \oplus \mathcal{R}(\mathbf{P_B}) = \mathbb{C}^n$ (see [17, Theorem 4.1]). From Corollary 3.6 and by using that $\mathcal{R}(\mathbf{P_A}) = \mathcal{R}(\mathbf{A}) = \mathcal{R}(\mathbf{A}^*)$, $\mathcal{R}(\mathbf{P_B}) = \mathcal{R}(\mathbf{B}) = \mathcal{R}(\mathbf{B}^*)$ (because $\mathbf{A}$ and $\mathbf{B}$ are EP matrices), we have

$$[\mathcal{N}(a\mathbf{A} + b\mathbf{B})]^\perp = [\mathcal{N}(\mathbf{A}) \cap \mathcal{N}(\mathbf{B})]^\perp = \mathcal{N}(\mathbf{A})^\perp + \mathcal{N}(\mathbf{B})^\perp = \mathcal{R}(\mathbf{A}^*) + \mathcal{R}(\mathbf{B}^*) = \mathbb{C}^n.$$

Thus, $\mathcal{N}(a\mathbf{A} + b\mathbf{B}) = \{\mathbf{0}\}$. ∎

In the next results, we give a representation and the norm of the projector onto $\mathcal{R}(\mathbf{A})$ along $\mathcal{N}(\mathbf{B})$ and the projector onto $\mathcal{R}(\mathbf{A})$ along $\mathcal{N}(\mathbf{A})$.

Corollary 3.8. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{n,n}$ be two EP matrices such that $\mathbf{P_A}\mathbf{P_B} - \mathbf{P_B}\mathbf{P_A}$ is nonsingular. Then $(\mathbf{AB})(\mathbf{AB})^\#$ is the oblique projector onto $\mathcal{R}(\mathbf{A})$ along $\mathcal{N}(\mathbf{B})$ and $\|\mathbf{AB}(\mathbf{AB})^\#\|_2 = 1/\cos\theta_r$, where θ_r is the greatest canonical angle between $\mathcal{R}(\mathbf{A})$ and $\mathcal{R}(\mathbf{B})$.*

Proof. Recall that the nonsingularity of $\mathbf{P_A}\mathbf{P_B} - \mathbf{P_B}\mathbf{P_A}$ implies that $\mathbf{P_A} - \mathbf{P_B}$ is nonsingular and $\mathrm{rk}(\mathbf{A}) = \mathrm{rk}(\mathbf{B})$. We represent the matrices $\mathbf{A}$ and $\mathbf{B}$ as in (3.1). Corollary 3.2 yields that matrix $\mathbf{C}$ is nonsingular, and thus the representation (3.1) leads to $\mathrm{rk}(\mathbf{AB}) = \mathrm{rk}(\mathbf{A}) = \mathrm{rk}(\mathbf{B})$. From the obvious $\mathcal{R}(\mathbf{AB}) \subseteq \mathcal{R}(\mathbf{A})$ and $\mathcal{N}(\mathbf{B}) \subseteq \mathcal{N}(\mathbf{AB})$ we obtain, respectively, $\mathcal{R}(\mathbf{AB}) = \mathcal{R}(\mathbf{A})$ and $\mathcal{N}(\mathbf{B}) = \mathcal{N}(\mathbf{AB})$. By recalling that $(\mathbf{AB})(\mathbf{AB})^\#$ is the oblique projector onto $\mathcal{N}(\mathbf{AB})$ along $\mathcal{N}(\mathbf{AB})$, we finish the proof of the first part of the corollary.

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From (3.6) and (3.10), it is easy to get

$$\mathbf{AB}(\mathbf{AB})^{\#} = \mathbf{U}\begin{bmatrix} \mathbf{I}_r & \mathbf{C}^{-1}\mathbf{S} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{U}^* = \mathbf{U}\begin{bmatrix} \mathbf{C}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{C} & \mathbf{S} \\ -\mathbf{S} & \mathbf{C} \end{bmatrix}\mathbf{U}^*.$$

Thus, $\|\mathbf{AB}(\mathbf{AB})^{\#}\|_2 = \|\mathbf{C}^{-1}\|_2 = 1/\cos\theta_r$. ☐

Corollary 3.9. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{n,n}$ *be EP matrices such that* $\mathrm{rk}(\mathbf{A}) = \mathrm{rk}(\mathbf{B})$ *and* $\mathbf{P_A} - \mathbf{P_B}$ *is nonsingular. Then the projector onto* $\mathcal{R}(\mathbf{A})$ *along* $\mathcal{R}(\mathbf{B})$ *is* $\mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}$*. The norm of this projector equals to* $1/\sin\theta_1$*, where* θ_1 *is the smallest canonical angle between* $\mathcal{R}(\mathbf{A})$ *and* $\mathcal{R}(\mathbf{B})$*.*

Proof. Let us represent $\mathbf{A}$ and $\mathbf{B}$ as in (3.1). Observe that $\mathbf{S} = \mathbf{S}^*$ since $\mathbf{S}$ is real and diagonal. The matrix $\mathbf{P}$ defined as

$$\mathbf{P} = \mathbf{U}\begin{bmatrix} \mathbf{I}_{r+s} & -\mathbf{CS}^{-1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{U}^* \tag{3.13}$$

obviously satisfies

$$\mathbf{P}^2 = \mathbf{P}, \qquad \mathbf{PA} = \mathbf{A}, \qquad \mathbf{PB} = \mathbf{0}, \qquad \mathrm{rk}(\mathbf{P}) = r + s. \tag{3.14}$$

The second equality of (3.14) yields $\mathcal{R}(\mathbf{A}) \subset \mathcal{R}(\mathbf{P})$. Since $\mathrm{rk}(\mathbf{A}) = \mathrm{rk}(\mathbf{P})$, we deduce $\mathcal{R}(\mathbf{A}) = \mathcal{R}(\mathbf{P})$. The third equality of (3.14) implies $\mathcal{R}(\mathbf{B}) \subset \mathcal{N}(\mathbf{P})$. Since $\dim \mathcal{N}(\mathbf{P}) = n - \mathrm{rk}(\mathbf{P}) = r + s = \mathrm{rk}(\mathbf{B})$ we deduce $\mathcal{R}(\mathbf{B}) = \mathcal{N}(\mathbf{P})$. Thus, the matrix $\mathbf{P}$ defined in (3.13) is the oblique projector onto $\mathcal{R}(\mathbf{A})$ along $\mathcal{R}(\mathbf{B})$. From (3.14) we get $\mathbf{P}(\mathbf{A} + \mathbf{B}) = \mathbf{A}$, and from Corollary 3.7 we obtain $\mathbf{P} = \mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}$. From (3.13) we have $\|\mathbf{P}\|_2^2 = \|\mathbf{PP}^*\|_2 = \|\mathbf{I}_{r+s} + \mathbf{C}^2\mathbf{S}^{-2}\|_2$, but $\mathbf{I}_{r+s} + \mathbf{C}^2\mathbf{S}^{-2} = \mathbf{S}^{-2}(\mathbf{S}^2 + \mathbf{C}^2) = \mathbf{S}^{-2}$. By using $\|\mathbf{S}^{-2}\| = 1/\sin^2\theta_1$, the proof is finished. Let us notice that by using [17, Theorem 3.1] we can prove also the assertion on the norm. ☐

The expressions concerning the norm of the oblique projectors appearing in Corollaries 3.8 and 3.9 are not only conceptually simple, but, as illustrated in Figure 3.2, there is also a particularly nice picture that accompanies it.

Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{n,n}$ be EP matrices such that $\mathbf{P_A P_B} - \mathbf{P_B P_A}$ is nonsingular. As we have proved, $\mathbf{P_A} - \mathbf{P_B}$ is nonsingular. Let $\mathbf{P} = \mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}$ be the oblique projector onto $\mathcal{R}(\mathbf{A})$ along $\mathcal{R}(\mathbf{B})$ and $\mathbf{Q} = \mathbf{AB}(\mathbf{AB})^{\#}$ be the oblique projector onto $\mathcal{R}(\mathbf{A})$ along $\mathcal{N}(\mathbf{B})$. In Figure 3.2, $\mathbf{x}_1$ and $\mathbf{x}_2$ are vectors of the unit ball that maximize $\|\mathbf{Px}\|$ and $\|\mathbf{Qx}\|$, respectively, when $\|\mathbf{x}\| \leq 1$, and thus, $\|\mathbf{P}\| = \|\mathbf{Px}_1\|$ and $\|\mathbf{Q}\| = \|\mathbf{Qx}_2\|$. But, as we can see in the right triangle of Figure 3.2, we have $\sin\theta = 1/\|\mathbf{Px}_1\|$. In addition, from the left triangle of Figure 3.2, we get $\cos\theta = 1/\|\mathbf{Qx}_2\|$.

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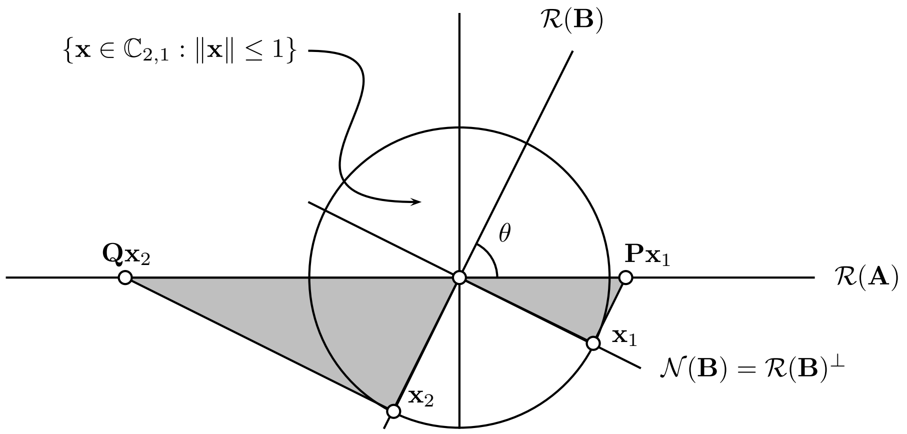

FIG. 3.2.

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