

GRADED TRIANGULAR ALGEBRAS*

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Abstract. The structure of graded triangular algebras $\mathfrak{T}$ of arbitrary dimension are studied in this paper. This is motivated in part for the important role that triangular algebras play in the study of oriented graphs, upper triangular matrix algebras or nest algebras. It is shown that $\mathfrak{T}$ decomposes as $\mathfrak{T} = \mathcal{U} + (\sum_{i \in I} \mathfrak{T}_i)$, where $\mathcal{U}$ is an $\mathcal{R}$ -submodule contained in the 0-homogeneous component and any $\mathfrak{T}_i$ a well-described (graded) ideal satisfying $\mathfrak{T}_i \mathfrak{T}_j = 0$ if $i \neq j$. Since any $\mathfrak{T}$ is not simple as associative algebra, the concept of quasi-simple triangular algebra is introduced as those $\mathfrak{T}$ which are as near to simplicity as possible. Under mild conditions, the quasi-simplicity of $\mathfrak{T}$ is characterized and it is proven that $\mathfrak{T}$ is the direct sum of quasi-simple graded triangular algebras which are also ideals.

Key words. Triangular algebra, Graded algebra, Simplicity, Structure theory.

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1. Introduction and preliminaries. Let us fix an arbitrary commutative ring of scalars $\mathcal{R}$. Given two associative $\mathcal{R}$ -algebras $\mathfrak{A}$ and $\mathfrak{B}$ and a non-zero faithful $(\mathfrak{A}, \mathfrak{B})$ -bimodule $\mathfrak{M}$, the *triangular algebra* consisting of $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{M}$ is defined as the associative $\mathcal{R}$ -algebra

$$\mathfrak{T} = \begin{pmatrix} \mathfrak{A} & \mathfrak{M} \\ 0 & \mathfrak{B} \end{pmatrix} = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in \mathfrak{A}, b \in \mathfrak{B}, m \in \mathfrak{M} \right\}$$

endowed with the matrix-like $\mathcal{R}$ -module structure and the matrix-like product such that $\text{Ann}(\mathfrak{T}) = 0$. We note that in the literature, see for instance the seminal papers [12] and [13], the algebras $\mathfrak{A}$ and $\mathfrak{B}$ are supposed to be unital. However, we just impose in our definition that the set $\text{Ann}(\mathfrak{T}) := \{t \in \mathfrak{T} : t\mathfrak{T} = 0\} \cup \{t \in \mathfrak{T} : \mathfrak{T}t = 0\} = 0$ (observe that in case $\mathfrak{A}$ and $\mathfrak{B}$ are unital, then $\text{Ann}(\mathfrak{T}) = 0$).

This kind of algebra plays an important role in the study of path algebras associated to a finite oriented graph, of upper triangular matrix algebras, of block upper triangular matrix algebras and of nest algebras over a real or complex Banach space

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B or Hilbert space H , respectively (see [3, 5, 22, 27]). Triangular algebras have been intensively studied in the last years, especially the problem of describing different types of morphisms in this category (see [17, 18, 19, 25, 28, 29, 30]).

In the present paper, we are interested in studying the structure of graded triangular algebras. The study of gradings on different classes of algebras has been remarkable in recent years, especially those gradings in which $(G, +)$ is an abelian group (see [8, 16, 20, 23, 24]). In particular, graded matrix algebras have been considered in [1, 4, 7, 15, 21], not only for the interest by themselves but also because we can derive from them many examples of graded Lie algebras, which play an important role in the theory of strings, color supergravity, Walsh functions, and electroweak interactions [11, 14, 26].

We recall that an associative $\mathcal{R}$ -algebra $\mathfrak{A}$ and an abelian group G is said to be a *graded associative algebra*, by means of G , if

$$\mathfrak{A} = \bigoplus_{g \in G} \mathfrak{A}_g,$$

where $\mathfrak{A}_g$ is an $\mathcal{R}$ -submodule satisfying $\mathfrak{A}_g \mathfrak{A}_h \subset \mathfrak{A}_{g+h}$ for any $h \in G$. We call $\Sigma_{\mathfrak{A}} = \{g \in G : \mathfrak{A}_g \neq 0\}$ the *support* of the grading.

In this paper, we also consider graded bimodules. Given two G -graded associative $\mathcal{R}$ -algebras $\mathfrak{A} = \bigoplus_{g \in G} \mathfrak{A}_g$ and $\mathfrak{B} = \bigoplus_{g \in G} \mathfrak{B}_g$, an $(\mathfrak{A}, \mathfrak{B})$ -bimodule $\mathfrak{M}$ is said to be *G -graded* if

$$\mathfrak{M} = \bigoplus_{g \in G} \mathfrak{M}_g,$$

where each $\mathfrak{M}_g$ is an $\mathcal{R}$ -submodule such that $\mathfrak{A}_h \mathfrak{M}_g \subset \mathfrak{M}_{h+g}$ and $\mathfrak{M}_g \mathfrak{B}_h \subset \mathfrak{M}_{g+h}$ for any $h \in G$.

The concept of graded triangular algebra appears in a natural way as those triangular algebras $\mathfrak{T}$ in which $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{M}$ are graded by the same abelian group G .

DEFINITION 1.1. A triangular algebra $\mathfrak{T} = \begin{pmatrix} \mathfrak{A} & \mathfrak{M} \\ 0 & \mathfrak{B} \end{pmatrix}$ is said to be *graded* by the abelian group G if $\mathfrak{A} = \bigoplus_{g \in G} \mathfrak{A}_g$ and $\mathfrak{B} = \bigoplus_{g \in G} \mathfrak{B}_g$ are G -graded algebras and $\mathfrak{M} = \bigoplus_{g \in G} \mathfrak{M}_g$ is a G -graded $(\mathfrak{A}, \mathfrak{B})$ -bimodule.

Observe that Definition 1.1 is equivalent to assert that we can decompose $\mathfrak{T}$ as the direct sum of $\mathcal{R}$ -submodules

$$\mathfrak{T} = \bigoplus_{g \in G} \mathfrak{T}_g$$

such that

$$\mathfrak{T}_g \mathfrak{T}_h \subset \mathfrak{T}_{g+h} \text{ for any } g, h \in G,$$

and each $\mathfrak{T}_g$ is of the form

$$\mathfrak{T}_g = \begin{pmatrix} \mathfrak{A}_g & \mathfrak{M}_g \\ 0 & \mathfrak{B}_g \end{pmatrix}.$$

As usual, the *support* of the grading of $\mathfrak{T}$ will be the set $\Sigma_{\mathfrak{T}} = \{g \in G \setminus 0 : \mathfrak{T}_g \neq 0\}$. Since the fact $\mathfrak{T}_g \neq 0$ implies either $\mathfrak{A}_g \neq 0$ or $\mathfrak{B}_g \neq 0$ or $\mathfrak{M}_g \neq 0$, we clearly have

$$(1.1) \quad \Sigma_{\mathfrak{T}} = \Sigma_{\mathfrak{A}} \cup \Sigma_{\mathfrak{B}} \cup \Sigma_{\mathfrak{M}}.$$

The regularity concepts will be understood by looking at $\mathfrak{T}$ as an associative algebra with a grading. A *subalgebra* S of a graded triangular algebra $\mathfrak{T} = \bigoplus_{g \in G} \mathfrak{T}_g$ is a G -graded $\mathcal{R}$ -submodule $S = \bigoplus_{g \in G} S_g$ closed under multiplication, that is, satisfying $SS \subset S$. This is equivalent to assert that there exist a graded subalgebra $S_{\mathfrak{A}} = \bigoplus_{g \in G} (S_{\mathfrak{A}})_g$ of $\mathfrak{A}$, a graded subalgebra $S_{\mathfrak{B}} = \bigoplus_{g \in G} (S_{\mathfrak{B}})_g$ of $\mathfrak{B}$, and a graded $(S_{\mathfrak{A}}, S_{\mathfrak{B}})$ -subbimodule $S_{\mathfrak{M}} = \bigoplus_{g \in G} (S_{\mathfrak{M}})_g$ of $\mathfrak{M}$, (that is, any $(S_{\mathfrak{M}})_g$ is an $\mathcal{R}$ -submodule of $\mathfrak{M}$ and $S_{\mathfrak{M}}$ is closed under the actions of $(S_{\mathfrak{A}}, S_{\mathfrak{B}})$ induced by the ones of $(\mathfrak{A}, \mathfrak{B})$ over $\mathfrak{M}$), such that S is the graded associative algebra

$$(1.2) \quad S = \begin{pmatrix} \bigoplus_{g \in G} (S_{\mathfrak{A}})_g & \bigoplus_{g \in G} (S_{\mathfrak{M}})_g \\ 0 & \bigoplus_{g \in G} (S_{\mathfrak{B}})_g \end{pmatrix}.$$

A subalgebra $\mathfrak{J}$ is said to be an *ideal* if $\mathfrak{J}\mathfrak{T} + \mathfrak{T}\mathfrak{J} \subset \mathfrak{J}$. Observe that $\mathfrak{J}$ is of the form given by (1.2), but in this case, $S_{\mathfrak{A}}$ and $S_{\mathfrak{B}}$ are graded ideals of $\mathfrak{A}$ and $\mathfrak{B}$, respectively, and $S_{\mathfrak{M}}$ is a graded $(\mathfrak{A}, \mathfrak{B})$ -subbimodule of $\mathfrak{M}$. It is important to note that we can find ideals $\mathfrak{J}$ in $\mathfrak{T}$ which of course are ideals of $\mathfrak{T}$ as graded associative algebras but are not triangular algebras by themselves because either the bimodule component is not faithful or $\text{Ann}(\mathfrak{J}) \neq 0$, (see examples in the paragraph below). It is interesting to see that the decomposition of $\mathfrak{T}$ by a family of ideals is given by ideals which are also (graded) triangular algebras by themselves. See Corollary 2.6 and Theorem 3.4.

It is necessary to discuss for a while about the concept of simplicity in the framework of triangular algebras $\mathfrak{T} = \begin{pmatrix} \mathfrak{A} & \mathfrak{M} \\ 0 & \mathfrak{B} \end{pmatrix}$. We have that $\mathfrak{H} := \begin{pmatrix} 0 & \mathfrak{M} \\ 0 & 0 \end{pmatrix}$,

$\mathfrak{K} := \begin{pmatrix} \mathfrak{A} & \mathfrak{M} \\ 0 & 0 \end{pmatrix}$ and $\mathfrak{P} := \begin{pmatrix} 0 & \mathfrak{M} \\ 0 & \mathfrak{B} \end{pmatrix}$ are always ideals of $\mathfrak{T}$, so the usual concept of simplicity of algebras is not of interest in our framework since $\mathfrak{T}$ is not a simple associative algebra. However, it is interesting to study those triangular algebras which are as near as possible to simplicity that we will call *quasi-simple*.

DEFINITION 1.2. A graded triangular algebra $\mathfrak{T}$ is said to be *quasi-simple* if $\mathfrak{T} \neq 0$ and its only ideals are $\{0\}$, $\mathfrak{H}$, $\mathfrak{K}$, $\mathfrak{P}$ and $\mathfrak{T}$. Finally, observe that if $\mathfrak{T}$ is quasi-simple then the only non-zero ideal which is also a triangular algebra is $\mathfrak{T}$.

The paper is organized as follows. In Section 2, we develop connection techniques on the support of $\mathfrak{T}$ in order to show that $\mathfrak{T}$ is of the form $\mathfrak{T} = \mathcal{U} + (\sum_{i \in I} \mathfrak{T}_i)$, where $\mathcal{U}$ is an $\mathcal{R}$ -submodule contained in the 0-homogeneous component and $\mathfrak{T}_i$ is an (graded) ideal satisfying $\mathfrak{T}_i \mathfrak{T}_j = 0$ if $i \neq j$. If furthermore $\mathfrak{T}_0$ is tight, $\mathfrak{T}_i$ becomes a triangular algebra for adequate associative $\mathcal{R}$ -algebras $\mathfrak{A}_j$ and $\mathfrak{B}_j$ and an adequate $(\mathfrak{A}_j, \mathfrak{B}_j)$ -bimodule $\mathfrak{M}_j$. In Section 3, under certain conditions, the quasi-simplicity of $\mathfrak{T}$ is characterized and it is shown that $\mathfrak{T}$ is the direct sum of ideals which are quasi-simple graded triangular algebras.

Finally, we would like to note that throughout this paper, there is no restriction on the dimensions of $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{M}$.

2. Connections in the support techniques. Decompositions of $\mathfrak{T}$. From now on, $\mathfrak{T}$ denotes a graded triangular algebra by means of an abelian group G , where

$$\begin{aligned} \mathfrak{T} = \bigoplus_{g \in G} \mathfrak{T}_g &= \mathfrak{T}_0 \oplus \left(\bigoplus_{g \in \Sigma_{\mathfrak{T}}} \mathfrak{T}_g \right) = \begin{pmatrix} \mathfrak{A}_0 \oplus \left(\bigoplus_{g \in \Sigma_{\mathfrak{A}}} \mathfrak{A}_g \right) & \mathfrak{M}_0 \oplus \left(\bigoplus_{h \in \Sigma_{\mathfrak{M}}} \mathfrak{M}_h \right) \\ 0 & \mathfrak{B}_0 \oplus \left(\bigoplus_{k \in \Sigma_{\mathfrak{B}}} \mathfrak{B}_k \right) \end{pmatrix} \\ &= \begin{pmatrix} \mathfrak{A}_0 & \mathfrak{M}_0 \\ 0 & \mathfrak{B}_0 \end{pmatrix} \oplus \begin{pmatrix} \bigoplus_{g \in \Sigma_{\mathfrak{A}}} \mathfrak{A}_g & \bigoplus_{h \in \Sigma_{\mathfrak{M}}} \mathfrak{M}_h \\ 0 & \bigoplus_{k \in \Sigma_{\mathfrak{B}}} \mathfrak{B}_k \end{pmatrix} \end{aligned}$$

is the corresponding grading.

For any non-empty subset H of a given group G , we will denote by $-H := \{-h : h \in H\} \subset G$.

DEFINITION 2.1. Let g and h be two elements in $\Sigma_{\mathfrak{M}}$. We say that g is *connected* to h if there exist $g_1, g_2, \dots, g_n \in \pm \Sigma_{\mathfrak{T}} = \pm \Sigma_{\mathfrak{A}} \cup \pm \Sigma_{\mathfrak{B}} \cup \pm \Sigma_{\mathfrak{M}}$ (see (1.1)), such that

1. $g_1 = g$,
2. $\{g_1 + g_2, g_1 + g_2 + g_3, \dots, g_1 + g_2 + g_3 + \dots + g_{n-1}\} \subset \pm \Sigma_{\mathfrak{T}}$,
3. $g_1 + g_2 + g_3 + \dots + g_{n-1} + g_n \in \{\pm h\}$.

We also say that $\{g_1, g_2, \dots, g_n\}$ is a *connection* from g to h .

PROPOSITION 2.2. *The relation $\sim$ in $\Sigma_{\mathfrak{M}}$ defined by $g \sim h$ if and only if g is connected to h is an equivalence relation.*

Proof. Clearly, the set $\{g\}$ is a connection from g to itself and so the relation is reflexive.

If $g \sim h$, then there exists a connection $\{g_1, g_2, \dots, g_n\}$ from g to h , being so

$$\{g_1 + g_2, g_1 + g_2 + g_3, \dots, g_1 + g_2 + \dots + g_{n-1}\} \subset \pm \Sigma_{\mathfrak{A}}$$

and $g_1 + g_2 + \dots + g_n \in \{\pm h\}$. Hence, we have two possibilities. In the first one $g_1 + g_2 + \dots + g_n = h$, and in the second one $g_1 + g_2 + \dots + g_n = -h$. Now observe that the set $\{h, -g_n, -g_{n-1}, \dots, -g_2\}$ gives us a connection from h to g for the first possibility and $\{h, g_n, g_{n-1}, \dots, g_2\}$ for the second. Hence, $\sim$ is symmetric.

Finally, suppose $g \sim h$ and $h \sim k$, and write $\{g_1, g_2, \dots, g_n\}$ for a connection from g to h and $\{h_1, h_2, \dots, h_m\}$ for a connection from h to k . If $h \notin \{\pm k\}$, then $m \geq 1$ and so $\{g_1, g_2, \dots, g_n, h_2, \dots, h_m\}$ is a connection from g to k if $g_1 + g_2 + \dots + g_n = h$, and $\{g_1, g_2, \dots, g_n, -h_2, \dots, -h_m\}$ if $g_1 + g_2 + \dots + g_n = -h$. If $h \in \{\pm k\}$ then $\{g_1, g_2, \dots, g_n\}$ is a connection from g to k . Therefore, $g \sim k$. $\square$

Consider the quotient set $\Sigma_{\mathfrak{M}}/\sim = \{[g] : g \in \Sigma_{\mathfrak{M}}\}$. Clearly, if $h \in [g]$ and $-h \in \Sigma_{\mathfrak{M}}$, then $-h \in [g]$.

Our next goal is to associate an $(\mathfrak{A}, \mathfrak{B})$ -subbimodule $\mathfrak{M}_{[g]}$ of $\mathfrak{M}$ to any $[g]$. For each $[g]$, $g \in \Sigma_{\mathfrak{M}}$, we define the following graded $\mathcal{R}$ -submodule of $\mathfrak{M}$:

$$\mathfrak{M}_{0,[g]} := \sum_{h \in [g], -h \in \Sigma_{\mathfrak{A}} \cup \Sigma_{\mathfrak{B}}} (\mathfrak{A}_{-h} \mathfrak{M}_h + \mathfrak{M}_h \mathfrak{B}_{-h}) \subset \mathfrak{M}_0$$

and

$$\mathfrak{M}_{[g]} := \mathfrak{M}_{0,[g]} \oplus \left(\bigoplus_{h \in [g]} \mathfrak{M}_h \right).$$

LEMMA 2.3. *The following assertions hold:*

1. *For any $g \in \Sigma_{\mathfrak{M}}$ and $a \in \Sigma_{\mathfrak{A}}$ with $a \neq -g$, if $\mathfrak{A}_a \mathfrak{M}_g \neq 0$, then $a + g \in \Sigma_{\mathfrak{M}}$ with $a + g \sim g$.*
2. *If $g \in \Sigma_{\mathfrak{M}}$ satisfies that $-g \in \Sigma_{\mathfrak{A}}$, then for any $c \in \Sigma_{\mathfrak{A}}$ such that*

$$\mathfrak{A}_c(\mathfrak{A}_{-g} \mathfrak{M}_g) \neq 0$$

(resp., $d \in \Sigma_{\mathfrak{B}}$ such that $(\mathfrak{A}_{-g} \mathfrak{M}_g) \mathfrak{B}_d \neq 0$), we have $c \in \Sigma_{\mathfrak{M}}$ with $g \sim c$ (resp., $d \in \Sigma_{\mathfrak{M}}$ with $g \sim d$).

3. For any $g \in \Sigma_{\mathfrak{M}}$ and $b \in \Sigma_{\mathfrak{B}}$ with $b \neq -g$, if $\mathfrak{M}_g \mathfrak{B}_b \neq 0$, then $g + b \in \Sigma_{\mathfrak{M}}$ with $g + b \sim g$.
4. If $g \in \Sigma_{\mathfrak{M}}$ satisfies that $-g \in \Sigma_{\mathfrak{B}}$, then for any $d \in \Sigma_{\mathfrak{B}}$ such that

$$(\mathfrak{M}_g \mathfrak{B}_{-g}) \mathfrak{B}_d \neq 0$$

(resp., $c \in \Sigma_{\mathfrak{A}}$ such that $\mathfrak{A}_c(\mathfrak{M}_g \mathfrak{B}_{-g}) \neq 0$), we have $d \in \Sigma_{\mathfrak{M}}$ with $g \sim d$ (resp., $c \in \Sigma_{\mathfrak{M}}$ with $g \sim c$).

Proof. 1. The fact $\mathfrak{A}_a \mathfrak{M}_g \neq 0$ with $a \neq -g$ ensures $a + g \in \Sigma_{\mathfrak{M}}$. Hence, we just consider the connection $\{g, a\}$.

2. If $c = \pm g$, then it is clear. Hence, suppose $c \neq \pm g$. From $\mathfrak{A}_c(\mathfrak{A}_{-g} \mathfrak{M}_g) \neq 0$ we have $(\mathfrak{A}_c \mathfrak{A}_{-g}) \mathfrak{M}_g \neq 0$ and so $c - g \in \Sigma_{\mathfrak{A}}$. From here $\{g, c - g\}$ is a connection from g to c . We can argue as above if $d \in \Sigma_{\mathfrak{B}}$ with $(\mathfrak{A}_{-g} \mathfrak{M}_g) \mathfrak{B}_d \neq 0$.

3. and 4. Similar to 1. and 2., respectively. $\square$

PROPOSITION 2.4. The set $\mathfrak{M}_{[g]}$ is an $(\mathfrak{A}, \mathfrak{B})$ -subbimodule of $\mathfrak{M}$ for each $[g] \in \Sigma_{\mathfrak{M}} / \sim$.

Proof. We have

$$\begin{aligned} \mathfrak{A}_0 \mathfrak{M}_{0,[g]} &= \mathfrak{A}_0 \left(\sum_{a \in [g], -a \in \Sigma_{\mathfrak{A}}} \mathfrak{A}_{-a} \mathfrak{M}_a + \sum_{b \in [g], -b \in \Sigma_{\mathfrak{B}}} \mathfrak{M}_b \mathfrak{B}_{-b} \right) \subset \\ &\sum_{a \in [g], -a \in \Sigma_{\mathfrak{A}}} (\mathfrak{A}_0 \mathfrak{A}_{-a}) \mathfrak{M}_a + \sum_{b \in [g], -b \in \Sigma_{\mathfrak{B}}} (\mathfrak{A}_0 \mathfrak{M}_b) \mathfrak{B}_{-b} \subset \\ &\sum_{a \in [g], -a \in \Sigma_{\mathfrak{A}}} \mathfrak{A}_{-a} \mathfrak{M}_a + \sum_{b \in [g], -b \in \Sigma_{\mathfrak{B}}} \mathfrak{M}_b \mathfrak{B}_{-b} \subset \mathfrak{M}_{0,[g]} \end{aligned}$$

and $\mathfrak{A}_0 \mathfrak{M}_h \subset \mathfrak{M}_h$ for any $h \in [g]$. Hence,

$$(2.1) \quad \mathfrak{A}_0 \mathfrak{M}_{[g]} = \mathfrak{A}_0 (\mathfrak{M}_{0,[g]} \oplus (\bigoplus_{h \in [g]} \mathfrak{M}_h)) \subset \mathfrak{M}_{[g]}.$$

Now observe that when $\mathfrak{A}_c(\mathfrak{A}_{-g} \mathfrak{M}_g + \mathfrak{M}_g \mathfrak{B}_{-g}) \neq 0$ for some $c \in \Sigma_{\mathfrak{A}}$ and $g \in \Sigma_{\mathfrak{M}}$, from Lemma 2.3-2, $c \in [g]$ so we can assert

$$\mathfrak{A}_c \mathfrak{M}_{0,[g]} \subset \mathfrak{M}_{[g]} \text{ for any } c \in \Sigma_{\mathfrak{A}}.$$

This fact together with Lemma 2.3-1 allow us to get

$$(2.2) \quad \mathfrak{A}_c \mathfrak{M}_{[g]} = \mathfrak{A}_c (\mathfrak{M}_{0,[g]} \oplus (\bigoplus_{h \in [g]} \mathfrak{M}_h)) \subset \mathfrak{M}_{[g]}$$

for any $c \in \Sigma_{\mathfrak{A}}$. From (2.1) and (2.2), we finally conclude

$$\mathfrak{A}\mathfrak{M}_{[g]} = (\mathfrak{A}_0 \oplus (\bigoplus_{c \in \Sigma_{\mathfrak{A}}} \mathfrak{A}_c))\mathfrak{M}_{[g]} \subset \mathfrak{M}_{[g]}.$$

In a similar way we have

$$\mathfrak{M}_{[g]}\mathfrak{B} \subset \mathfrak{M}_{[g]}$$

so $\mathfrak{M}_{[g]}$ is an $(\mathfrak{A}, \mathfrak{B})$ -subbimodule of $\mathfrak{M}$. $\square$

Consider now $\mathfrak{C} \in \{\mathfrak{A}, \mathfrak{B}\}$ which satisfies $\mathfrak{C} \neq 0$, and as consequence, $\text{Ann}(\mathfrak{T}) = 0$. Since $\mathfrak{C}$ is a G -graded $(\mathfrak{C}, \mathfrak{C})$ -bimodule, we can introduce a concept of connection in $\Sigma_{\mathfrak{C}}$ similar to Definition 2.1 and obtain results analogous to the above ones so that we can consider the quotient set $\Sigma_{\mathfrak{C}} / \sim = \{[c] : c \in \Sigma_{\mathfrak{C}}\}$ and that for any $[c] \in \Sigma_{\mathfrak{C}}$, the graded $\mathcal{R}$ -submodule of $\mathfrak{C}$ given by

$$\mathfrak{C}_{[c]} := \mathfrak{C}_{0,[c]} \oplus (\bigoplus_{b \in [c]} \mathfrak{C}_b),$$

where

$$\mathfrak{C}_{0,[c]} := \sum_{b \in [c]} \mathfrak{C}_b \mathfrak{C}_{-b} \subset \mathfrak{C}_0$$

is actually an (graded) ideal of $\mathfrak{C}$.

Summarizing, we have the quotient sets $\Sigma_{\mathfrak{M}} / \sim$, $\Sigma_{\mathfrak{A}} / \sim$ and $\Sigma_{\mathfrak{B}} / \sim$ such that $\mathfrak{M}_{[g]}$, $[g] \in \Sigma_{\mathfrak{M}}$, is an $(\mathfrak{A}, \mathfrak{B})$ -subbimodule of $\mathfrak{M}$, any $\mathfrak{A}_{[a]}$, $[a] \in \Sigma_{\mathfrak{A}}$, an ideal of $\mathfrak{A}$, and any $\mathfrak{B}_{[b]}$, $[b] \in \Sigma_{\mathfrak{B}}$, an ideal of $\mathfrak{B}$. To get an adequate decomposition of $\mathfrak{T}$ from these ingredients we need to relate all of these elements.

For any $[g] \in \Sigma_{\mathfrak{M}} / \sim$, since $\text{Ann}(\mathfrak{T}) \neq 0$ and $\mathfrak{A}_0, \mathfrak{B}_0$ are tight, there exists $[a] \in \Sigma_{\mathfrak{A}} / \sim$ such that $\mathfrak{A}_{[a]}\mathfrak{M}_{[g]} \neq 0$. Let us show that the element $[a]$ is unique. To do that, consider $c \in \Sigma_{\mathfrak{A}}$ satisfying $\mathfrak{A}_{[c]}\mathfrak{M}_{[g]} \neq 0$. From the facts $\mathfrak{A}_{[a]}\mathfrak{M}_{[g]} \neq 0$ and $\mathfrak{A}_{[c]}\mathfrak{M}_{[g]} \neq 0$ we can take $a' \in [a]$, $c' \in [c]$ and $g', g'' \in [g]$ such that $\mathfrak{A}_{a'}\mathfrak{M}_{g'} \neq 0$ and $\mathfrak{A}_{c'}\mathfrak{M}_{g''} \neq 0$. Since $g', g'' \in [g]$, we can fix a connection

$$\{g', g_1, \dots, g_n\}$$

from g' to g'' .

Let us distinguish four cases. Firstly, if $a' + g' \neq 0$, $c' + g'' \neq 0$ and so $a' + g'$, $c' + g'' \in \Sigma_{\mathfrak{M}}$, we have that $\{a', g', -a', g_1, \dots, g_n, c', -g''\}$ is a connection from a' to c' if $g' + g_1 + \dots + g_n = g''$ while $\{a', g', -a', g_1, \dots, g_n, -c', g''\}$ gives us the same connection if $g' + g_1 + \dots + g_n = -g''$. From here $a' \sim c'$ and so $[a] = [c]$.

Secondly, if $a' + g' = 0$, $c' + g'' \neq 0$ and so $a' = -g'$, $c' + g'' \in \pm\Sigma_{\mathfrak{M}}$, we have that $\{-g', -g_1, \dots, -g_n, -c', g''\}$ is a connection between a' and c' if $g' + g_1 + \dots + g_n = g''$ while $\{-g', -g_1, \dots, -g_n, c', -g''\}$ if $g' + g_1 + \dots + g_n = -g''$. From here, $[a] = [c]$. Thirdly, if $a' + g' \neq 0$, $c' + g'' = 0$ we can argue as in the previous case to get $[a] = [c]$. Finally, in the fourth case we suppose $a' + g' = 0$, $c' + g'' = 0$ and so $a' = -g'$, $c' = -g''$. Then $\{-g', -g_1, \dots, -g_n\}$ is a connection between a' and b' which implies $[a] = [c]$. We conclude that the element $[a] \in \Sigma_{\mathfrak{A}}/\sim$ such that $\mathfrak{A}_{[a]}\mathfrak{M}_{[g]} \neq 0$ is unique and so, by denoting $\phi_{\mathfrak{A}}([g]) := [a]$, it makes sense to consider the mapping

$$\begin{aligned}\phi_{\mathfrak{A}} : (\Sigma_{\mathfrak{M}}/\sim) &\rightarrow (\Sigma_{\mathfrak{A}}/\sim) \\ [g] &\mapsto [a].\end{aligned}$$

Let us verify that $\phi_{\mathfrak{A}}$ is bijective. On the one hand, if $\phi_{\mathfrak{A}}([g]) = \phi_{\mathfrak{A}}([h])$ for some $[g], [h] \in \Sigma_{\mathfrak{M}}/\sim$, then $\mathfrak{A}_{[a]}\mathfrak{M}_{[g]} \neq 0$ and $\mathfrak{A}_{[a]}\mathfrak{M}_{[h]} \neq 0$. A similar argument to the above one gives us $[g] = [h]$ and so $\phi_{\mathfrak{A}}$ is injective. On the other hand, the faithful character of $\mathfrak{M}$ as $\mathfrak{A}$ -module implies that for any $[a] \in \Sigma_{\mathfrak{A}}/\sim$, there exists $[g] \in \Sigma_{\mathfrak{M}}/\sim$ such that $\mathfrak{A}_{[a]}\mathfrak{M}_{[g]} \neq 0$ so $\phi_{\mathfrak{A}}([g]) = [a]$ and $\phi_{\mathfrak{A}}$ is surjective.

Similarly, we can define the bijective mapping

$$\begin{aligned}\phi_{\mathfrak{B}} : (\Sigma_{\mathfrak{M}}/\sim) &\rightarrow (\Sigma_{\mathfrak{B}}/\sim) \\ [g] &\mapsto [b],\end{aligned}$$

where $\phi_{\mathfrak{B}}([g])$ is the only $[b] \in \Sigma_{\mathfrak{B}}/\sim$ such that $\mathfrak{M}_{[g]}\mathfrak{B}_{[b]} \neq 0$.

Now, for any $[g] \in \Sigma_{\mathfrak{M}}/\sim$ we introduce the following ideal of $\mathfrak{T}$, (see (1.2) and the comments below),

$$(2.3) \quad \mathfrak{T}_{[g]} := \begin{pmatrix} \mathfrak{A}_{\phi_{\mathfrak{A}}([g])} & \mathfrak{M}_{[g]} \\ 0 & \mathfrak{B}_{\phi_{\mathfrak{B}}([g])} \end{pmatrix}$$

and denote by

$$\mathfrak{T}_{0, \Sigma_{\mathfrak{A}}/\sim} := \begin{pmatrix} \sum_{[a] \in \Sigma_{\mathfrak{A}}/\sim} \mathfrak{A}_{0, [a]} & \sum_{[g] \in \Sigma_{\mathfrak{M}}/\sim} \mathfrak{M}_{0, [g]} \\ 0 & \sum_{[b] \in \Sigma_{\mathfrak{B}}/\sim} \mathfrak{B}_{0, [b]} \end{pmatrix} \subset \mathfrak{T}_{[0]}.$$

THEOREM 2.5. *The graded triangular algebra $\mathfrak{T}$ decomposes as*

$$\mathfrak{T} = \mathcal{U} + \left(\sum_{[g] \in \Sigma_{\mathfrak{M}}/\sim} \mathfrak{T}_{[g]} \right),$$

where $\mathcal{U}$ is a linear complement of $\mathfrak{T}_{0, \Sigma_{\mathfrak{A}}/\sim}$ in $\mathfrak{T}_0$ and any $\mathfrak{T}_{[g]}$ is one of the ideals given by (2.3). Furthermore, $\mathfrak{T}_{[g]}\mathfrak{T}_{[h]} = 0$ if $[g] \neq [h]$.

Proof. Since $\mathfrak{D} = \mathfrak{D}_0 \oplus (\bigoplus_{d \in \Sigma_{\mathfrak{D}}} \mathfrak{D}_d)$ and $\sim$ is an equivalence relation on $\Sigma_{\mathfrak{D}}$, we have $\mathfrak{D} = \mathcal{U}_{\mathfrak{D}} \oplus (\sum_{[d] \in \Sigma_{\mathfrak{D}}/\sim} \mathfrak{D}_{[d]})$, where $\mathcal{U}_{\mathfrak{D}}$ is a linear complement of $\sum_{[d] \in \Sigma_{\mathfrak{D}}/\sim} \mathfrak{D}_{0, [d]}$ in $\mathfrak{D}_0$ for any $\mathfrak{D} \in \{\mathfrak{A}, \mathfrak{M}, \mathfrak{B}\}$, being any $\mathfrak{D}_{[d]}$ an $(\mathfrak{A}, \mathfrak{B})$ -subbimodule of $\mathfrak{M}$ if $\mathfrak{D} = \mathfrak{M}$ and an ideal of $\mathfrak{D}$ if $\mathfrak{D}$ is either $\mathfrak{A}$ or $\mathfrak{B}$. From here we can write

$$\mathfrak{T} = \mathcal{U} + \begin{pmatrix} \sum_{[a] \in \Sigma_{\mathfrak{A}}/\sim} \mathfrak{A}_{[a]} & \sum_{[g] \in \Sigma_{\mathfrak{M}}/\sim} \mathfrak{M}_{[g]} \\ 0 & \sum_{[b] \in \Sigma_{\mathfrak{B}}/\sim} \mathfrak{B}_{[b]} \end{pmatrix}.$$

Since $\phi_{\mathfrak{A}}$ and $\phi_{\mathfrak{B}}$ are bijective, we have

$$\mathfrak{T} = \mathcal{U} + \sum_{[g] \in \Sigma_{\mathfrak{M}}/\sim} \begin{pmatrix} \mathfrak{A}_{\phi_{\mathfrak{A}}([g])} & \mathfrak{M}_{[g]} \\ 0 & \mathfrak{B}_{\phi_{\mathfrak{B}}([g])} \end{pmatrix} = \mathcal{U} + \sum_{[g] \in \Sigma_{\mathfrak{M}}/\sim} \mathfrak{T}_{[g]}.$$

Now, let us verify $\mathfrak{T}_{[g]}\mathfrak{T}_{[h]} = 0$ if $[g] \neq [h]$. We begin by observing that

$$(2.4) \quad \mathfrak{C}_{[c]}\mathfrak{C}_{[d]} = 0 \text{ for any } \mathfrak{C} \in \{\mathfrak{A}, \mathfrak{B}\} \text{ and } [c] \neq [d].$$

Indeed, we have

$$(2.5) \quad \begin{aligned} \mathfrak{C}_{[c]}\mathfrak{C}_{[d]} &= (\mathfrak{C}_{0, [c]} \oplus \bigoplus_{c' \in [c]} \mathfrak{C}_{c'}) (\mathfrak{C}_{0, [d]} \oplus \bigoplus_{d' \in [d]} \mathfrak{C}_{d'}) \subset \\ &= \mathfrak{C}_{0, [c]}\mathfrak{C}_{0, [d]} + \mathfrak{C}_{0, [c]}(\bigoplus_{d' \in [d]} \mathfrak{C}_{d'}) + (\bigoplus_{c' \in [c]} \mathfrak{C}_{c'})\mathfrak{C}_{0, [d]} + (\bigoplus_{c' \in [c]} \mathfrak{C}_{c'}) (\bigoplus_{d' \in [d]} \mathfrak{C}_{d'}). \end{aligned}$$

Consider the fourth summand $(\bigoplus_{c' \in [c]} \mathfrak{C}_{c'}) (\bigoplus_{d' \in [d]} \mathfrak{C}_{d'})$ and suppose that there exist $c' \in [c]$ and $d' \in [d]$ such that $\mathfrak{C}_{c'}\mathfrak{C}_{d'} \neq 0$. As necessarily $c' \neq -d'$, then $c' + d' \in \Sigma_{\mathfrak{C}}$. So $\{c', d', -c'\}$ is a connection between c' and d' . By the transitivity of the connection relation, we have $d' \in [c]$, a contradiction. Hence, $\mathfrak{C}_{c'}\mathfrak{C}_{d'} = 0$ and so

$$(2.6) \quad (\bigoplus_{c' \in [c]} \mathfrak{C}_{c'}) (\bigoplus_{d' \in [d]} \mathfrak{C}_{d'}) = 0.$$

Consider now the first summand $\mathfrak{C}_{0, [c]}\mathfrak{C}_{0, [d]}$ in (2.5) and suppose there exist $c' \in [c]$ and $d' \in [d]$ such that $(\mathfrak{C}_{c'}\mathfrak{C}_{-c'}) (\mathfrak{C}_{d'}\mathfrak{C}_{-d'}) \neq 0$. We have $\mathfrak{C}_{c'}(\mathfrak{C}_{-c'}\mathfrak{C}_{d'})\mathfrak{C}_{-d'} \neq 0$ and so $\mathfrak{C}_{-c'}\mathfrak{C}_{d'} \neq 0$, which contradicts (2.6). Hence,

$$\mathfrak{C}_{0, [c]}\mathfrak{C}_{0, [d]} = 0.$$

Finally, we note that the same argument leads to

$$\mathfrak{C}_{0,[c]}(\bigoplus_{d' \in [d]} \mathfrak{C}_{d'}) + (\bigoplus_{c' \in [c]} \mathfrak{C}_{c'})\mathfrak{C}_{0,[d]} = 0$$

and so (2.5) gives us $\mathfrak{C}_{[c]}\mathfrak{C}_{[d]} = 0$. Taking now into account (2.4) and the fact when $[g] \neq [h]$, $\mathfrak{A}_{\phi_{\mathfrak{A}}([h])}\mathfrak{M}_{[g]} = \mathfrak{M}_{[g]}\mathfrak{B}_{\phi_{\mathfrak{B}}([h])} = 0$, then if $[g] \neq [h]$, we get $\mathfrak{T}_{[g]}\mathfrak{T}_{[h]} = 0$. $\square$

It is said that $\mathfrak{T}_0$ is *tight*, whence $\mathfrak{T}_0 = \mathfrak{T}_{0,\Sigma_{\mathfrak{T}}/\sim}$.

COROLLARY 2.6. *Suppose $\mathfrak{T}_0$ is tight. Then the graded triangular algebra $\mathfrak{T}$ decomposes as the direct sum of the ideals*

$$\mathfrak{T} = \bigoplus_{[g] \in \Sigma_{\mathfrak{M}}/\sim} \mathfrak{T}_{[g]},$$

where any $\mathfrak{T}_{[g]}$ is a graded triangular algebra. Furthermore, $\mathfrak{T}_{[g]}\mathfrak{T}_{[h]} = 0$, whence $[g] \neq [h]$.

Proof. Clearly, we have $\mathfrak{T} = \sum_{[g] \in \Sigma_{\mathfrak{M}}/\sim} \mathfrak{T}_{[g]}$. To show the direct character of the sum, take some $x \in \mathfrak{T}_{[g]} \cap \sum_{[h] \neq [g]} \mathfrak{T}_{[h]}$. By using the fact $\mathfrak{T}_{[g]}\mathfrak{T}_{[h]} = 0$, if $[g] \neq [h]$ we obtain $x\mathfrak{T} = 0$, that is, $x \in \text{Ann}(\mathfrak{T}) = 0$, as desired. Hence, we just have to prove that each of the ideals $\mathfrak{T}_{[g]} = \begin{pmatrix} \mathfrak{A}_{\phi_{\mathfrak{A}}([g])} & \mathfrak{M}_{[g]} \\ 0 & \mathfrak{B}_{\phi_{\mathfrak{B}}([g])} \end{pmatrix}$, (see (2.3)), is a triangular algebra. That is, $\mathfrak{M}_{[g]}$ is faithful as $(\mathfrak{A}_{\phi_{\mathfrak{A}}([g])}, \mathfrak{B}_{\phi_{\mathfrak{B}}([g])})$ -bimodule and $\text{Ann}(\mathfrak{T}_{[g]}) = 0$. But these facts are easy to verify if we take into account that $\mathfrak{T}_{[g]}\mathfrak{T}_{[h]} = 0$ when $[g] \neq [h]$, that $\mathfrak{T}_0$ is tight, that $\mathfrak{M}$ is a faithful $(\mathfrak{A}, \mathfrak{B})$ -bimodule, and that $\text{Ann}(\mathfrak{T}) = 0$. $\square$

3. The simple components. In this section, we will show that, under mild conditions, the decomposition of $\mathfrak{T}$ stated in Corollary 2.6 can be given by means of quasi-simple ideals. We will say that the support of $\mathfrak{T}$ is *symmetric* if $\Sigma_{\mathfrak{D}} = -\Sigma_{\mathfrak{D}}$ for any $\mathfrak{D} \in \{\mathfrak{A}, \mathfrak{B}, \mathfrak{M}\}$. From now on, the support of $\mathfrak{T}$ will be symmetric. We assume $\Sigma_{\mathfrak{M}} = \Sigma_{\mathfrak{A}} \cup \Sigma_{\mathfrak{B}}$ and that the ring of scalar is an arbitrary field $\mathbb{F}$.

We recall, see [2] and [6], that a graded associative algebra $\mathfrak{C} = \mathfrak{C}_0 \oplus (\bigoplus_{g \in \Sigma_{\mathfrak{C}}} \mathfrak{C}_g)$ is called of *maximal length* if $\mathfrak{C}_0 \neq 0$ and $\dim \mathfrak{C}_g = 1$ for any $g \in \Sigma_{\mathfrak{C}}$. The graded associative algebra $\mathfrak{C}$ is said to be $\Sigma_{\mathfrak{C}}$ -*multiplicative* if given $c, c' \in \Sigma_{\mathfrak{C}} \cup \{0\}$ such that $c + c' \in \Sigma_{\mathfrak{C}}$, then $\mathfrak{C}_c\mathfrak{C}_{c'} \neq 0$. A graded $(\mathfrak{A}, \mathfrak{B})$ -bimodule $\mathfrak{M} = \mathfrak{M}_0 \oplus (\bigoplus_{g \in \Sigma_{\mathfrak{M}}} \mathfrak{M}_g)$ is of *maximal length* if $\mathfrak{M}_0 \neq 0$ and $\dim \mathfrak{M}_g = 1$ for any $g \in \Sigma_{\mathfrak{M}}$. Finally $\mathfrak{M}$ is said to be $\Sigma_{\mathfrak{M}}$ -*multiplicative* if given $a \in \Sigma_{\mathfrak{A}} \cup \{0\}$ and $g \in \Sigma_{\mathfrak{M}}$ such that $a + g \in \Sigma_{\mathfrak{M}}$, then $\mathfrak{A}_a\mathfrak{M}_g \neq 0$; and if $b \in \Sigma_{\mathfrak{B}} \cup \{0\}$ and $g \in \Sigma_{\mathfrak{M}}$ are such that $g + b \in \Sigma_{\mathfrak{M}}$, then $\mathfrak{M}_g\mathfrak{B}_b \neq 0$ (see [2, 6, 7, 9, 10, 21] for discussion and examples on these concepts).

From here, we can introduce natural notions of maximal length and multiplicativity in the support for triangular algebras as follows.

DEFINITION 3.1. A graded triangular algebra $\mathfrak{T} = \begin{pmatrix} \mathfrak{A} & \mathfrak{M} \\ 0 & \mathfrak{B} \end{pmatrix}$ is said to be of *maximal length* if $\mathfrak{D}$ is of maximal length for any $\mathfrak{D} \in \{\mathfrak{A}, \mathfrak{M}, \mathfrak{B}\}$.

Observe that in a graded triangular algebra of maximal length $\mathfrak{T}$ we have $\dim \mathfrak{T}_g \leq 3$ for any $g \in \Sigma_{\mathfrak{T}}$.

DEFINITION 3.2. A graded triangular algebra $\mathfrak{T} = \begin{pmatrix} \mathfrak{A} & \mathfrak{M} \\ 0 & \mathfrak{B} \end{pmatrix}$ is called $\Sigma_{\mathfrak{T}}$ -*multiplicative* if any $\mathfrak{D}$ is $\Sigma_{\mathfrak{D}}$ -multiplicative for $\mathfrak{D} \in \{\mathfrak{A}, \mathfrak{M}, \mathfrak{B}\}$. Furthermore, in case $\mathfrak{T}_0$ is tight then $\mathfrak{A}_{-g}\mathfrak{M}_g = \mathfrak{A}_g\mathfrak{M}_{-g}$.

We will also say that $\Sigma_{\mathfrak{T}}$ has all of its elements connected when any $\Sigma_{\mathfrak{D}}$, for $\mathfrak{D} \in \{\mathfrak{A}, \mathfrak{M}, \mathfrak{B}\}$, has all of its elements connected.

THEOREM 3.3. Let $\mathfrak{T}$ be a $\Sigma_{\mathfrak{T}}$ -multiplicative and of maximal length graded triangular algebra. Then $\mathfrak{T}$ is quasi-simple if and only if $\Sigma_{\mathfrak{T}}$ has all of its elements connected and $\mathfrak{T}_0$ is tight.

Proof. Suppose $\mathfrak{T}$ is quasi-simple. Given any $[g] \in \Sigma_{\mathfrak{M}}/\sim$, we consider the ideal $\mathfrak{T}_{[g]}$ of $\mathfrak{T}$ given by (2.3). Then by quasi-simplicity we get $\mathfrak{M}_{[g]} = \mathfrak{M}$ and $\mathfrak{C}_{\phi_{\mathfrak{C}}([g])} = \mathfrak{C}$ for $\mathfrak{C} \in \{\mathfrak{A}, \mathfrak{B}\}$, that is, $\Sigma_{\mathfrak{M}} = [g]$ and $\Sigma_{\mathfrak{C}} = [c]$. So $\Sigma_{\mathfrak{T}}$ has all of its elements connected. Finally, we also have as consequence of $\mathfrak{M}_{[g]} = \mathfrak{M}$ and $\mathfrak{C}_{\phi_{\mathfrak{C}}([g])} = \mathfrak{C}$ that $\mathfrak{T}_0$ is tight.

Conversely, consider $\mathfrak{I}$ a nonzero ideal of $\mathfrak{T}$. We can write

$$\mathfrak{I} = \begin{pmatrix} \mathfrak{I}_{\mathfrak{A}} & \mathfrak{I}_{\mathfrak{M}} \\ 0 & \mathfrak{I}_{\mathfrak{B}} \end{pmatrix},$$

where $\mathfrak{I}_{\mathfrak{A}}$ and $\mathfrak{I}_{\mathfrak{B}}$ are ideals of $\mathfrak{A}$ and $\mathfrak{B}$ respectively and $\mathfrak{I}_{\mathfrak{M}}$ an $(\mathfrak{A}, \mathfrak{B})$ -subbimodule of $\mathfrak{M}$, (see (1.2) and the comments below), with $\mathfrak{I}_{\mathfrak{D}} = (\mathfrak{I}_{\mathfrak{D}} \cap \mathfrak{D}_0) \oplus \left(\bigoplus_{g \in \Sigma_{\mathfrak{D}}^{\mathfrak{I}}} \mathfrak{D}_g \right)$, where

$\Sigma_{\mathfrak{D}}^{\mathfrak{I}} = \{g \in \Sigma_{\mathfrak{D}} : \mathfrak{I}_{\mathfrak{D}} \cap \mathfrak{D}_g \neq 0\}$ for any $\mathfrak{D} \in \{\mathfrak{A}, \mathfrak{M}, \mathfrak{B}\}$. Observe that $\Sigma_{\mathfrak{M}}^{\mathfrak{I}} \neq \emptyset$ if $\mathfrak{I}_{\mathfrak{M}} \neq 0$, since in the opposite case $\mathfrak{I}_{\mathfrak{M}} \subset \mathfrak{M}_0$ and taking into account $\text{Ann}(\mathfrak{T}) \neq 0$ and $\mathfrak{T}_0$ is tight, there exist $0 \neq v \in \mathfrak{I}_{\mathfrak{M}}$ and $a \in \Sigma_{\mathfrak{A}}$ such that $0 \neq \mathfrak{A}_a v \subset \mathfrak{I}_{\mathfrak{M}} \cap \mathfrak{M}_a \subset \mathfrak{M}_0 \cap \mathfrak{M}_a$, a contradiction. By considering $\mathfrak{C}$ as a $(\mathfrak{C}, \mathfrak{C})$ -bimodule for $\mathfrak{C} \in \{\mathfrak{A}, \mathfrak{B}\}$, we get in a similar way that $\Sigma_{\mathfrak{C}}^{\mathfrak{I}} \neq \emptyset$ if $\mathfrak{I}_{\mathfrak{C}} \neq 0$.

We can distinguish four possibilities:

- $\mathfrak{I}_{\mathfrak{A}} \neq 0$ and $\mathfrak{I}_{\mathfrak{B}} = 0$. By the above, we can take $a_0 \in \Sigma_{\mathfrak{A}}^{\mathfrak{I}}$ so that

$$(3.1) \quad 0 \neq \begin{pmatrix} \mathfrak{A}_{a_0} & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{I}.$$

For $c \in \Sigma_{\mathfrak{A}} \setminus \{\pm a_0\}$, a_0 is connected to c . Therefore, there exists a connection $\{a_0, a_1, \dots, a_n\} \subset \Sigma_{\mathfrak{A}}$ from a_0 to c such that

$$a_0, a_0 + a_1, \dots, a_0 + a_1 + \dots + a_{n-1} \in \Sigma_{\mathfrak{A}}$$

and

$$a_0 + a_1 + \dots + a_n \in \{\pm c\}.$$

Consider a_0 , a_1 and $a_0 + a_1$. The $\Sigma_{\mathfrak{X}}$ -multiplicativity of $\mathfrak{I}$ gives us $0 \neq \mathfrak{A}_{a_0}\mathfrak{A}_{a_1}$. Hence, the maximal length of $\mathfrak{I}$ implies $0 \neq \mathfrak{A}_{a_0}\mathfrak{A}_{a_1} = \mathfrak{A}_{a_0+a_1}$

and (3.1) implies $0 \neq \begin{pmatrix} \mathfrak{A}_{a_0+a_1} & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{I}$. We can argue in a similar way

from $a_0 + a_1$, a_2 and $a_0 + a_1 + a_2$ to get $0 \neq \begin{pmatrix} \mathfrak{A}_{a_0+a_1+a_2} & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{I}$.

Following this process we obtain $0 \neq \begin{pmatrix} \mathfrak{A}_{a_0+a_1+a_2+\dots+a_n} & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{I}$ and so

either $0 \neq \begin{pmatrix} \mathfrak{A}_c & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{I}$ or $0 \neq \begin{pmatrix} \mathfrak{A}_{-c} & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{I}$ for any $c \in \Sigma_{\mathfrak{A}}$. The fact $\mathfrak{A}_0 = \sum_{c \in \Sigma_{\mathfrak{A}}} \mathfrak{A}_c \mathfrak{A}_{-c}$ implies

$$(3.2) \quad 0 \neq \begin{pmatrix} \mathfrak{A}_0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{I}.$$

Given any $a \in \Sigma_{\mathfrak{A}}$, the $\Sigma_{\mathfrak{X}}$ -multiplicativity of $\mathfrak{I}$ together with its maximal length allow us to assert $\mathfrak{A}_a = \mathfrak{A}_0 \mathfrak{A}_a$ and so

$$(3.3) \quad 0 \neq \begin{pmatrix} \mathfrak{A}_0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathfrak{A}_a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathfrak{A}_a & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{I}$$

for any $a \in \Sigma_{\mathfrak{A}}$.

From (3.2) and (3.3), we get $\begin{pmatrix} \mathfrak{A} & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{I}$.

Let us show that we also have

$$(3.4) \quad \begin{pmatrix} 0 & \mathfrak{M} \\ 0 & 0 \end{pmatrix} \in \mathfrak{I}.$$

Indeed, for any $g \in \Sigma_{\mathfrak{M}}$ the $\Sigma_{\mathfrak{T}}$ -multiplicativity of $\mathfrak{T}$ gives us $\mathfrak{A}_0 \mathfrak{M}_g = \mathfrak{M}_g$, and by (3.2), we get $\begin{pmatrix} \mathfrak{A}_0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathfrak{M}_g \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathfrak{M}_g \\ 0 & 0 \end{pmatrix} \subset \mathfrak{J}$. The tightness of $\mathfrak{T}_0$ together with (3.3) implies $\begin{pmatrix} 0 & \mathfrak{M}_0 \\ 0 & 0 \end{pmatrix} \subset \mathfrak{J}$ and so we have showed that (3.4) holds. Consequently,

$$(3.5) \quad \mathfrak{J} = \begin{pmatrix} \mathfrak{A} & \mathfrak{M} \\ 0 & 0 \end{pmatrix}.$$

- $\mathfrak{J}_{\mathfrak{A}} = 0$ and $\mathfrak{J}_{\mathfrak{B}} \neq 0$. An argument similar to the previous case gives us

$$(3.6) \quad \mathfrak{J} = \begin{pmatrix} 0 & \mathfrak{M} \\ 0 & \mathfrak{B} \end{pmatrix}.$$

- $\mathfrak{J}_{\mathfrak{A}} \neq 0$ and $\mathfrak{J}_{\mathfrak{B}} \neq 0$. By arguing as in the two previous cases, we get

$$(3.7) \quad \mathfrak{J} = \begin{pmatrix} \mathfrak{A} & \mathfrak{M} \\ 0 & \mathfrak{B} \end{pmatrix}.$$

- $\mathfrak{J}_{\mathfrak{A}} = 0$ and $\mathfrak{J}_{\mathfrak{B}} = 0$. In this case, $\mathfrak{J} = \begin{pmatrix} 0 & \mathfrak{J}_{\mathfrak{M}} \\ 0 & 0 \end{pmatrix}$ with $\mathfrak{J}_{\mathfrak{M}}$ a non-zero $(\mathfrak{A}, \mathfrak{B})$ -subbimodule of $\mathfrak{M}$. We have as in the first case that we can fix $g_0 \in \Sigma_{\mathfrak{M}}^{\mathfrak{J}}$ with $0 \neq \begin{pmatrix} 0 & \mathfrak{M}_{g_0} \\ 0 & 0 \end{pmatrix} \subset \mathfrak{J}$ and so, by arguing with the fact that g_0 is connected to any $h \in \Sigma_{\mathfrak{M}} \setminus \{\pm g_0\}$ by means of a connection

$$\{g_0, g_1, \dots, g_n\}$$

from g_0 to h satisfying $g_0 + g_1 + \dots + g_n = \epsilon_h h$, that $0 \neq \begin{pmatrix} 0 & \mathfrak{M}_{\epsilon_h h} \\ 0 & 0 \end{pmatrix} \subset \mathfrak{J}$ for any $h \in \Sigma_{\mathfrak{M}}$ and some $\epsilon_h \in \{\pm 1\}$.

Let us observe that $-g_0 \in \Sigma_{\mathfrak{M}}^{\mathfrak{J}}$. Indeed, if $2g_0 = 0$ it is clear and in case $2g_0 \neq 0$, since $\text{Ann}(\mathfrak{T}) = 0$, $\mathfrak{T}_0$ is tight and $\mathfrak{T}$ is $\Sigma_{\mathfrak{T}}$ -multiplicative, we can find $a \in \Sigma_{\mathfrak{A}}$, $a \neq -g_0$, such that $0 \neq \mathfrak{A}_a \mathfrak{M}_{g_0} = \mathfrak{M}_{a+g_0} \subset \mathfrak{J}_{\mathfrak{M}}$. By $\Sigma_{\mathfrak{T}}$ -multiplicativity and the fact $\Sigma_{\mathfrak{M}} = \Sigma_{\mathfrak{A}} \cup \Sigma_{\mathfrak{B}}$ that either $0 \neq \mathfrak{A}_{-g_0} \mathfrak{M}_{g_0+a} = \mathfrak{M}_a \subset \mathfrak{J}_{\mathfrak{M}}$ or $0 \neq \mathfrak{M}_{g_0+a} \mathfrak{B}_{-g_0} = \mathfrak{M}_a \subset \mathfrak{J}_{\mathfrak{M}}$ and so either $0 \neq \mathfrak{A}_{-a-g_0} \mathfrak{M}_a = \mathfrak{M}_{-g_0} \subset \mathfrak{J}_{\mathfrak{M}}$ or $0 \neq \mathfrak{M}_a \mathfrak{B}_{-a-g_0} = \mathfrak{M}_{-g_0} \subset \mathfrak{J}_{\mathfrak{M}}$. Consequently, $-g_0 \in \Sigma_{\mathfrak{M}}^{\mathfrak{J}}$.

Now

$$\{-g_0, -g_1, \dots, -g_n\}$$

is a connection from $-g_0$ to h satisfying $-g_0 - g_1 - \cdots - g_n = -\epsilon_h h$. So we have $0 \neq \begin{pmatrix} 0 & \mathfrak{M}_{-\epsilon_h h} \\ 0 & 0 \end{pmatrix} \subset \mathfrak{J}$ and $\Sigma_{\mathfrak{M}}^{\mathfrak{J}} = \Sigma_{\mathfrak{M}}$. Since $\mathfrak{T}_0$ is tight, we also have $\begin{pmatrix} 0 & \mathfrak{M}_0 \\ 0 & 0 \end{pmatrix} \subset \mathfrak{J}$ and so $\mathfrak{J}_{\mathfrak{M}} = \mathfrak{M}$. Consequently,

$$(3.8) \quad \mathfrak{J} = \begin{pmatrix} 0 & \mathfrak{M} \\ 0 & 0 \end{pmatrix}.$$

Taking into account (3.5), (3.6), (3.7) and (3.8), $\mathfrak{T}$ is quasi-simple. $\square$

THEOREM 3.4. *Let $\mathfrak{T}$ be a $\Sigma_{\mathfrak{T}}$ -multiplicative and of maximal length graded triangular algebra with $\mathfrak{T}_0$ tight. Then $\mathfrak{T}$ decomposes as the direct sum of the quasi-simple ideals*

$$\mathfrak{T} = \bigoplus_{[g] \in \Sigma_{\mathfrak{M}} / \sim} \mathfrak{T}_{[g]},$$

where any $\mathfrak{T}_{[g]}$ is a graded triangular algebra. Furthermore, $\mathfrak{T}_{[g]}\mathfrak{T}_{[h]} = 0$, whence $[g] \neq [h]$.

Proof. By Corollary 2.6,

$$\mathfrak{T} = \bigoplus_{[g] \in \Sigma_{\mathfrak{M}} / \sim} \mathfrak{T}_{[g]}$$

is the direct sum of the ideals $\mathfrak{T}_{[g]}$, with support denoted by $\Sigma_{\mathfrak{T}}^{[g]}$, with all of its elements connected, where $\mathfrak{T}_{[g]}\mathfrak{T}_{[h]} = 0$ if $[g] \neq [h]$ and where $\mathfrak{T}_{[g]}$ is a triangular algebra. We also have that any of the $\mathfrak{T}_{[g]}$ is $\Sigma_{\mathfrak{T}}^{[g]}$ -multiplicative as consequence of the $\Sigma_{\mathfrak{T}}$ -multiplicativity of $\mathfrak{T}$ and clearly $\mathfrak{T}_{[g]}$ is of maximal length. Hence, we can apply Theorem 3.3 to any $\mathfrak{T}_{[g]}$ so as to conclude that $\mathfrak{T}_{[g]}$ is quasi-simple and so the proof is completed. $\square$

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