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IMAGE-KERNEL (P,Q)-INVERSES IN RINGS*

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Abstract. Elements with equal idempotents related to their image-kernel (p,q)-inverses are characterized, and applications to perturbations, reverse order law and commutativity are given.

Key words. Outer inverse, Idempotents, Ring.

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1. Introduction. Let $\mathcal{R}$ be an associative ring with unit 1. We use $\mathcal{R}^\bullet$ to denote the set of all idempotents of $\mathcal{R}$. For $a \in \mathcal{R}$, we define the following kernel ideals $a^\circ = \{x \in \mathcal{R} : ax = 0\}$, ${}^\circ a = \{x \in \mathcal{R} : xa = 0\}$, and image ideals $a\mathcal{R} = \{ax : x \in \mathcal{R}\}$ and $\mathcal{R}a = \{xa : x \in \mathcal{R}\}$.

Let $a \in \mathcal{R}$. If there exists $c \in \mathcal{R}$ such that $c = cac$ holds, then we say that c is an outer inverse for a, and write $c = a^{(2)}$. The outer inverse is not unique in general, but it is unique if we fix the idempotents ac and ca [4]: Let $p, q \in \mathcal{R}^\bullet$, the (p,q)-outer generalized inverse of a is the unique element $c \in \mathcal{R}$ (in the case when it exists) satisfying

$$cac = c, \quad ca = p, \quad 1 - ac = q.$$

In this case, we write $c = a^{(2)}_{p,q}$. Note that, for $a \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$, $a^{(2)}_{p,q}$ exists if and only if $(1-q)a = (1-q)ap$ and there exists some $c \in \mathcal{R}$ such that $pc = c$, $cq = 0$, $cap = p$ and $ac = 1 - q$ [4].

Instead of prescribing the idempotents ac and ca, we may prescribe certain kernel and image ideals related to these idempotents: Let $p, q \in \mathcal{R}^\bullet$, an element $c \in \mathcal{R}$ is the image-kernel (p,q)-inverse of a if

$$cac = c, \quad ca\mathcal{R} = p\mathcal{R} \quad \text{and} \quad (1 - ac)\mathcal{R} = q\mathcal{R}.$$

The image-kernel (p,q)-inverse c is unique if it exists [5], and it will be denoted by $a^\times$.

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Trying to mimic the operators case (since for the case of bounded linear operators on Banach spaces, the outer inverse is unique if we fix its range and kernel), the third author [5] has investigated sufficient algebraic conditions for the uniqueness of the outer inverse, several classes of outer inverses that can be found in the literature, and proved some relations between these different approaches.

An element $a \in \mathcal{R}$ is group invertible if there is $a^{\#} \in \mathcal{R}$ such that

$$(1)\ aa^{\#}a = a, \quad (2)\ a^{\#}aa^{\#} = a^{\#}, \quad (5)\ aa^{\#} = a^{\#}a.$$

Recall that $a^{\#}$ is uniquely determined by the equations as above and it is called the group inverse of a. We use $\mathcal{R}^{\#}$ to denote the set of all group invertible elements of $\mathcal{R}$.

If $\delta \subset \{1,2,5\}$ and c satisfies the equations (i) for all $i \in \delta$, then c is a δ–inverse of a. The set of all δ–inverses of a is denoted by $a\{\delta\}$. Obviously, $a\{1,2,5\} = \{a^{\#}\}$.

The Drazin and the Moore-Penrose inverses are particular classes of outer inverses. Castro-González, Koliha and Wei [3] characterized matrices with the same eigenprojections, i.e., the same projections corresponding to the Drazin inverses of these matrices. Koliha and Patrício [6] proved analogous results for Drazin invertible elements of a ring. A similar problem was considered for the Moore-Penrose inverse in [8]. In [7], these results were generalized to the weighted Moore-Penrose inverse in rings with involution. In [4], Wei and the second author characterized elements with equal idempotents related to their (p,q)-outer generalized inverses. In this paper, we characterize elements of a ring which have the same idempotents related to their particular image-kernel (p,q)-inverses.

2. Image-kernel (p,q)-inverse. For $u, v \in \mathcal{R}^{\bullet}$, notice that $u^{\circ} = (1-u)\mathcal{R}$ and ${}^{\circ}u = \mathcal{R}(1-u)$. Also, we have

$$u\mathcal{R} = v\mathcal{R} \ \Leftrightarrow\ {}^{\circ}u = {}^{\circ}v \tag{2.1}$$

and

$$\mathcal{R}u = \mathcal{R}v \ \Leftrightarrow\ u^{\circ} = v^{\circ}. \tag{2.2}$$

From these equations, we can verify the next result.

Lemma 2.1. *Let $a, c \in \mathcal{R}$ and $c = cac$.*

(i) *If $p \in \mathcal{R}^{\bullet}$, then the following conditions are equivalent:*

1. *$ca\mathcal{R} = p\mathcal{R}$;*
2. *$\mathcal{R}(1-ca) = \mathcal{R}(1-p)$;*

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3. ${}^{\circ}(ca) = {}^{\circ}p$;
4. $(1-ca)^{\circ} = (1-p)^{\circ}$.

(ii) *If $q \in \mathcal{R}^{\bullet}$, then the following conditions are equivalent:*

1. $(1-ac)\mathcal{R} = q\mathcal{R}$;
2. $\mathcal{R}ac = \mathcal{R}(1-q)$;
3. ${}^{\circ}(1-ac) = {}^{\circ}q$;
4. $(ac)^{\circ} = (1-q)^{\circ}$.

Proof. We only prove the part (i), because the part (ii) follows in the same way. Since $ca, p, 1-ca, 1-p \in \mathcal{R}^{\bullet}$, by (2.1) and (2.2), we obtain *1.* $\Leftrightarrow$ *3.* and *2.* $\Leftrightarrow$ *4.* The equalities ${}^{\circ}(ca) = \mathcal{R}(1-ca)$ and ${}^{\circ}p = \mathcal{R}(1-p)$ imply *2.* $\Leftrightarrow$ *3.* $\square$

All these conditions involve ideals, but we can also give a necessary and sufficient condition for the existence of the image-kernel (p,q)-inverse without explicit reference to ideals.

THEOREM 2.2. *Let $p, q \in \mathcal{R}^{\bullet}$ and let $a \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *$a^{\times}$ exists;*
(ii) *there exists some $c \in \mathcal{R}$ such that*

$$c = pc, \quad cap = p, \quad cq = 0, \quad 1-q = (1-q)ac.$$

Proof. (i) $\Rightarrow$ (ii): Take $c = a^{\times}$. Then $ca\mathcal{R} = p\mathcal{R}$ implies that $ca = pca$ and $p = cap$. Thus, $c = cac = pcac = pc$. In a similar way, $\mathcal{R}ac = \mathcal{R}(1-q)$ gives $(1-q) = (1-q)ac$ and $ac = ac(1-q)$, and we get $acq = 0$ which yields $cq = cacq = 0$.

(ii) $\Rightarrow$ (i): A direct calculation shows that

$$cac = capc = pc = c,$$

$$ca\mathcal{R} = pca\mathcal{R} \subset p\mathcal{R} = cap\mathcal{R} \subset ca\mathcal{R},$$

and

$$\mathcal{R}(1-q) = \mathcal{R}(1-q)ac \subset \mathcal{R}ac = \mathcal{R}ac(1-q) \subset \mathcal{R}(1-q).$$

So, $ca\mathcal{R} = p\mathcal{R}$ and $\mathcal{R}(1-q) = \mathcal{R}ac$. $\square$

By Theorem 2.2, if c and d are two image-kernel (p,q)-inverses of a, then $c = pc = dapc = dac = d(1-q)ac = d(1-q) = d$, i.e., the image-kernel (p,q)-inverse of a is unique if it exists [5].

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Now, we give an example where the second equivalent condition of Theorem 2.2 is applicable.

EXAMPLE 2.3. Let $\mathcal{R}$ be a ring of 2×2 matrices with real entries, and let $a,c,p,q\in\mathcal{R}$ be defined by

$$a=\begin{bmatrix}0&2\\1&0\end{bmatrix},\quad c=\begin{bmatrix}0&1\\0&0\end{bmatrix},\quad p=\begin{bmatrix}1&3\\0&0\end{bmatrix},\quad q=\begin{bmatrix}1&2\\0&0\end{bmatrix}.$$

Since $c=pc$, $cap=p$, $cq=0$ and $1-q=(1-q)ac$, we conclude that c is the image-kernel (p,q)-inverse of a.

3. Elements with equal idempotents related to image-kernel (p,q)-inverses. We will use the following auxiliary result, which we state without proof.

LEMMA 3.1. [9] *Let $c,s\in\mathcal{R}$ satisfy $cs=sc$ and $s\in\mathcal{R}^\bullet$. Then c is invertible in $\mathcal{R}$ if and only if cs is invertible in $s\mathcal{R}s$ and $c(1-s)$ is invertible in $(1-s)\mathcal{R}(1-s)$. In this case,*

$$c^{-1}=[cs]^{-1}_{s\mathcal{R}s}+[c(1-s)]^{-1}_{(1-s)\mathcal{R}(1-s)}.$$

LEMMA 3.2. *Let $p,q\in\mathcal{R}^\bullet$ and let $a,b\in\mathcal{R}$ be such that $a^\times$ and $b^\times$ exist. Then*

(i) $a^\times bb^\times a=a^\times a$,
(ii) $a^\times bb^\times ap=p$.

Proof. (i) Observe that

$$\begin{aligned}a^\times bb^\times a&=a^\times aa^\times bb^\times a=a^\times aa^\times(1-q)bb^\times a\\&=a^\times aa^\times(1-q)a=a^\times a.\end{aligned}$$

(ii) From the part (i), we get

$$a^\times bb^\times ap=a^\times ap=p.\qquad\square$$

Now, as our main result, we give a characterization of elements of $\mathcal{R}$ which have the same idempotents related to their particular image-kernel (p,q)-inverses.

THEOREM 3.3. *Let $p,q\in\mathcal{R}^\bullet$ and let $a\in\mathcal{R}$ be such that $a^\times$ exists. Then for $b\in\mathcal{R}$ the following statements are equivalent:*

(i) $b^\times$ *exists;*
(ii) $(ba^\times a)^\times$ *exists;*

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(iii) $1 - p + a^{\times}bp$ *is invertible;*
(iv) $q + (1 - q)ba^{\times}$ *is invertible.*

If any of the previous statements is valid, then

$$b^{\times} = (a^{\times}bp)^{-1}_{p\mathcal{R}p}a^{\times} = a^{\times}((1-q)ba^{\times})^{-1}_{(1-q)\mathcal{R}(1-q)}.$$

Proof. (i) $\Rightarrow$ (ii): We prove that $(ba^{\times}a)^{\times} = b^{\times}$, by

$$b^{\times}ba^{\times}ab^{\times} = b^{\times}ba^{\times}apb^{\times} = b^{\times}bpb^{\times} = b^{\times},$$

$$b^{\times}ba^{\times}a\mathcal{R} = b^{\times}bp\mathcal{R} = p\mathcal{R},$$

$$\mathcal{R}ba^{\times}ab^{\times} = \mathcal{R}ba^{\times}apb^{\times} = \mathcal{R}bpb^{\times} = \mathcal{R}bb^{\times} = \mathcal{R}(1-q).$$

(ii) $\Rightarrow$ (i): If $z = (ba^{\times}a)^{\times}$, then

$$bz = bpz = ba^{\times}apz = ba^{\times}az$$

which gives $zbz = zba^{\times}az = z$ and $\mathcal{R}bz = \mathcal{R}ba^{\times}az = \mathcal{R}(1-q)$. Since

$$p\mathcal{R} = zba^{\times}a\mathcal{R} = zbp\mathcal{R} \subset zb\mathcal{R} = pzb\mathcal{R} \subset p\mathcal{R},$$

we conclude that $p\mathcal{R} = zb\mathcal{R}$. Consequently, $z = b^{\times}$.

(i) $\Rightarrow$ (iii): Notice that, by Lemma 3.2(ii),

$$(1 - p + a^{\times}bp)(1 - p + b^{\times}ap) = 1 - p + a^{\times}bb^{\times}ap = 1.$$

In the same way, it follows that $(1-p+b^{\times}ap)(1-p+a^{\times}bp) = 1$. Hence, $1-p+a^{\times}bp$ is invertible and $(1 - p + a^{\times}bp)^{-1} = 1 - p + b^{\times}ap$.

(iii) $\Rightarrow$ (i): Since $x = 1-p+a^{\times}bp = 1-p+pa^{\times}bp$ is invertible and $px = a^{\times}bp = xp$, by Lemma 3.1,

$$x^{-1} = 1 - p + (a^{\times}bp)^{-1}_{p\mathcal{R}p}.$$

Denote by $y = x^{-1}a^{\times} = (a^{\times}bp)^{-1}_{p\mathcal{R}p}a^{\times}$. From

$$(a^{\times}bp)^{-1}_{p\mathcal{R}p} = p(a^{\times}bp)^{-1}_{p\mathcal{R}p} = (a^{\times}bp)^{-1}_{p\mathcal{R}p}p,$$

we get

$$yby = (a^{\times}bp)^{-1}_{p\mathcal{R}p}a^{\times}bp(a^{\times}bp)^{-1}_{p\mathcal{R}p}a^{\times} = (a^{\times}bp)^{-1}_{p\mathcal{R}p}a^{\times} = y$$

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and

$$yb\mathcal{R} = p(a^\times bp)^{-1}_{p\mathcal{R}p}a^\times b\mathcal{R} \subset p\mathcal{R} = (a^\times bp)^{-1}_{p\mathcal{R}p}a^\times bp\mathcal{R} \subset yb\mathcal{R}.$$

Thus, $yb\mathcal{R} = p\mathcal{R}$. By

$$\begin{aligned} 1 - q &= (1 - q)aa^\times = (1 - q)apa^\times \\ &= (1 - q)aa^\times bp(a^\times bp)^{-1}_{p\mathcal{R}p}a^\times = (1 - q)by, \end{aligned}$$

we deduce that $\mathcal{R}(1 - q) \subset \mathcal{R}by$. Now

$$\mathcal{R}by \subset \mathcal{R}a^\times = \mathcal{R}aa^\times = \mathcal{R}(1 - q)$$

implies $\mathcal{R}by = \mathcal{R}(1 - q)$. So, $y = b^\times$.

(i) $\Rightarrow$ (iv): Using

$$\begin{aligned} (1 - q)ba^\times ab^\times &= (1 - q)ba^\times apb^\times = (1 - q)bpb^\times \\ &= (1 - q)bb^\times = 1 - q, \end{aligned}$$

we obtain

$$(q + (1 - q)ba^\times)(q + (1 - q)ab^\times) = q + (1 - q)ba^\times ab^\times = 1.$$

Similarly, we can verify that $(q+(1-q)ab^\times)(q+(1-q)ba^\times) = 1$. Therefore, $q+(1-q)ba^\times$ is invertible and $(q + (1 - q)ba^\times)^{-1} = q + (1 - q)ab^\times$.

(iv) $\Rightarrow$ (i): As in the part (iii) $\Rightarrow$ (i), we show that $(q+(1-q)ba^\times)^{-1} = q+((1-q)ba^\times)^{-1}_{(1-q)\mathcal{R}(1-q)}$ and $b^\times = a^\times((1 - q)ba^\times)^{-1}_{(1-q)\mathcal{R}(1-q)}$. $\square$

Remark 3.4. In the similar manner as in the proof of Theorem 3.3, we can verify the following: Let $p, q \in \mathcal{R}^\bullet$ and let $a \in \mathcal{R}$ be such that $a^\times$ exists. Then for $b \in \mathcal{R}$ the following statements are equivalent:

(i) $b^\times$ exists;
(ii) $(ba^\times a)^\times$ exists;
(iii) $1 - p + a^\times bp$ is invertible and $(1 - p + a^\times bp)^{-1} = 1 - p + b^\times ap$;
(iv) $q + (1 - q)ba^\times$ is invertible and $(q + (1 - q)ba^\times)^{-1} = q + (1 - q)ab^\times$.

As a consequence of Theorem 3.3, we obtain the following result which gives the form of the image-kernel (p, q)-inverse of a perturbed element.

Corollary 3.5. *Let $p, q \in \mathcal{R}^\bullet$ and let $a, b, e \in \mathcal{R}$ be such that $b = a + e$. If $a^\times$ exists, then the following statements are equivalent:*

(i) $b^\times$ *exists;*
(ii) $1 + a^\times ep$ *is invertible;*

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(iii) $1 + (1 - q)ea^\times$ *is invertible.*

If any of the previous statements is valid, then

$$b^\times = (p + a^\times ep)^{-1}_{p\mathcal{R}p} a^\times = a^\times (1 - q + (1 - q)ea^\times)^{-1}_{(1-q)\mathcal{R}(1-q)}.$$

We can prove the next result.

THEOREM 3.6. *Let $p, q \in \mathcal{R}^\bullet$ and let $a, b \in \mathcal{R}$ be such that $a^\times$ and $b^\times$ exist. Then*

(i) $1 + a^\times bp - a^\times a$ *is invertible and* $(1 + a^\times bp - a^\times a)^{-1} = 1 - p + b^\times a$,
(ii) $1 + (1 - q)ba^\times - aa^\times$ *is invertible and* $(1 + (1 - q)ba^\times - aa^\times)^{-1} = q + ab^\times$.

In addition,

$$b^\times = (1 + a^\times bp - a^\times a)^{-1} a^\times = a^\times (1 + (1 - q)ba^\times - aa^\times)^{-1}.$$

Proof. (i) By Lemma 3.2, we have $a^\times bb^\times a = a^\times a$ and $b^\times aa^\times bp = p$. Also, by $(1 - aa^\times)\mathcal{R} = q\mathcal{R}$, we have $(1 - aa^\times) = q(1 - aa^\times)$. Set $x = 1 + a^\times bp - a^\times a$ and $y = 1 - p + b^\times a$. Then

$$\begin{aligned} xy &= 1 - p + b^\times a + a^\times bb^\times a - a^\times a + p - a^\times ab^\times a \\ &= 1 + b^\times a - a^\times apb^\times a = 1 + b^\times a - b^\times a = 1 \end{aligned}$$

and

$$\begin{aligned} yx &= 1 - p + b^\times a + b^\times aa^\times bp - b^\times aa^\times a \\ &= 1 + b^\times (1 - aa^\times)a = 1 + b^\times q(1 - aa^\times)a = 1. \end{aligned}$$

So, x is invertible, $x^{-1} = y$ and

$$\begin{aligned} x^{-1}a^\times &= (1 - p + b^\times a)a^\times = b^\times aa^\times \\ &= b^\times bb^\times aa^\times = b^\times bb^\times (1 - q)aa^\times \\ &= b^\times bb^\times (1 - q) = b^\times. \end{aligned}$$

The part (ii) follows in a similar way. $\square$

In the following theorem, we give necessary and sufficient conditions for $aa^\times = bb^\times$.

THEOREM 3.7. *Let $p, q \in \mathcal{R}^\bullet$ and let $a, b \in \mathcal{R}$ be such that $a^\times$ and $b^\times$ exist. Then, the following statements are equivalent:*

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(i) $aa^\times = bb^\times$;
(ii) $ab^\times ba^\times = ba^\times ab^\times$;
(iii) $aa^\times bb^\times = bb^\times aa^\times$;
(iv) $ab^\times \in \mathcal{R}^\#$ *and* $(ab^\times)^\# = ba^\times$;
(v) $ba^\times \in \mathcal{R}^\#$ *and* $(ba^\times)^\# = ab^\times$;
(vi) $ba^\times \in (ab^\times)\{1,5\}$;
(vii) $ab^\times \in (ba^\times)\{1,5\}$.

If any of the previous statements is valid, then

$$a^\times = b^\times (ab^\times)^\#.$$

Proof. (i) $\Leftrightarrow$ (ii): The equalities

$$aa^\times = apa^\times = ab^\times bpa^\times = ab^\times ba^\times, \tag{3.1}$$

$$bb^\times = bpb^\times = ba^\times ab^\times, \tag{3.2}$$

imply that the part (i) is equivalent to the part (ii).

(i) $\Leftrightarrow$ (iii): From

$$aa^\times = aa^\times(1-q) = aa^\times(1-q)bb^\times = aa^\times bb^\times,$$

$$bb^\times = bb^\times aa^\times,$$

we conclude that (i) $\Leftrightarrow$ (iii).

(ii) $\Leftrightarrow$ (iv): Applying the equalities (3.1) and (3.2), we obtain

$$ab^\times ba^\times ab^\times = ab^\times bb^\times = ab^\times,$$

$$ba^\times ab^\times ba^\times = ba^\times aa^\times = ba^\times.$$

Thus, $ab^\times ba^\times = ba^\times ab^\times$ is equivalent to $(ab^\times)^\# = ba^\times$.

(iv) $\Leftrightarrow$ (v): It follows by the fact $(a^\#)^\# = a$ for $a \in \mathcal{R}^\#$.

By the part (iv), we have

$$a^\times = b^\times bpa^\times = b^\times ba^\times = b^\times (ab^\times)^\#.$$

(iv) $\Rightarrow$ (vi): Obvious.

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(vi) $\Rightarrow$ (iv): Notice that $ba^\times \in (ab^\times)\{2\}$. Indeed, by (3.1),

$$ba^\times ab^\times ba^\times = ba^\times aa^\times = ba^\times.$$

So, (iv) holds.

(v) $\Leftrightarrow$ (vii): Similar to (iv) $\Leftrightarrow$ (vi). $\square$

Now, we present some equivalent conditions for $a^\times a = b^\times b$.

THEOREM 3.8. *Let $p, q \in \mathcal{R}^\bullet$ and let $a, b \in \mathcal{R}$ be such that $a^\times$ and $b^\times$ exist. Then the following statements are equivalent:*

(i) $a^\times a = b^\times b$;
(ii) $a^\times bb^\times a = b^\times aa^\times b$;
(iii) $a^\times ab^\times b = b^\times ba^\times a$;
(iv) $a^\times b \in \mathcal{R}^\#$ *and* $(a^\times b)^\# = b^\times a$;
(v) $b^\times a \in \mathcal{R}^\#$ *and* $(b^\times a)^\# = a^\times b$;
(vi) $b^\times a \in (a^\times b)^\#\{1,5\}$;
(vii) $a^\times b \in (b^\times a)^\#\{1,5\}$.

If any of the previous statements is valid, then

$$b^\times = (a^\times b)^\# a^\times.$$

Proof. (i) $\Leftrightarrow$ (ii): This equivalence follows from

$$a^\times a = a^\times aa^\times(1-q)bb^\times a = a^\times bb^\times a$$

and $b^\times b = b^\times aa^\times b$.

The rest of the proof can be verified similarly as in the proof of Theorem 3.7. $\square$

In the following theorem, we consider equivalent conditions which ensure that the reverse order law $(ab)^\times = b^\times_{p,r}a^\times_{s,q}$ holds, where $a^\times_{s,q}$ is the image-kernel (s,q)-inverse of a and $b^\times_{p,r}$ is the image-kernel (p,r)-inverse of b.

THEOREM 3.9. *Let $p, q, r, s \in \mathcal{R}^\bullet$ and let $a, b \in \mathcal{R}$ be such that the image-kernel (s,q)-inverse $a^\times_{s,q}$ of a and the image-kernel (p,r)-inverse $b^\times_{p,r}$ of b exist. Then the following statements are equivalent:*

(i) $bb^\times_{p,r} = a^\times_{s,q}a$;
(ii) $b^\times_{p,r} = b^\times_{p,r}a^\times_{s,q}a$ *and* $a^\times_{s,q} = bb^\times_{p,r}a^\times_{s,q}$.

If any of the previous statements is valid, then

$$(ab)^\times = b^\times_{p,r}a^\times_{s,q}.$$

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Proof. (i) $\Rightarrow$ (ii): We see that $b^\times_{p,r} = b^\times_{p,r} b b^\times_{p,r} = b^\times_{p,r} a^\times_{s,q} a$ and $a^\times_{s,q} = a^\times_{s,q} a a^\times_{s,q} = b b^\times_{p,r} a^\times_{s,q}$.

(ii) $\Rightarrow$ (i): Observe that $bb^\times_{p,r} = bb^\times_{p,r} a^\times_{s,q} a = a^\times_{s,q} a$.

Hence, (i) $\Leftrightarrow$ (ii). In order to prove that $(ab)^\times = b^\times_{p,r} a^\times_{s,q}$, we assume that (ii) holds. Then

$$b^\times_{p,r} a^\times_{s,q} ab b^\times_{p,r} a^\times_{s,q} = b^\times_{p,r} b b^\times_{p,r} a^\times_{s,q} = b^\times_{p,r} a^\times_{s,q},$$

$$b^\times_{p,r} a^\times_{s,q} ab\mathcal{R} = b^\times_{p,r} b\mathcal{R} = p\mathcal{R},$$

$$\mathcal{R} abb^\times_{p,r} a^\times_{s,q} = \mathcal{R} aa^\times_{s,q} = \mathcal{R}(1-q).$$

So, $(ab)^\times = b^\times_{p,r} a^\times_{s,q}$. ◻

Remark 3.10. In the same way as in Theorems 3.7–3.9, we can show the following: Let $p, q \in \mathcal{R}^\bullet$ and let $a, b \in \mathcal{R}$ be such that $a^\times$ and $b^\times$ exist. Then the following statements are equivalent:

(i) $bb^\times = a^\times a$;
(ii) $b^\times = b^\times a^\times a$ and $a^\times = bb^\times a^\times$;
(iii) $a^\times ab^\times = b^\times a^\times a$ and $a^\times bb^\times = bb^\times a^\times$;
(iv) $ba^\times ab^\times = a^\times bb^\times a$;
(v) $bb^\times aa^\times = b^\times ba^\times a$.

If any of the previous statements is valid, then

$$(ab)^\times = b^\times a^\times.$$

We investigate necessary and sufficient conditions for $aa^\times = a^\times a$ in the next result.

Theorem 3.11. *Let $p, q \in \mathcal{R}^\bullet$ and let $a \in \mathcal{R}$ be such that $a^\times$ exists. Then the following statements are equivalent:*

(i) $aa^\times = a^\times a$;
(ii) $aa^\times \mathcal{R} = p\mathcal{R}$ *and* $\mathcal{R} a^\times a = \mathcal{R}(1-q)$;
(iii) $a^\times = a(a^\times)^2 = (a^\times)^2 a$;
(iv) $(aa^\times)^\times = aa^\times$ *and* $(a^\times a)^\times = a^\times a$.

If any of the previous statements is valid, then

$$a^\times = a^\times (aa^\times)^\times = (a^\times a)^\times a^\times.$$

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Proof. (i) $\Rightarrow$ (ii): Clear.

(ii) $\Rightarrow$ (i): Since $aa^\times\mathcal{R} = a^\times a\mathcal{R}$ and $\mathcal{R}a^\times a = \mathcal{R}aa^\times$, there exist $x, y \in \mathcal{R}$ such that

$$aa^\times = a^\times ax = a^\times aa^\times ax = a^\times aaa^\times$$

and

$$a^\times a = yaa^\times = yaa^\times aa^\times = a^\times aaa^\times.$$

Hence, we conclude that $aa^\times = a^\times a$.

(i) $\Leftrightarrow$ (iii): This is trivial.

(ii) $\Leftrightarrow$ (iv): First, note that $(aa^\times)^3 = aa^\times$ and $(a^\times a)^3 = a^\times a$. Also, $\mathcal{R}(aa^\times)^2 = \mathcal{R}aa^\times = \mathcal{R}(1-q)$ and $(a^\times a)^2\mathcal{R} = a^\times a\mathcal{R} = p\mathcal{R}$. By $(aa^\times)^2\mathcal{R} = aa^\times\mathcal{R}$ and $\mathcal{R}(a^\times a)^2 = \mathcal{R}a^\times a$, we deduce that (iv) holds if and only if (ii) is satisfied. $\square$

4. Final remarks. A ring $\mathcal{R}$ is called a Rickart ring if for every $a \in \mathcal{R}$, there exist some idempotent elements $p, q \in \mathcal{R}$ such that $a^\circ = p\mathcal{R}$ and ${}^\circ a = \mathcal{R}q$.

Let $\mathcal{R}$ be a (von Neumann) regular ring. Then, for every $b \in \mathcal{R}$ there exists $a \in \mathcal{R}$ such that $b = a^\times$, where $p = ba$ and $q = ab$, and we have ${}^\circ b = \mathcal{R}(1-p)$ and $b^\circ = q\mathcal{R}$. Thus, every regular ring is a Rickart ring and every element of a regular ring is the image-kernel inverse of some element of that ring.

Notice that the image-kernel (p, q)-inverse coincides with the (p, q, l)-outer generalized inverse of Cao and Xue [2]. In [2], the authors discussed the existence of $a^{(2)}_{p,q}$ and the (p, q, l)-outer generalized inverse, and gave some explicit representations for these inverses by the group inverses and $(1, 5)$-inverses in Banach algebras. However, we have proved different results.

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