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INEQUALITIES OF GENERALIZED MATRIX FUNCTIONS VIA TENSOR PRODUCTS*

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Abstract. By an embedding approach and through tensor products, some inequalities for generalized matrix functions (of positive semidefinite matrices) associated with any subgroup of the permutation group and any irreducible character of the subgroup are obtainned.

Key words. Determinant, Generalized matrix function, Permanent, Positive semidefinite matrix, Tensor product.

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1. Introduction. Let H be a subgroup of S_n, the permutation group on n letters, and let χ be an irreducible character of H. For any $n \times n$ complex matrix $A = (a_{ij})$, we define

$$d_\chi^H(A) = \sum_{\sigma \in H} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

and call the mapping d_χ^H from the matrix space to the complex number field *a generalized matrix function* (also known as *immanant*) associated with the subgroup H and the irreducible character χ.

Specifying the subgroup H and the character χ gives some familiar functions on matrices. If $H = S_n$ and χ is the signum function with values ± 1, then the generalized matrix function becomes the usual matrix determinant; setting $\chi(\sigma) = 1$ for each $\sigma \in H = S_n$ defines the permanent of the matrix; and by taking $H = \{e\} \subseteq S_n$, we have the product of the main diagonal entries of the matrix (also known as the Hadamard matrix function). We write $A \geq 0$ if A is a positive semidefinite matrix. It is known that $A \geq 0$ implies $d_\chi^H(A) \geq 0$. One may refer to, e.g., [2], [3], [5], and [8] for definitions, available techniques, and existing results on generalized matrix functions.

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Let A and B be $n\times n$ positive semidefinite matrices (which are necessarily Hermitian over the complex number field [9, p. 80]). A classical result (see, e.g., [5, p. 228]) states that

$$d_\chi^H(A+B) \geq d_\chi^H(A) + d_\chi^H(B). \tag{1.1}$$

Recall the fact (see, e.g., [1, p. 441]) that every positive semidefinite matrix is a Gram matrix. By embedding the vectors of Gram matrices into a "sufficiently large" inner product space and by using tensor products, we extend (1.1) to multiple matrices (in a stronger form). We first show that for three $n\times n$ positive semidefinite matrices A, B, and C,

$$d_\chi^H(A+B+C) + d_\chi^H(A) \geq d_\chi^H(A+B) + d_\chi^H(A+C). \tag{1.2}$$

We then generalize this to any finite number of positive semidefinite matrices. Nevertheless, our main effort is to prove (1.2), as the general case of more matrices reduces to that of triple matrices. Our approach to establish (1.2) is algebraic as well as combinatorial.

We organize the paper as follows: In Section 2, we evolve our idea of embedding, with which we present a direct proof for (1.1). In Section 3, we decompose a tensor product T_{A+B+C} of 1-forms (linear functionals) into a sum of tensor products T_{A+B}, T_{A+C}, and T_A. Carefully examining each term in the summation, we conclude (1.2) and obtain some existing results as its special cases. In Section 4, we extend (1.2) to any finite number of positive semidefinite matrices.

2. Some preliminaries. Let $A=(a_{ij})$ and $B=(b_{ij})$ be $n\times n$ positive semidefinite matrices, $n\geq 2$. Since every positive semidefinite matrix is a Gram matrix, we can write

$$a_{ij} = \langle x_j, x_i\rangle \qquad \text{and} \qquad b_{ij} = \langle y_j, y_i\rangle\,, \quad 1\leq i,j\leq n,$$

where $x_i = (x_{i1},\dots,x_{in_A}) \in \mathbb{C}^{n_A}$, $y_i = (y_{i1},\dots,y_{in_B}) \in \mathbb{C}^{n_B}$, $\langle\cdot,\cdot\rangle$ is the standard inner product, and n_A and n_B are the ranks of A and B, respectively. We can also embed the vectors x_i, y_j of the Gram matrices into $\mathbb{C}^{n_A+n_B}$ in such a way that

$$x_i = (x_{i1},\dots,x_{in_A},0,\dots,0), \qquad y_i = (0,\dots,0,y_{i1},\dots,y_{in_B})$$

with the appropriate number of zero coordinates for each. As a result of this embedding, we have $\langle x_i, y_j\rangle = 0$ for all $i,j=1,\dots,n$, i.e., vectors x_i and y_j (or simply x and y) are orthogonal for all i,j. In what follows, we assume $x_i, y_j \in \mathbb{C}^{n_A+n_B}$ (or even in a "larger" space in Section 3).

LEMMA 2.1. *In the set-up above, for the (i,j)-entry of $A+B$, we have*

$$(A+B)_{ij} = a_{ij} + b_{ij} = \langle z_j, z_i\rangle\,,$$

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where $z_i = x_i + y_i = (x_{i1}, \ldots, x_{in_A}, y_{i1}, \ldots, y_{in_B})$, $i = 1, \ldots, n$.

Proof. Using the orthogonality of x_i and y_j, we compute

$$\begin{aligned}\langle z_j, z_i\rangle &= \langle x_j + y_j, x_i + y_i\rangle \\ &= \langle x_j, x_i\rangle + \langle x_j, y_i\rangle + \langle y_j, x_i\rangle + \langle y_j, y_i\rangle \\ &= \langle x_j, x_i\rangle + \langle y_j, y_i\rangle = a_{ij} + b_{ij}. \quad \square\end{aligned}$$

As usual, if x is a vector in a vector space V, the associated 1-form x^* of x in the dual space V^* is defined as $x^*(y) = \langle y, x\rangle$ for any $y \in V$. Moreover, the dualizing operation $*$ is additive. That is, $(x + y)^* = x^* + y^*$.

For $n \times n$ positive semidefinite matrices A and B given as before, we obtain the elements (tensors) $T_A, T_B \in V^* \otimes \cdots \otimes V^*$ with $V = \mathbb{C}^{n_A+n_B}$ as

$$T_A = x_1^* \otimes \cdots \otimes x_n^*, \qquad T_B = y_1^* \otimes \cdots \otimes y_n^*.$$

Similarly, by Lemma 2.1, we also have

$$T_{A+B} = z_1^* \otimes \cdots \otimes z_n^* = \bigotimes_{i=1}^{n} (x_i^* + y_i^*) = T_A + T_B + \sum_{i=1}^{2^n-2} \Theta_i, \tag{2.1}$$

in which each Θ_i is a tensor product containing both x^* and y^* vectors. More explicitly, if we let $X = \{x_1^*, \ldots, x_n^*\}$, $Y = \{y_1^*, \ldots, y_n^*\}$, and $\Theta_i = \omega_1^* \otimes \cdots \otimes \omega_n^*$, then there exist distinct $1 \le i, j \le n$ such that $\omega_i^* \in X$ and $\omega_j^* \in Y$. We denote

$$\Theta_{xy} = \sum_{i=1}^{2^n-2} \Theta_i.$$

Let H be any subgroup of S_n and χ be any irreducible character of H. Let χ act on the space of $*$-tensor products as

$$\chi \cdot w_1^* \otimes \cdots \otimes w_n^* = \sum_{\sigma\in H} \chi(\sigma) w_{\sigma^{-1}(1)}^* \otimes \cdots \otimes w_{\sigma^{-1}(n)}^* = \mathcal{T}(w_1^* \otimes \cdots \otimes w_n^*),$$

where $\mathcal{T}$ is actually the (linear) "symmetry operator" defined in [2, p. 317] (see also, e.g., [3, p. 77] or "symmetrizer" in [5, p. 153] for any degree $\chi(e)$ of the character). It is known that $\mathcal{T}^* = \mathcal{T}$ and $\mathcal{T}^2 = h\mathcal{T}$. Here h is the order of the subgroup H. Observe that the action permutes the vectors within the tensor product. Additionally, let $u_1, \ldots, u_n, v_1, \ldots, v_n$ be vectors in an inner product space W. Then the space $(W^*)^{\otimes n}$ of tensor products is naturally equipped with the inner product

$$\langle u_1^* \otimes \cdots \otimes u_n^*, v_1^* \otimes \cdots \otimes v_n^*\rangle = \prod_{i=1}^{n} \langle u_i^*, v_i^*\rangle = \prod_{i=1}^{n} \langle v_i, u_i\rangle.$$

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Let A be an $n \times n$ positive semidefinite matrix and χ be an irreducible character of H. Then

$$d_\chi^H(A) = \langle \chi \cdot T_A, T_A \rangle \quad \text{(see the proof of Lemma 1 in [6, p. 878] or see [5, p. 226])}$$

Our idea of embedding gives a direct proof for (1.1). We demonstrate the proof here. This approach will also be used in the next section for three positive semidefinite matrices.

Proposition 2.2. ([5, p. 228]) *Let $A, B \geq 0$. Then $d_\chi^H(A+B) \geq d_\chi^H(A) + d_\chi^H(B)$.*

Proof. With $T_{A+B} = T_A + T_B + \Theta_{xy}$, letting $M = d_\chi^H(A+B)$, we have

$$\begin{aligned}
M &= \langle \chi \cdot T_{A+B}, T_{A+B} \rangle \\
&= \langle \chi \cdot (T_A + T_B + \Theta_{xy}), T_A + T_B + \Theta_{xy} \rangle \\
&= \langle \chi \cdot T_A, T_A \rangle + \langle \chi \cdot T_A, T_B \rangle + \langle \chi \cdot T_A, \Theta_{xy} \rangle \\
&\quad + \langle \chi \cdot T_B, T_A \rangle + \langle \chi \cdot T_B, T_B \rangle + \langle \chi \cdot T_B, \Theta_{xy} \rangle \\
&\quad + \langle \chi \cdot \Theta_{xy}, T_A \rangle + \langle \chi \cdot \Theta_{xy}, T_B \rangle + \langle \chi \cdot \Theta_{xy}, \Theta_{xy} \rangle .
\end{aligned}$$

There are nine terms in the last equality. We study each of them. First, we have $\langle \chi \cdot T_A, T_A \rangle = d_\chi^H(A)$ and $\langle \chi \cdot T_B, T_B \rangle = d_\chi^H(B)$. Note that $T_A = x_1^* \otimes \cdots \otimes x_n^*$ and $T_B = y_1^* \otimes \cdots \otimes y_n^*$. Observe that when χ acts on a tensor product, it only permutes the vectors in the tensor product. By the orthogonality of x^* and y^* vectors, we have

$$\langle \chi \cdot T_A, T_B \rangle = \langle \chi \cdot T_B, T_A \rangle = 0.$$

Θ_{xy} is a sum of tensor products each of which contains at least one component from $\{x_1^*, \ldots, x_n^*\}$ and at least one component from $\{y_1^*, \ldots, y_n^*\}$. $\chi \cdot T_A$ consists solely of x^* vectors. Hence, in the expanded product $\langle \chi \cdot T_A, \Theta_{xy} \rangle$, there is always a y^* vector that will be paired with some x^* vector coming from $\chi \cdot T_A$. Once again, by the orthogonality of x^* and y^*, the product $\langle \chi \cdot T_A, \Theta_{xy} \rangle$ vanishes. A similar reasoning can be applied to the other mixed inner products. Namely, we have

$$\langle \chi \cdot T_A, \Theta_{xy} \rangle = \langle \chi \cdot T_B, \Theta_{xy} \rangle = \langle \chi \cdot \Theta_{xy}, T_A \rangle = \langle \chi \cdot \Theta_{xy}, T_B \rangle = 0.$$

So, we can write

$$d_\chi^H(A+B) = d_\chi^H(A) + d_\chi^H(B) + \langle \chi \cdot \Theta_{xy}, \Theta_{xy} \rangle .$$

Now it suffices to show that the last term is nonnegative. Since $\chi \cdot \Theta_{xy} = \mathcal{T}(\Theta_{xy})$, we compute

$$\langle \chi \cdot \Theta_{xy}, \Theta_{xy} \rangle = \langle \mathcal{T}(\Theta_{xy}), \Theta_{xy} \rangle = \frac{1}{h} \langle h\mathcal{T}(\Theta_{xy}), \Theta_{xy} \rangle$$

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$$= \frac{1}{h}\left\langle \mathcal{T}^2(\Theta_{xy}), \Theta_{xy}\right\rangle = \frac{1}{h}\left\langle \mathcal{T}(\Theta_{xy}), \mathcal{T}^*(\Theta_{xy})\right\rangle$$
$$= \frac{1}{h}\left\langle \mathcal{T}(\Theta_{xy}), \mathcal{T}(\Theta_{xy})\right\rangle \geq 0. \quad \square$$

We point out that our results in the paper are presented for linear characters, i.e., $\chi(e) = 1$. They are in fact true for irreducible characters of any degree $\chi(e)$. The proofs are essentially the same up to a positive multiple (see [5, p. 153]).

3. Main theorem (for three matrices). Let A, B, and C be $n \times n$ positive semidefinite matrices. We write

$$A = (a_{ij}) = \langle x_j, x_i\rangle, \quad B = (b_{ij}) = \langle y_j, y_i\rangle, \quad C = (c_{ij}) = \langle z_j, z_i\rangle, \quad 1 \leq i, j \leq n,$$

where x, y, and z are mutually orthogonal vectors in some $\mathbb{C}^K$. (One may take K to be the sum of the ranks of A, B, and C.) Let $X = \{x_1^*, \ldots, x_n^*\}$, $Y = \{y_1^*, \ldots, y_n^*\}$, and $Z = \{z_1^*, \ldots, z_n^*\}$.

LEMMA 3.1. *With the set-up above, we can write T_{A+B+C} as*

$$T_{A+B+C} = T_{A+B} + T_{A+C} - T_A + \Gamma_{yz},$$

where Γ_{yz} is the sum of tensor products each of which contains at least one vector from the set Y and at least one vector from the set Z.

Proof. Using the distributive property of tensor products, we can write

$$T_{A+B+C} = \sum_{w_i \in X\cup Y\cup Z} w_1 \otimes w_2 \otimes \cdots \otimes w_n,$$
$$T_{A+B} = \sum_{w_i \in X\cup Y} w_1 \otimes w_2 \otimes \cdots \otimes w_n,$$
$$T_{A+C} = \sum_{w_i \in X\cup Z} w_1 \otimes w_2 \otimes \cdots \otimes w_n,$$
$$T_A = \sum_{w_i \in X} w_1 \otimes w_2 \otimes \cdots \otimes w_n.$$

For each $w_1 \otimes w_2 \otimes \cdots \otimes w_n$, the following chart gives its coefficient in the expressions above. The left hand side exploits all possible appearances of individual w_i's within the tensor product $w_1 \otimes w_2 \otimes \cdots \otimes w_n$ and the numerical values on the right hand side are the coefficients of $w_1 \otimes w_2 \otimes \cdots \otimes w_n$ (in T_{A+B+C}, T_{A+B}, etc) with these choices of w_i's.

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Type of w_i	T_{A+B+C}	T_{A+B}	T_{A+C}	T_A	Γ_{yz}
$w_i \in X$ for all i	1	1	1	1	0
$w_i \in Y$ for all i	1	1	0	0	0
$w_i \in Z$ for all i	1	0	1	0	0
$w_i \in X \cup Y$ for all i, but not all in X nor in Y	1	1	0	0	0
$w_i \in X \cup Z$ for all i, but not all in X nor in Z	1	0	1	0	0
$w_i \in Y \cup Z$ for all i, but not all in Y nor in Z	1	0	0	0	1
$\exists i, j, k$ with $w_i \in X$, $w_j \in Y$ and, $w_k \in Z$	1	0	0	0	1

One can obtain matrix W from the right hand side for each $w_1 \otimes w_2 \otimes \cdots \otimes w_n$. Namely, $W = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$. Moreover, the vector given by $\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$ is a null vector for the matrix W. Therefore, considering the charts for all possible $w_1 \otimes w_2 \otimes \cdots \otimes w_n$, we obtain $T_{A+B+C} = T_{A+B} + T_{A+C} - T_A + \Gamma_{yz}$ as desired. $\square$

Now we are ready to show our main result.

THEOREM 3.2. *Let A, B, and C be $n \times n$ positive semidefinite matrices. Then*

$$d_\chi^H(A+B+C) + d_\chi^H(A) \geq d_\chi^H(A+B) + d_\chi^H(A+C).$$

Proof. Let $N = d_\chi^H(A+B+C)$. Then by Lemma 3.1, we have

$$\begin{aligned} N &= \langle \chi \cdot (T_{A+B} + T_{A+C} - T_A + \Gamma_{yz}), T_{A+B} + T_{A+C} - T_A + \Gamma_{yz} \rangle \\ &= \langle \chi \cdot T_{A+B}, T_{A+B} \rangle + \langle \chi \cdot T_{A+B}, T_{A+C} \rangle - \langle \chi \cdot T_{A+B}, T_A \rangle + \langle \chi \cdot T_{A+B}, \Gamma_{yz} \rangle \\ &\quad + \langle \chi \cdot T_{A+C}, T_{A+B} \rangle + \langle \chi \cdot T_{A+C}, T_{A+C} \rangle - \langle \chi \cdot T_{A+C}, T_A \rangle + \langle \chi \cdot T_{A+C}, \Gamma_{yz} \rangle \\ &\quad - \langle \chi \cdot T_A, T_{A+B} \rangle - \langle \chi \cdot T_A, T_{A+C} \rangle + \langle \chi \cdot T_A, T_A \rangle - \langle \chi \cdot T_A, \Gamma_{yz} \rangle \\ &\quad + \langle \chi \cdot \Gamma_{yz}, T_{A+B} \rangle + \langle \chi \cdot \Gamma_{yz}, T_{A+C} \rangle - \langle \chi \cdot \Gamma_{yz}, T_A \rangle + \langle \chi \cdot \Gamma_{yz}, \Gamma_{yz} \rangle . \end{aligned}$$

We inspect each of the above terms. Note that

$$\langle \chi \cdot T_{A+B}, T_{A+B} \rangle = d_\chi^H(A+B),$$

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$$\begin{aligned}\langle \chi \cdot T_{A+C}, T_{A+C}\rangle &= d_\chi^H(A+C),\\ \langle \chi \cdot T_A, T_A\rangle &= d_\chi^H(A).\end{aligned}$$

We also know (by (2.1)) that $T_{A+B} = T_A + T_B + \Theta_{xy}$ and $T_{A+C} = T_A + T_C + \Theta_{xz}$. Thus,

$$\begin{aligned}\langle \chi \cdot T_{A+B}, T_{A+C}\rangle &= \langle \chi \cdot T_A + \chi \cdot T_B + \chi \cdot \Theta_{xy}, T_A + T_C + \Theta_{xz}\rangle\\ &= d_\chi^H(A) + \langle \chi \cdot T_A, T_C\rangle + \langle \chi \cdot T_A, \Theta_{xz}\rangle\\ &\quad + \langle \chi \cdot T_B, T_A\rangle + \langle \chi \cdot T_B, T_C\rangle + \langle \chi \cdot T_B, \Theta_{xz}\rangle\\ &\quad + \langle \chi \cdot \Theta_{xy}, T_A\rangle + \langle \chi \cdot \Theta_{xy}, T_C\rangle + \langle \chi \cdot \Theta_{xy}, \Theta_{xz}\rangle .\end{aligned}$$

By the orthogonality of the sets X, Y, Z and the reasoning elaborated in the proof of Proposition 2.2, the identity above reduces to $\langle \chi \cdot T_{A+B}, T_{A+C}\rangle = d_\chi^H(A)$. Note also that

$$\langle \chi \cdot T_{A+B}, T_A\rangle = d_\chi^H(A) + \langle \chi \cdot T_B, T_A\rangle + \langle \chi \cdot \Theta_{xy}, T_A\rangle = d_\chi^H(A).$$

Next, we have

$$\langle \chi \cdot T_{A+B}, \Gamma_{yz}\rangle = \langle \chi \cdot T_A, \Gamma_{yz}\rangle + \langle \chi \cdot T_B, \Gamma_{yz}\rangle + \langle \chi \cdot \Theta_{xy}, \Gamma_{yz}\rangle .$$

From Lemma 3.1 we know that Γ_{yz} is the sum of tensors containing at least one vector from each set Y and Z. Considering the structure of T_A, T_B, Θ_{xy} and the fact that χ action permutes the vectors within the tensor product, the orthogonality of X, Y, and Z results in

$$\langle \chi \cdot T_A, \Gamma_{yz}\rangle = \langle \chi \cdot T_B, \Gamma_{yz}\rangle = \langle \chi \cdot \Theta_{xy}, \Gamma_{yz}\rangle = 0,$$

which, in turn, imply $\langle \chi \cdot T_{A+B}, \Gamma_{yz}\rangle = 0$.

In a similar way, $\langle \chi \cdot T_{A+C}, T_{A+B}\rangle = d_\chi^H(A)$. In addition, we also have the following equalities:

$$\langle \chi \cdot T_{A+C}, T_A\rangle = \langle \chi \cdot T_A, T_{A+B}\rangle = \langle \chi \cdot T_A, T_{A+C}\rangle = d_\chi^H(A),$$

$$\langle \chi \cdot T_{A+C}, \Gamma_{yz}\rangle = \langle \chi \cdot T_A, \Gamma_{yz}\rangle = 0,$$

$$\langle \chi \cdot \Gamma_{yz}, T_{A+B}\rangle = \langle \chi \cdot \Gamma_{yz}, T_{A+C}\rangle = \langle \chi \cdot \Gamma_{yz}, T_A\rangle = 0.$$

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By a similar argument as in the proof of Proposition 2.2, we have $\langle \chi \cdot \Gamma_{yz}, \Gamma_{yz} \rangle \geq 0$. Combining all of the above computations, we arrive at

$$\begin{aligned} d_\chi^H(A+B+C) &= d_\chi^H(A+B) + d_\chi^H(A) - d_\chi^H(A) + d_\chi^H(A) + d_\chi^H(A+C) - d_\chi^H(A) \\ &\quad - d_\chi^H(A) - d_\chi^H(A) + d_\chi^H(A) + \langle \chi \cdot \Gamma_{yz}, \Gamma_{yz} \rangle \\ &= d_\chi^H(A+B) + d_\chi^H(A+C) - d_\chi^H(A) + \langle \chi \cdot \Gamma_{yz}, \Gamma_{yz} \rangle \\ &\geq d_\chi^H(A+B) + d_\chi^H(A+C) - d_\chi^H(A). \quad \square \end{aligned}$$

A special case of the above inequality is the well known determinant inequality $\det(A+B) \geq \det(A) + \det(B)$ for positive semidefinite matrices A and B (see, e.g., [4, p. 117] or [1, p. 490]). The following inequalities on determinant (det) and permanent (per) that have appeared in [7] are also immediate consequences of our theorem.

Corollary 3.3. *Let A, B, and C be $n \times n$ positive semidefinite matrices. Then*

$$\begin{aligned} \det(A+B+C) + \det(A) &\geq \det(A+B) + \det(A+C); \\ \operatorname{per}(A+B+C) + \operatorname{per}(A) &\geq \operatorname{per}(A+B) + \operatorname{per}(A+C). \end{aligned}$$

Proof. Put $H = S_n$. We specify $\chi = sgn$, the signum function, for the determinant and $\chi = 1$ for the permanent, respectively, then apply Theorem 3.2. $\square$

4. The inequality for more positive semidefinite matrices. Now we extend Theorem 3.2 to any finite number of positive semidefinite matrices.

Theorem 4.1. *Let $A_1, \ldots, A_m$, $m \geq 3$, be $n \times n$ positive semidefinite matrices. Then*

$$d_\chi^H\Big(\sum_{j=1}^m A_j\Big) \geq \sum_{j \neq i}^m d_\chi^H(A_i + A_j) - (m-2) d_\chi^H(A_i), \quad i = 1, \ldots, m.$$

Proof. We use induction on m. For $m = 3$, it is Theorem 3.2. Assume that the assertion holds true for m (≥ 3) matrices. We show that it holds true for $m+1$ matrices. Without loss of generality, we take $A_i = A_1$.

Case I: m is even. Let $m = 2k, k \geq 2$. For simplicity, set $B_1 = A_2 + A_3$, $B_2 = A_4 + A_5$, $\ldots$, $B_k = A_m + A_{m+1}$. Then

$$d_\chi^H(A_1 + A_2 + A_3 + \cdots + A_m + A_{m+1}) = d_\chi^H(A_1 + B_1 + \cdots + B_k).$$

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By induction hypotheses, we obtain

$$d_\chi^H(A_1+B_1+\cdots+B_k) \geq \sum_{j=1}^{k} d_\chi^H(A_1+B_j)-(k-1)d_\chi^H(A_1).$$

It follows that

$$\begin{aligned}
&d_\chi^H(A_1+B_1+\cdots+B_k)+(m-1)d_\chi^H(A_1)\\
&\geq \sum_{j=1}^{k} d_\chi^H(A_1+B_j)-(k-1)d_\chi^H(A_1)+(m-1)d_\chi^H(A_1)\\
&= d_\chi^H(A_1+A_2+A_3)+\cdots+d_\chi^H(A_1+A_m+A_{m+1})+(m-k)d_\chi^H(A_1)\\
&\geq d_\chi^H(A_1+A_2)+d_\chi^H(A_1+A_3)+\cdots+d_\chi^H(A_1+A_{m+1})-kd_\chi^H(A_1)\\
&\quad +(m-k)d_\chi^H(A_1)\\
&= d_\chi^H(A_1+A_2)+d_\chi^H(A_1+A_3)+\cdots+d_\chi^H(A_1+A_{m+1})+(m-2k)d_\chi^H(A_1)\\
&= d_\chi^H(A_1+A_2)+d_\chi^H(A_1+A_3)+\cdots+d_\chi^H(A_1+A_{m+1}).
\end{aligned}$$

Therefore,

$$d_\chi^H(A_1+A_2+\cdots+A_m+A_{m+1}) \geq \sum_{j=2}^{m+1} d_\chi^H(A_1+A_j)-(m-1)d_\chi^H(A_1).$$

Case II: m is odd. Let $m=2k-1$ with $k \geq 3$. Put $A=A_1+A_2$. Then

$$\begin{aligned}
&d_\chi^H(A_1+A_2+\cdots+A_{m+1})+(m-1)d_\chi^H(A_1)\\
&= d_\chi^H(A+A_3+\cdots+A_{m+1})+(m-1)d_\chi^H(A_1)\\
&\geq d_\chi^H(A+A_3)+\cdots+d_\chi^H(A+A_{m+1})-(m-2)d_\chi^H(A)+(m-1)d_\chi^H(A_1)\\
&= d_\chi^H(A_1+A_2+A_3)+\cdots+d_\chi^H(A_1+A_2+A_{m+1})-(m-2)d_\chi^H(A)\\
&\quad +(m-1)d_\chi^H(A_1)\\
&\geq d_\chi^H(A)+d_\chi^H(A_1+A_3)+\cdots+d_\chi^H(A)+d_\chi^H(A_1+A_{m+1})-(m-1)d_\chi^H(A_1)\\
&\quad -(m-2)d_\chi^H(A)+(m-1)d_\chi^H(A_1)\\
&= (m-1)d_\chi^H(A)+d_\chi^H(A_1+A_3)+\cdots+d_\chi^H(A_1+A_{m+1})-(m-2)d_\chi^H(A)\\
&= d_\chi^H(A)+d_\chi^H(A_1+A_3)+\cdots+d_\chi^H(A_1+A_{m+1})\\
&= d_\chi^H(A_1+A_2)+d_\chi^H(A_1+A_3)+\cdots+d_\chi^H(A_1+A_{m+1}), \quad \text{as desired.} \quad \square
\end{aligned}$$

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