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ON THE LOG-CONCAVITY OF LAPLACIAN CHARACTERISTIC POLYNOMIALS OF GRAPHS*

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Abstract. Let G be a graph and $L(G)$ be the Laplacian matrix of G. In this article, we first point out that the sequence of the moduli of Laplacian coefficients of G is log-concave and hence unimodal. Using this fact, we provide an upper bound for the partial sums of the Laplacian eigenvalues of G, based on coefficients of its Laplacian characteristic polynomial. We then obtain some lower bounds on the algebraic connectivity of G. Finally, we investigate the mode of such sequences.

Key words. Laplacian matrix, Laplacian characteristic polynomial, Log-Concave sequence, Unimodal sequence, Mode.

AMS subject classifications. 05C50, 05C90.

1. Introduction. Throughout this paper, we consider simple undirected graphs having n vertices and m edges. For a given graph G, let $V(G)$ and $E(G)$ denote the vertex and the edge set of G, respectively. Let $A(G)$ be the adjacency matrix of G. The Laplacian matrix and signless Laplacian matrix of G are defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G)+A(G)$, respectively, where $D(G)$ is a diagonal matrix whose diagonal entries are vertex degrees of G. The eigenvalues of matrices $L(G)$ and $Q(G)$ are denoted by $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$ and $\nu_1(G) \geq \nu_2(G) \geq \cdots \geq \nu_n(G)$, respectively. As it is well-known, $L(G)$ and $Q(G)$ are positive semi-definite, and they have the same characteristic polynomial if and only if the graph G is bipartite. The second smallest eigenvalue of $L(G)$, $\mu_{n-1}(G)$, is called the algebraic connectivity of G, and it is positive if and only if the graph G is connected. For bibliographies on the graph Laplacian, the reader is referred to [11].

The Laplacian characteristic polynomial of G is denoted by $L_G(x) = det(xI - L(G)) = \sum_{i=0}^{n}(-1)^{n-i}c_i x^i$. By the following theorem attributed to Kel'mans, the Laplacian coefficient, c_k, can be expressed in terms of subtree structures of G, for $0 \leq k \leq n$ (see e.g. [3, 11]). Let F be a spanning forest of G with components T_i of

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order n_i $(1 \leq i \leq k)$, the weight of F is defined as $\gamma(F) = \prod_{i=1}^{k} n_i$.

THEOREM 1.1. [3, 11] *The Laplacian coefficient c_k of a graph G of order n is given by*

$$c_k = \sum_{F \in \mathfrak{F}_k} \gamma(F),$$

where $\mathfrak{F}_k$ is the set of all spanning forests of G with exactly k components.

In particular, we have $c_n = 1$, $c_{n-1} = 2m$, $c_0 = 0$, and

$$c_1 = n\tau(G), \tag{1.1}$$

in which $\tau(G)$ denotes the number of spanning trees of G (see e.g. [11, Theorem 4.3]).

Let $Q_G(x) = \sum_{i=0}^{n} (-1)^{n-i} \zeta_i x^i$ be the characteristic polynomial of $Q(G)$. Using the terminology and notation from [4], a spanning subgraph of G whose connected components are trees or odd unicyclic graphs is called a *TU-subgraph* of G. Suppose that a TU-subgraph H of G contains c odd unicyclic graphs and s trees such as $T_1, \ldots, T_s$. The weight of H is defined as $W(H) = 4^c \prod_{i=1}^{s} n_i$, in which n_i is the order of T_i. Note that if H contains no tree, then $W(H) = 4^c$. According to the following theorem, ζ_i can be expressed in terms of TU-subgraphs of G.

THEOREM 1.2. [4, Theorem 4.4] *Let G be a connected graph. Then we have $\zeta_n = 1$ and*

$$\zeta_{n-i} = \sum_{H_i} W(H_i), \quad i = 1, \ldots, n,$$

where the summation is over all TU-subgraphs H_i of G with i edges.

In particular, we have $\zeta_{n-1} = 2m$ and

$$\zeta_{n-2} = a + \frac{3}{2}m(m-1), \tag{1.2}$$

where a is the number of pairs of non-adjacent edges in G [4, Corollary 4.5]. Moreover, according to the definition, $c_{n-2} = \zeta_{n-2}$.

A finite sequence of real numbers $\{a_0, a_1, a_2, \ldots, a_n\}$ is said to be

(i) *unimodal* if there is some $k \in \{0, 1, \ldots, n\}$, called the *mode* of the sequence, such that

$$a_0 \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \cdots \geq a_n,$$

(ii) *logarithmically concave* (or simply, *log-concave*) if the inequality

$$a_i^2 \geq a_{i-1}a_{i+1},$$

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holds for every $i \in \{1, 2, \ldots, n-1\}$.

For instance, one may check that the Laplacian characteristic polynomial of the path P_4 is $L_{P_4}(x) = x^4 - 6x^3 + 10x^2 - 4x$. So that the sequence of the moduli of coefficients of $L_{P_4}(x)$, i.e., $\{0, 4, 10, 6, 1\}$, is log-concave and unimodal as well.

Also, a polynomial $\sum_{k=0}^{n} a_k x^k$ is called unimodal (resp., log-concave) if the sequence of its coefficients $\{a_0, a_1, \ldots, a_n\}$ is unimodal (resp., log-concave). Unimodality problems of graph polynomials have always been of great interest to researchers in graph theory. For instance, it was conjectured that the chromatic polynomial of a graph is unimodal [14, p. 68] and even log-concave [16, p. 266]. There has been an extensive literature in recent years on the unimodality problems of independence and matching polynomials of graphs, we refer the reader to [15] and the references therein. In the present paper, we are interested in studying this problem for the (signless) Laplacian characteristic polynomial. First, we show that the sequences of the moduli of (signless) Laplacian coefficients of graphs are log-concave, and hence unimodal. As a consequence, we obtain some lower and upper bounds on the algebraic connectivity and the least eigenvalue of the signless Laplacian matrix, which is studied in [7] as a measure of non-bipartiteness of a graph. Moreover, we obtain upper bounds for the partial sums of the Laplacian eigenvalues. Finally, we investigate the mode of such sequences.

2. Unimodality of the Laplacian characteristic polynomial.

Before proving our results, we state some theorems which are useful in the sequel of the paper.

A basic approach to unimodality problems is to use Newton's inequalities, as you see in the following theorem.

Theorem 2.1. [2, Theorem B, p. 270] *If the generating polynomial*

$$P(x) = \sum_{i=0}^{n} a_i x^i, \quad a_n \neq 0,$$

of a finite sequence $0 \leq a_i$ *for* $0 \leq i \leq n$*, has only real roots* (≤ 0)*, then*

$$a_i^2 \geq a_{i-1}a_{i+1}\,\frac{i}{i-1}\,\frac{n-i+1}{n-i}, \quad 2 \leq i \leq n-1,$$

and hence, $\{a_i\}_{i=0}^{n}$ *is unimodal, either with a peak or with a plateau of* 2 *points.*

Theorem 2.2. *Let* G *be a graph on* n *vertices. Then the sequences* $\{c_i\}_{i=0}^{n}$ *and* $\{\zeta_i\}_{i=0}^{n}$ *are log-concave. In particular, these two sequences are unimodal, either with a unique mode or with two consecutive modes.*

Proof. Since the Laplacian matrix is a positive semi-definite matrix, all non-zero

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coefficients of the polynomial

$$\begin{aligned}(-1)^n L_G(-x) &= (-1)^{2n}\sum_{i=0}^{n} c_i x^i \\ &= (-1)^{2n}\prod_{i=1}^{n}(x+\mu_i(G)),\end{aligned}$$

are positive. Moreover, all zeros of this polynomial are real and negative. So, Theorem 2.1 implies that the sequence $\{c_i\}_{i=0}^n$ is log-concave and unimodal (with a peak or a plateau of two points) as well. In a similar way, the assertion holds for the sequence $\{\zeta_i\}_{i=0}^n$. ☐

EXAMPLE 2.3. The Laplacian characteristic polynomial of the graph $G = K_4 - e$, where $e \in E(K_4)$, is $L_G(x) = x^4 - 10x^3 + 32x^2 - 32x$. Thus, the mode of the sequence of the moduli of coefficients of $L_G(x)$, i.e., $\{0, 32, 32, 10, 1\}$, is not necessarily unique (Note that in a unimodal sequence $\{a_i\}_{i=0}^n$, the mode is any index m for which $a_m = \max_{0\le i\le n} a_i$).

It is not hard to see that if a sequence $\{a_i\}_{i=0}^n$ is positive, then it is log-concave if and only if the sequence $\{\frac{a_i}{a_{i+1}}\}_{i=0}^{n-1}$ is non-decreasing. Applying this fact, we have the following corollary.

COROLLARY 2.4. *Let G be a graph on n vertices with the Laplacian characteristic polynomial $L_G(x) = \sum_{i=0}^n (-1)^{n-i} c_i x^i$, and let $l_i = \frac{c_{n-i}}{c_{n-i+1}}$, for $1 \le i \le n$. Then the sequence $\{l_i\}_{i=1}^n$ is a decreasing sequence.*

3. Partial sums of the Laplacian eigenvalues. In this section, we would like to obtain an upper bound for the sum of k largest eigenvalues of $L(G)$. We first state some definitions and theorems.

A nonnegative matrix $A = (a_{ij})_{n\times n}$ is called *doubly stochastic* if $\sum_{i=1}^n a_{ik} = 1$ and $\sum_{i=1}^n a_{ki} = 1$, for all $1 \le k \le n$.

THEOREM 3.1. [13, Birkhoff Theorem] *The set of all doubly stochastic matrices of order n is a convex polyhedron with permutation matrices as its vertices.*

THEOREM 3.2. [13, Theorem 38.3.1] *Let $X = (x_1, \ldots, x_n)^t$ and $Y = (y_1, \ldots, y_n)^t$ be two vectors in $\mathbb{R}^n$ such that*

$$x_n \le \cdots \le x_1,$$

$$y_n \le \cdots \le y_1,$$

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$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \quad \textit{for all} \quad k < n,$$

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.$$

Then there exists a doubly stochastic matrix S such that $SY = X$.

LEMMA 3.3. *Let G be a graph on n vertices and m edges such that $4 \leq n \leq m$. Then we have $\mu_1\mu_2 \leq \frac{1}{\sqrt{2}}c_{n-2}$.*

Proof. From the inequality of arithmetic and geometric means, we conclude that $\mu_1\mu_2 \leq \left(\frac{\mu_1+\mu_2}{2}\right)^2$. On the other hand, [9, Theorem 7] states that if G is a graph with at least two vertices, then $\mu_1 + \mu_2 \leq m + 3$. Let a be the number of pairs of non-adjacent edges in G. Using equation (1.2), we find that for any $m(\geq n \geq 4)$

$$\begin{aligned}
\mu_1\mu_2 &\leq \left(\frac{m+3}{2}\right)^2 \\
&\leq \frac{3}{2\sqrt{2}}\,(m^2 - m) && \text{since } m \geq 4 \\
&\leq \frac{1}{\sqrt{2}}\left(a + \frac{3}{2}m(m-1)\right) && \text{since } a \geq 0 \\
&= \frac{1}{\sqrt{2}}\; c_{n-2}. \qquad \square
\end{aligned}$$

THEOREM 3.4. *Let G be a graph on n vertices and m edges, where $4 \leq n \leq m$. Then*

(i) $\mu_1 \leq m$;

(ii) $\mu_1 + \mu_2 \leq m + \sqrt{2}\,\dfrac{c_{n-2}}{c_{n-1}}$;

(iii) $\mu_1 + \mu_2 + \mu_3 \leq m + \sqrt{2}\,\dfrac{c_{n-2}}{c_{n-1}} + \sqrt{2}\,\dfrac{c_{n-3}}{c_{n-2}}$;

(iv) $\displaystyle\sum_{i=1}^{k} \mu_i \leq m + \sqrt{2}\,\frac{c_{n-2}}{c_{n-1}} + \sqrt{2}\,\frac{c_{n-3}}{c_{n-2}} + \sum_{i=4}^{k} \frac{c_{n-i}}{c_{n+1-i}}$, *for* $4 \leq k \leq n-1$.

Proof. (i) Theorem 2.2(c) in [11] states that for any graph of order n, we have $\mu_1 \leq n$. So, by the assumption on m, we obtain $\mu_1 \leq m$. To prove the other parts, let $l_i = \frac{c_{n-i}}{c_{n-i+1}}$, for $1 \leq i \leq n-1$. Consider two vectors

$$X = (\mu_1, \ldots, \mu_{n-1})^t,$$
$$Y = \left(\frac{1}{2}\,l_1, \sqrt{2}\,l_2, \sqrt{2}\,l_3, l_4, \ldots, l_{n-1}\right)^t.$$

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First, we want to show that coordinates of the vector Y (except the first two) are in non-increasing order. Applying Theorem 2.1 to the polynomial $(-1)^n L_G(-x)$ we find that

$$c_{n-1}^2 \geq c_{n-2} c_n \frac{2(n-1)}{n-2},$$

and

$$c_{n-2}^2 \geq c_{n-1} c_{n-3} \frac{3(n-2)}{2(n-3)}.$$

Therefore,

$$l_1 \geq \frac{2(n-1)}{n-2} l_2, \tag{3.1}$$

$$l_2 \geq \frac{3(n-2)}{2(n-3)} l_3. \tag{3.2}$$

Using equations (3.1) and (3.2), one may obtain that $\frac{1}{2}\, l_1 \geq \sqrt{2}\, l_3$. Moreover, Corollary 2.4 implies that $l_2 \geq l_3 \geq l_4 \geq \cdots \geq l_{n-1}$. Hence,

$$\frac{1}{2}\, l_1 \geq \sqrt{2}\, l_3 \geq l_4 \geq \cdots \geq l_{n-1}, \tag{3.3}$$

$$\sqrt{2}\, l_2 \geq \sqrt{2}\, l_3 \geq l_4 \geq \cdots \geq l_{n-1}. \tag{3.4}$$

In order to sort coordinates of Y in non-increasing order, if $\sqrt{2}\, l_2 > \frac{1}{2}\, l_1$, then we could change the first two coordinates of the vector Y without any effect in our subsequent calculations, because case (i) is proved.

Also, Lemma 3.3 implies that $\mu_1 \mu_2 \leq \frac{1}{\sqrt{2}}\, c_{n-2} = \frac{\sqrt{2}}{2}\, l_1 l_2$. For $3 \leq k \leq n-1$, the moduli of the $(n-k)$th coefficient of $L_G(x)$ equals

$$c_{n-k} = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I|=k}} \prod_{i \in I} \mu_i(G).$$

Then

$$\prod_{i=1}^{k} \mu_i \leq c_{n-k} = \prod_{i=1}^{k} l_i. \tag{3.5}$$

On the other hand, $\mu_n(G) = 0$ follows that

$$\prod_{i=1}^{n-1} \mu_i = c_1 = \prod_{i=1}^{n-1} l_i. \tag{3.6}$$

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Next, in order to use Theorem 3.2, we need to define the following vectors

$$X' = (\ln(\mu_1), \ldots, \ln(\mu_{n-1}))^t, \tag{3.7}$$

$$Y' = \left(\ln\left(\frac{1}{2}\,l_1\right), \ln\left(\sqrt{2}\,l_2\right), \ln\left(\sqrt{2}\,l_3\right), \ln\left(l_4\right), \ldots, \ln\left(l_{n-1}\right)\right)^t. \tag{3.8}$$

Now, applying Theorem 3.2 together with Lemma 3.3 and Equations (3.5) and (3.6), there exists a doubly stochastic matrix S such that $SY' = X'$.

For any $2 \le k \le n-1$, consider the function

$$\begin{aligned} f : \mathbb{R}^{n-1} &\longrightarrow \mathbb{R} \\ (x_1, \ldots, x_{n-1})^t &\longmapsto \sum_{i=1}^{k} \exp(x_i). \end{aligned}$$

Since the exponential function is convex, one may see that f is a convex function on a set of positive vectors. Let Ω_n denote the set of all doubly stochastic matrices of order n.

For the vector $Y' \in \mathbb{R}^{n-1}$ defined in equation (3.8), let

$$\begin{aligned} g : \Omega_{n-1} &\longrightarrow \mathbb{R} \\ S &\longmapsto f(SY'). \end{aligned}$$

Again the function g is a convex function, because for each $\lambda \in [0,1]$

$$\begin{aligned} g(\lambda S_1 + (1-\lambda)S_2) &= f(\lambda S_1 Y' + (1-\lambda)S_2 Y') \\ &\le \lambda f(S_1 Y') + (1-\lambda) f(S_2 Y') \\ &= \lambda g(S_1) + (1-\lambda) g(S_2). \end{aligned}$$

A convex function defined on a convex polytope takes its maximal value at one of its vertices. Therefore, by Theorem 3.1 $g(S) \le g(P)$, where P is the matrix related to the permutation π. Then

$$\begin{aligned} f(X') = f(SY') &= g(S) \\ &\le g(P). \end{aligned} \tag{3.9}$$

Moreover,

$$\begin{aligned} \sum_{i=1}^{k} \mu_i = \sum_{i=1}^{k} \exp(\ln(\mu_i)) &= f(X') \\ &\le f(PY') \qquad \text{by equation (3.9)} \end{aligned}$$

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$$
\begin{aligned}
&= \sum_{i=1}^{k} \exp(y'_{\pi(i)}) \\
&= \sum_{i=1}^{k} \exp(\ln(y_{\pi(i)})) \\
&= \sum_{i=1}^{k} y_{\pi(i)}.
\end{aligned}
$$

Finally, by equations (3.4) and (3.3), for $2 \leq k \leq n-1$, we have $\sum_{i=1}^{k} y_{\pi(i)} \leq \sum_{i=1}^{k} y_i$. This completes the proof. $\square$

EXAMPLE 3.5. Suppose that G is the graph shown in Fig. 1. One may check that $L_G(x) = x^4 - 8x^3 + 19x^2 - 12x$. Using the notation of the previous theorem, we have

$$
\begin{aligned}
X &= (\mu_1(G), \mu_2(G), \mu_3(G))^t \\
&= (4, 3, 1)^t, \\
Y &= \left(\frac{1}{2}\, l_1, \sqrt{2}\, l_2, \sqrt{2}\, l_3\right)^t \\
&= \left(\frac{c_3}{2c_4}, \frac{\sqrt{2}\, c_2}{c_3}, \frac{\sqrt{2}\, c_1}{c_2}\right)^t \\
&\approx (4, 3.36, 0.89)^t,
\end{aligned}
$$

where c_i is the moduli of ith Laplacian coefficient. According to the previous theorem, we have

$$
\begin{aligned}
\mu_1(G) &\leq 4, \\
\mu_1(G) + \mu_2(G) &\leq 7.36, \\
\mu_1(G) + \mu_2(G) + \mu_3(G) &\leq 8.25.
\end{aligned}
$$

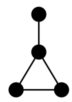

Fig. 1

4. Some eigenvalue bounds. In this section, applying unimodality property, we are seeking some lower bounds for the (signless) Laplacian eigenvalues. In this way, we first state the follwoing theorem which is an equivalent statement of the classical

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theorem due to Eneström and Kakeya for finding bounds for the moduli of the zeros of polynomials having positive real coefficients.

THEOREM 4.1. [1, Eneström-Kakeya Theorem] *Let $P(x) = \sum_{i=0}^{n} a_i x^i$, $n \geq 1$, be any polynomial with $a_i > 0$ for all $0 \leq i \leq n$. Setting*

$$\alpha = \min_{0 \leq i < n} \frac{a_i}{a_{i+1}} \quad \text{and} \quad \beta = \max_{0 \leq i < n} \frac{a_i}{a_{i+1}},$$

all the zeros of $P(x)$ are contained in the interval $\alpha \leq |x| \leq \beta$.

As it is well-known, the multiplicity of 0 as an eigenvalue of $L(G)$ is equal to the number of components of G. Now, using the above theorem, we find a lower bound on the algebraic connectivity of G, based on Laplacian coefficients.

THEOREM 4.2. *Let G be a graph on n vertices, then $\frac{c_\alpha}{c_{\alpha+1}} \leq \mu_{n-\alpha}$, where α is the number of connected components of G. In particular, if G is a connected graph of order n, then $\frac{n\tau(G)}{c_2} \leq \mu_{n-1}$.*

Proof. Suppose that $L_G(-x) = (-1)^n x^\alpha \prod_{i=1}^{n-\alpha}(x+\mu_i(G)) = (-1)^n x^\alpha P(x)$, where α is the number of connected components of G. Since the sequence $\{c_i\}_{i=0}^n$ is log-concave, applying Eneström-Kakeya Theorem and Corollary 2.4 to the polynomial $P(x)$, the result follows. ∎

If G is a tree, the coefficient c_2 is equal to its *Wiener index*, which is the sum of distances between all pairs of vertices [17]. In other words, we have

$$c_2 = \text{wien}(G) = \sum_{u,v \in V(G)} d_G(u,v).$$

COROLLARY 4.3. *Let T be a tree of order n. Then we have $\frac{n\tau(T)}{\text{wien(T)}} \leq \mu_{n-1}(G)$, in which $\tau(T)$ and* wien(T) *denote the number of spanning trees and Wiener index of T.*

Let η denote the nullity of the matrix $Q(G)$. The following theorem explicitly expresses the connection between η and the structure of G.

THEOREM 4.4. [4, Corollary 2.2] *In any graph, the multiplicity of the eigenvalue 0 of the signless Laplacian matrix is equal to the number of bipartite components.*

In [7], the least eigenvalue of $Q(G)$ was studied as a measure of non-bipartiteness of a graph. Here, we obtain a lower bound for this quantity.

THEOREM 4.5. *Let G be a graph on n vertices. Then we have $\frac{\zeta_\eta}{\zeta_{\eta+1}} \leq \nu_{n-\eta}$. In particular, if G is a graph without any bipartite connected component, then $\frac{\zeta_0}{\zeta_1} \leq \nu_n(G)$.*

Proof. The proof is similar to Theorem 4.2. ∎

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5. Mode of the Laplacian coefficients. In this section, we are interested in determining the mode of the sequence $\{c_i\}_{i=0}^n$, which is denoted by $mode_L(G)$.

Let $\{a_0,\ldots,a_n\}$ be a sequence of non-negative real numbers whose generating polynomial $A(x) = \sum_{i=0}^n a_i x^i$ has only real zeros and $A(1) > 0$. Suppose that P denotes the probability distribution on $\{0,1,\ldots,n\}$ defined by normalization of $\{a_0,\ldots,a_n\}$. Let μ denote the mean of the probability distribution P, that is $\mu := \frac{A'(1)}{A(1)}$ in which $A'(x)$ is the first derivative of the polynomial $A(x)$. As a well-known consequence of Newton's inequality, the sequence $\{a_0,\ldots,a_n\}$ has either a unique mode or two consecutive modes. Darroch showed that each mode, denoted by m, satisfies $\lfloor\mu\rfloor \leq m \leq \lceil\mu\rceil$. This remarkable result seems to be quite unknown to combinatorialists, although it has numerous combinatorial applications. To be more precise, according to [12, p. 284], we have:

Theorem 5.1. [6, Theorem 4] *Let $A(x) = \sum_{i=0}^n a_i x^i$ be a polynomial that has real roots only and satisfies $A(1) > 0$. Let m be an index so that $a_m = \max_{0\leq i\leq n} a_i$. Let $\mu = \frac{A'(1)}{A(1)}$. Then we have $|m-\mu| < 1$. Precisely, for any integer k with $0 \leq k \leq n$, the mode of the sequence $\{a_0,\ldots,a_n\}$ equals*

$$m = \begin{cases} k, & \text{if } k \leq \mu < k + \frac{1}{k+2}; \\ k,\ k+1, \textit{or both}, & \text{if } k + \frac{1}{k+2} \leq \mu \leq k + 1 - \frac{1}{n-k+1}; \\ k+1, & \text{if } k + 1 - \frac{1}{n-k+1} < \mu \leq k+1. \end{cases}$$

Remark 5.2. Suppose that $A(x) = (-1)^n L_G(-x) = (-1)^{2n}\prod_{i=1}^n (x + \mu_i(G))$. According to the definition, the mean of the probability distribution on $\{0,1,\ldots,n\}$ defined by normalization of $\{c_0,\ldots,c_n\}$ equals

$$\mu = \frac{A'(1)}{A(1)} = \frac{\sum_{j=1}^n \frac{\prod_{i=1}^n (1+\mu_i(G))}{1+\mu_j(G)}}{A(1)} = \frac{\sum_{j=1}^n \frac{A(1)}{1+\mu_j(G)}}{A(1)} = \sum_{j=1}^n \frac{1}{1+\mu_j(G)}.$$

So, using Darroch's Rule for the mode, we have $|mode_L(G) - \mu| < 1$.

Theorem 5.3. *Let G be a graph on n vertices and $e \in E(G)$. Then we have*

$$mode_L(G) - mode_L(G-e) \in \{-2,-1,0,1\}.$$

Proof. By [11, Theorem 3.2], we have

$$\mu_1(G) \geq \mu_1(G-e) \geq \mu_2(G) \geq \mu_2(G-e) \geq \cdots \geq \mu_n(G) \geq \mu_n(G-e).$$

Therefore, for each $1 \leq i \leq n-1$,

$$\frac{1}{1+\mu_i(G)} \leq \frac{1}{1+\mu_i(G-e)} \leq \frac{1}{1+\mu_{i+1}(G)}.$$

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By summation (over i) of each side of the above inequalities, we find that

$$\begin{aligned} 0 &< \sum_{i=1}^{n} \frac{1}{1+\mu_i(G-e)} - \sum_{i=1}^{n} \frac{1}{1+\mu_i(G)} \\ &\leq \frac{1}{1+\mu_n(G-e)} - \frac{1}{1+\mu_1(G)} \\ &< 1. \end{aligned}$$

Also, Darroch's Rule says that

$$-1 < mode_L(G) - \sum_{i=1}^{n} \frac{1}{1+\mu_i(G)} < 1.$$

Combining these two later inequalities, we obtain that

$$-2 < mode_L(G) - \sum_{i=1}^{n} \frac{1}{1+\mu_i(G-e)} < 1.$$

Again applying Darroch's Rule to the graph $G-e$, we have

$$-1 < mode_L(G-e) - \sum_{i=1}^{n} \frac{1}{1+\mu_i(G-e)} < 1.$$

Consequently, $-3 < mode_L(G) - mode_L(G-e) < 2$, and this completes the proof. ∎

Moreover, a similar statement holds for signless Laplacian matrix.

Recall that the kth elementary symmetric function of n real numbers $x_1, x_2, \ldots,$ x_n ($k \leq n$), is defined as

$$\sigma_k(x_1, \ldots, x_n) = \sum_{S \subseteq \{1,\ldots,n\}, |S|=k} \prod_{i \in S} x_i.$$

One may see that

$$L_G(x) = x^n - \sigma_1(\mu_1, \ldots, \mu_n)x^{n-1} + \sigma_2(\mu_1, \ldots, \mu_n)x^{n-2} - \cdots \pm \sigma_n(\mu_1, \ldots, \mu_n).$$

So, we have $c_{n-i}(G) = \sigma_i(\mu_1, \ldots, \mu_n)$. For other properties of elementary symmetric functions, the reader is referred to [10]. Also, $[x]$ denotes the integral part of x.

Theorem 5.4. [8, Proposition 3.4] *Let G be a graph with n vertices and without isolated vertices. Then we have $\sum_{k=1}^{i} \mu_k(G) \leq 2m - n + 2i$, for $1 \leq i \leq n$.*

Theorem 5.5. *Let G be a connected graph with n vertices. Then for each $0 \leq i \leq [\frac{n-1}{3}]$,*

$$c_{n-i-1}(G) > c_{n-i}(G).$$

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Proof. For abbreviation, we write $\sigma_i(\mu_1,\ldots,\mu_n)$ by σ_i, where $1 \le i \le n-1$. Consider any subset $S \subseteq \{1,\ldots,n-1\}$ such that $|S| = i+1$. Obviously, there exists exactly $i+1$ subsets $X \subseteq \{1,\ldots,n-1\}$ for which $X \subset S$ and $|X| = i$. Therefore, using this fact that $\sum_{i=1}^{n-1}\mu_i = 2m$, we have

$$\begin{aligned}
(i+1)\sigma_{i+1} &= \sum_{\substack{S\subseteq\{1,\ldots,n-1\}\\ |S|=i+1}} (i+1)\prod_{j\in S}\mu_j \\
&= \sum_{\substack{X\subseteq\{1,\ldots,n-1\}\\ |X|=i}} \left(\left(\prod_{j\in X}\mu_j\right)\cdot\left(\sum_{j\notin X}\mu_j\right)\right) \\
&= \sum_{\substack{X\subseteq\{1,\ldots,n-1\}\\ |X|=i}} \left(\left(\prod_{j\in X}\mu_j\right)\cdot\left(2m-\sum_{j\in X}\mu_j\right)\right) \\
&> (2m-\mu_1-\cdots-\mu_i)\sum_{\substack{X\subseteq\{1,\ldots,n-1\}\\ |X|=i}}\left(\prod_{j\in X}\mu_j\right) \\
&= (2m-\mu_1-\cdots-\mu_i)\sigma_i, \qquad (5.1)
\end{aligned}$$

where the fourth inequality holds provided $\sum_{j\in X}\mu_j \le \mu_1+\cdots+\mu_{|X|}$.

On the other hand, Theorem 5.4 implies that for $1 \le i \le \left[\frac{n-1}{3}\right]$,

$$\begin{aligned}
\sum_{k=1}^{i}\mu_k &\le 2m-n+2i \\
&\le 2m-i-1. \qquad (5.2)
\end{aligned}$$

Now using equation (5.2), we have for $1 \le i \le \left[\frac{n-1}{3}\right]$

$$(i+1)\sigma_{i+1} > (2m-\mu_1-\cdots-\mu_i)\sigma_i \ge (i+1)\sigma_i.$$

This completes the proof. □

Corollary 5.6. *Let G be a graph on n vertices. Then we have*

$$\sum_{i=1}^{n}\frac{1}{1+\mu_i(G)} < n-\left[\frac{n-1}{3}\right].$$

Proof. According to the previous theorem, one may see that

$$c_{n-[\frac{n-1}{3}]-1}(G) > c_{n-[\frac{n-1}{3}]}(G) > \cdots > c_{n-2}(G) > c_{n-1}(G) > c_n(G).$$

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Thus, $mode_L(G) \leq n-1-[\frac{n-1}{3}]$. Now, using Remark 5.2 together with Darroch's Rule for the mode, the result follows. ☐

Moreover, a similar statement holds for signless Laplacian matrix.

REMARK 5.7. Let G be a connected graph of order n and size m. Then, we have $\sum_{i=1}^{n-1} \mu_i = 2m$. Now, proceeding the proof of Theorem 5.5, we get that for each $1 \leq i \leq n$,

$$\begin{aligned}(i+1)\sigma_{i+1} &> (2m - \mu_1(G) - \cdots - \mu_i(G))\sigma_i \\ &= (\mu_{i+1} + \cdots + \mu_{n-1})\sigma_i \\ &\geq (n-1-i)\mu_{n-1}\sigma_i.\end{aligned}$$

Let $f(x) = \frac{n-1-x}{x+1}$. One may check that $f(x)$ is a decreasing function on $x \in \mathbb{R}\setminus\{-1\}$. Obviously, $f(x) > 1$ when $0 \leq x < [\frac{n-1}{2}]$. Therefore,

$$\frac{\sigma_{i+1}}{\sigma_i} > f(i)\mu_{n-1} > \mu_{n-1},$$

when $0 \leq i < [\frac{n-1}{2}]$. So, by Corollary 2.4 we have

$$\frac{c_{n-k-1}}{c_{n-k}} = \frac{\sigma_{k+1}}{\sigma_k} > \mu_{n-1}, \quad where\ k = \left[\frac{n-1}{2}\right] - 1.$$

On the other hand, we find that for each $1 \leq i \leq n$,

$$\begin{aligned}(i+1)\sigma_{i+1} &= \sum_{\substack{S\subseteq\{1,\ldots,n-1\}\\|S|=i+1}} \left((i+1)\prod_{j\in S}\mu_j\right) \\ &= \sum_{\substack{X\subseteq\{1,\ldots,n-1\}\\|X|=i}} \left(\left(\prod_{j\in X}\mu_j\right)\cdot\left(\sum_{j\notin X}\mu_j\right)\right) \\ &= \sum_{\substack{X\subseteq\{1,\ldots,n-1\}\\|X|=i}} \left(\left(\prod_{j\in X}\mu_j\right)\cdot\left(2m - \sum_{j\in X}\mu_j\right)\right) \\ &< (2m - \mu_{n-1} - \cdots - \mu_{n-1-i+1}) \sum_{\substack{X\subseteq\{1,\ldots,n-1\}\\|X|=i}} \prod_{j\in X}\mu_j \\ &= (2m - \mu_{n-1} - \cdots - \mu_{n-1-i+1})\sigma_i \\ &= (\mu_1 + \cdots + \mu_{n-1-i})\sigma_i \\ &\leq (n-1-i)\mu_1\sigma_i.\end{aligned}$$

In a similar way as above, we find that

$$\frac{\sigma_{k+1}}{\sigma_k} < \mu_1, \quad when\ k = \left[\frac{n-1}{2}\right].$$

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Finally, we conclude this article with the following question.

QUESTION 5.8. *Let G be a graph on n vertices with α connected components. Is it true to say that*

$$mode_L(G) \leq \left[\frac{n-\alpha}{2}\right],$$

i.e., $c_{n-i}(G) < c_{n-i-1}(G)$, for $0 \leq i < [\frac{n-\alpha}{2}]$?

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