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POSSIBLE NUMBERS OF NONZERO ENTRIES IN A MATRIX WITH A GIVEN TERM RANK*

YUN ZHANG† AND JIN ZHONG‡

Abstract. The possible numbers of nonzero entries in a matrix with a given term rank are determined respectively in the generic case, the symmetric case and the symmetric case with 0's on the main diagonal. The matrices that attain the largest number of nonzero entries are also determined.

Key words. Term rank, $(0,1)$-Matrices, Symmetric matrix, Adjacency matrix of a graph.

AMS subject classifications. 15A15, 15B57, 15B34, 05B20, 05D15.

1. Introduction. Let A be a matrix. We call a row or a column of A a *line*. The maximal number of nonzero entries of A with no two of these entries on a line is the *term rank* of A, and denoted by $\tau(A)$. This concept is important in matrix theory [6]. A set of lines of A is said to *cover* A if the lines in the set contain all the nonzero entries of A. If a set of lines covers A, then this set is called a *covering* of A. The minimal number of lines in a covering of A is called the *line rank* of A, denoted by $\delta(A)$. A covering of A with $\delta(A)$ lines is called a *minimal covering*. A $(0,1)$-matrix is a matrix whose entries are either 0 or 1. Such matrices arise frequently in combinatorics and graph theory. Clearly, to study term rank or line rank we need only consider $(0,1)$-matrices.

In [5], Hu, Li, and Zhan determined the possible numbers of ones in a $(0,1)$-matrix with a given rank in the generic case and in the symmetric case. In this paper, we consider a parallel problem: What are the possible numbers of ones in a $(0,1)$-matrix with a given term rank? We will answer this question in three cases: The generic case, the symmetric case, and the symmetric case with 0's on the main diagonal. Although the term rank is a purely combinatorial concept which is related

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†School of Mathematical Sciences, Huaibei Normal University, Huaibei 235000, and Department of Mathematics, East China Normal University, Shanghai 200241, P.R. China (zhangyunmaths@163.com). The research of this author was supported by the NSFC grant 11371145 and Anhui Provincial Natural Science Foundation (1408085MA08).

‡Faculty of Science, Jiangxi University of Science and Technology, Ganzhou 341000, P.R. China (zhongjin1984@126.com). The research of this author was supported by the PhD Research Startup Foundation of Jiangxi University of Science and Technology (jxxjbs12012).

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to the matching number of a bipartite graph, it is also related to the classical rank (see [1], [2] and [4]).

Let m and n be positive integers. For two $m \times n$ nonnegative matrices $A = (a_{ij})$ and $B = (b_{ij})$, the notation $A \leq B$ means that $a_{ij} \leq b_{ij}$ for all $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. We denote by $\mathcal{A}_{m,n}$ the set of all $m \times n$ $(0,1)$-matrices and $\mathcal{A}_n$ the set of all square $(0,1)$-matrices of order n. Let $J_{m,n}$ be the $m \times n$ matrix of all 1's, $J_n = J_{n,n}$ and let $E_{i,j}$ be the $m \times n$ matrix with its entry in (i,j) being 1 and all other entries being 0. Let I_k be the identity matrix of order k. Denote by $|S|$ the cardinality of a set S and $\emptyset$ the empty set. Let α be a subset of the set $M := \{1, 2, \ldots, m\}$ and let β be a subset of $N := \{1, 2, \ldots, n\}$. We denote by $A[\alpha|\beta]$ the submatrix of A with rows indexed by α and columns indexed by β. Also, $\alpha^c = M \backslash \alpha$ and $\beta^c = N \backslash \beta$. Then we denote $A(\alpha|\beta) = A[\alpha^c|\beta^c]$.

2. Main results. For $A \in \mathcal{A}_{m,n}$, let $\sharp(A)$ denote the number of ones in A. Let k be a positive integer with $k \leq \min\{m, n\}$. Denote by $\Omega(m,n,k)$ the set of all $m \times n$ $(0,1)$-matrices of term rank k. Denote by $O_{s,\,t}$ the $s \times t$ zero matrix. The following lemma is well known [3].

LEMMA 2.1. *For every matrix A, $\delta(A) = \tau(A)$.*

By the definition of the term rank, we immediately have the following lemma.

LEMMA 2.2. *Let $A, B, C \in \mathcal{A}_{m,n}$. If $A \leq B$, then $\tau(A) \leq \tau(B)$. In particular, if $A \leq B \leq C$ and $\tau(A) = \tau(C)$, then $\tau(A) = \tau(B) = \tau(C)$.*

Now, we give the following theorem, which determines

$$\Theta(m,n,k) := \max\{\sharp(A) \mid A \in \Omega(m,n,k)\}$$

and the $(0,1)$-matrices that attain $\Theta(m,n,k)$. Since the proof is trivial, it is left as an exercise.

THEOREM 2.3. *Let m, n, k be positive integers with $k \leq m \leq n$. Then*

$$\Theta(m,n,k) = kn.$$

If $m < n$ and $A \in \Omega(m,n,k)$, then $\sharp(A) = kn$ if and only if there exists a permutation matrix P such that $PA = \begin{bmatrix} J_{k,n} \\ O \end{bmatrix}$; if $m = n$ and $A \in \Omega(n,n,k)$, then $\sharp(A) = kn$ if and only if there exists a permutation matrix P such that $PA = \begin{bmatrix} J_{k,n} \\ O \end{bmatrix}$ or $AP = \begin{bmatrix} J_{n,k} & O \end{bmatrix}$.

Next, we determine the possible numbers of ones in the general $(0,1)$-matrices

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with a given term rank.

THEOREM 2.4. *Let k, m, n, d be positive integers with $k \leq m \leq n$. Then there exists a matrix $A \in \Omega(m, n, k)$ with exactly d 1's if and only if $k \leq d \leq kn$.*

Proof. Suppose there exists a matrix $A \in \Omega(m, n, k)$ with exactly d 1's. By Theorem 3.3, we have $k \leq d \leq kn$. Hence, we need only show the "if" part.

Let $A_1 = I_k \oplus O_{m-k,n-k}$ and $A_2 = \begin{bmatrix} J_{k,n} \\ O_{m-k,n} \end{bmatrix}$. It is clear that $A_1, A_2 \in \Omega(m, n, k)$ and $\sharp(A_1) = k, \sharp(A_2) = kn$.

Let $\Gamma = \{(i, j) | i \in \{1, 2, \ldots, k\}, j \in \{1, 2, \ldots, n\}$ and $i \neq j\}$. Then $|\Gamma| = kn - k$. For any given positive integer d with $k < d < kn$, we construct a matrix

$$A_0 := A_1 + \sum_{(i,j) \in \Gamma_1} E_{i,j},$$

where $\Gamma_1 \subseteq \Gamma$ and $|\Gamma_1| = d - k$.

It is clear that $A_1 \leq A_0 \leq A_2$. Hence, by Lemma 2.2, $A_0 \in \Omega(m, n, k)$ and $\sharp(A) = \sharp(A_1) + \sum_{(i,j) \in \Gamma_1} \sharp(E_{i,j}) = d$. ∎

Next we turn to the study of symmetric $(0, 1)$-matrices. Let

$$S(n) = \{A \in \mathcal{A}_n | A^T = A\}$$

and

$$\Delta(n, k) = \{A \in S(n) | \tau(A) = k\}.$$

In the following theorem, we will determine

$$\Phi(n, k) = \max\{\sharp(A) | A \in \Delta(n, k)\}$$

and the symmetric $(0, 1)$-matrices that attain the maximum.

THEOREM 2.5.

$$\Phi(n,k) = \begin{cases} nk - \frac{k^2}{4}, & \textit{if } \ 2 \leq k \leq \frac{4n}{5} \ \textit{ and } k \textit{ is even}, \\ (k-1)(n - \frac{k-1}{4}) + 1, & \textit{if } \ 2 \leq k \leq \frac{4n-3}{5} \ \textit{ and } k \textit{ is odd}, \\ k^2, & \textit{otherwise}. \end{cases}$$

Furthermore, a matrix $A \in \Delta(n, k)$ has exactly $\Phi(n, k)$ 1's if and only if A is permutation similar to one of the following matrices:
(1)

$$\begin{bmatrix} J_{\frac{k}{2}} & J_{\frac{k}{2},\ n-\frac{k}{2}} \\ J_{n-\frac{k}{2},\ \frac{k}{2}} & O \end{bmatrix},$$

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where k is even;
(2)

$$\begin{bmatrix} J_{\frac{k-1}{2}} & J_{\frac{k-1}{2},\ n-\frac{k-1}{2}} \\ J_{n-\frac{k-1}{2},\ \frac{k-1}{2}} & O \end{bmatrix} + E_{\frac{k+1}{2},\frac{k+1}{2}},$$

where k is odd;
(3)

$$\begin{bmatrix} J_k & O_{k,\ n-k} \\ O_{n-k,\ k} & O \end{bmatrix}.$$

Proof. For $A \in \Delta(n,k)$, let $R_{N_1} \cup C_{N_2}$ be a minimal covering of A, where N_1, N_2 are two subsets of $N := \{1, 2, \dots, n\}$ with $|N_1| + |N_2| = k$, R_i and C_j are the i-th row and the j-th column of A, $R_{N_1} = \{R_i | i \in N_1\}$, $C_{N_2} = \{C_j | j \in N_2\}$.

We distinguish the following two cases.

Case 1. $N_1 \cap N_2 = \emptyset$.

Subcase 1.1. $N_1 = \emptyset$, or $N_2 = \emptyset$. We only consider the case $N_1 = \emptyset$, since the case $N_2 = \emptyset$ can be proved similarly. Now A has a minimal covering C_{N_2} with $|N_2| = k$. Thus, $A[N|N_2^c] = O$. By symmetry, we also have $A[N_2^c|N] = O$. Therefore, A is permutation similar to a matrix of the form

$$\begin{bmatrix} A_0 & O \\ O & O \end{bmatrix},$$

where $A_0 \in S(k)$. It is obvious that $\sharp(A) = \sharp(A_0) \le k^2$.

Subcase 1.2. $N_1 \neq \emptyset$ and $N_2 \neq \emptyset$. In this subcase, we have $A(N_1|N_2) = O$. Since A is symmetric, we also have $A(N_2|N_1) = O$. Suppose $|N_1| = t$, $1 \le t \le k-1$. Then $|N_2| = k - t$. Therefore, A is permutation similar to a matrix of the form

$$\begin{bmatrix} A_1 & O & O \\ O & A_2 & O \\ O & O & O \end{bmatrix},$$

where $A_1 \in S(t)$ and $A_2 \in S(k-t)$. Now we have $\sharp(A) = \sharp(A_1) + \sharp(A_2)$ and

$$\sharp(A) \le t^2 + (k-t)^2 < k^2.$$

Case 2. $N_1 \cap N_2 \neq \emptyset$. We also consider two subcases.

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Subcase 2.1. $N_1 \cap N_2 = N_1$, or $N_1 \cap N_2 = N_2$. By symmetry, we need only consider the case $N_1 \cap N_2 = N_1$, i.e., $N_1 \subseteq N_2$. Then $A(N_1|N_2) = O$. Since A is symmetric, we have $A(N_2|N_1) = O$. Suppose $|N_1| = t$, $1 \le t \le \frac{k}{2}$. Then $|N_2| = k - t$. Therefore, A is permutation similar to a matrix of the form

$$\begin{bmatrix} A_1 & U^T & V^T \\ U & A_2 & O \\ V & O & O \end{bmatrix},$$

where $A_1 \in S(t)$ and $A_2 \in S(k-2t)$. Thus, $\sharp(A) = \sharp(A_1) + \sharp(A_2) + 2\left(\sharp(U) + \sharp(V)\right)$ and

$$\sharp(A) \le g(t) := k^2 + 3t^2 + (2n-4k)t.$$

A direct computation shows that

$$\max_{1 \le t \le \frac{k}{2}} g(t) = \begin{cases} g\left(\frac{k}{2}\right) = nk - \frac{k^2}{4}, & \text{if } k \text{ is even,} \\ g\left(\frac{k-1}{2}\right) = (k-1)(n - \frac{k-1}{4}) + 1, & \text{if } k \text{ is odd.} \end{cases}$$

Moreover, if k is even, a matrix $A \in \Delta(n,k)$ attains the largest number of ones with $\sharp(A) = nk - \frac{k^2}{4}$ if and only if $N_1 = N_2$. If k is odd, a matrix $A \in \Delta(n,k)$ attains the largest number of ones with $\sharp(A) = (k-1)(n - \frac{k-1}{4}) + 1$ if and only if $N_1 \subseteq N_2$ and $|N_2 \backslash N_1| = 1$.

Subcase 2.2. $N_1 \cap N_2 \ne N_1$ and $N_1 \cap N_2 \ne N_2$. Suppose $|N_1| = t$ and $|N_1 \cap N_2| = m$. Then,

$$|N_1 \backslash (N_1 \cap N_2)| = t - m > 0, \ |N_2 \backslash (N_1 \cap N_2)| = k - t - m > 0.$$

It is not difficult to see that A is permutation similar to a matrix of the form

$$\begin{bmatrix} A_1 & U & V & W \\ U^T & A_2 & O & O \\ V^T & O & A_3 & O \\ W^T & O & O & O \end{bmatrix},$$

where $A_1 \in S(m)$, $A_2 \in S(t-m)$ and $A_3 \in S(k-t-m)$.

It follows that

$$\begin{aligned} \sharp(A) &= \sharp(A_1) + \sharp(A_2) + \sharp(A_3) + 2(\sharp(U) + \sharp(V) + \sharp(W)) \\ &< k^2 + 3m^2 + (2n-4k)m := g(m). \end{aligned}$$

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Comparing $g(m)$ with $g(t)$ in Subcase 2.1 and noting that $1 \le m \le \frac{k}{2} - 1$, we have

$$\sharp(A) < \begin{cases} nk - \frac{k^2}{4}, & \text{if } k \text{ is even,} \\ (k-1)(n - \frac{k-1}{4}) + 1, & \text{if } k \text{ is odd.} \end{cases}$$

Now we can conclude that if k is even, then the possible value of $\Phi(n,k)$ is $nk - \frac{k^2}{4}$ or k^2. Similarly, if k is odd, then the possible value of $\Phi(n,k)$ is $(k-1)(n - \frac{k-1}{4}) + 1$ or k^2. After comparing these two pairs of numbers we have

$$\Phi(n,k) = \begin{cases} nk - \frac{k^2}{4}, & \text{if } 2 \le k \le \frac{4n}{5} \text{ and } k \text{ is even,} \\ (k-1)(n - \frac{k-1}{4}) + 1, & \text{if } 2 \le k \le \frac{4n-3}{5} \text{ and } k \text{ is odd,} \\ k^2, & \text{otherwise.} \end{cases}$$

Furthermore, if k is even with $2 \le k \le \frac{4n}{5}$ and $\sharp(A) = nk - \frac{k^2}{4}$, then we can see from Subcase 2.1 that A is permutation similar to the matrix

$$\begin{bmatrix} J_{\frac{k}{2}} & J_{\frac{k}{2},\ n-\frac{k}{2}} \\ J_{n-\frac{k}{2},\ \frac{k}{2}} & O \end{bmatrix}.$$

If k is odd with $2 \le k \le \frac{4n-3}{5}$ and $\sharp(A) = (k-1)(n - \frac{k-1}{4}) + 1$, then there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} J_{\frac{k-1}{2}} & J_{\frac{k-1}{2},\ n-\frac{k-1}{2}} \\ J_{n-\frac{k-1}{2},\ \frac{k-1}{2}} & O \end{bmatrix} + E_{\frac{k+1}{2},\frac{k+1}{2}}.$$

Finally, if $\sharp(A) = k^2$, then there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} J_k & O_{k,\ n-k} \\ O_{n-k,k} & O \end{bmatrix}. \quad \square$$

THEOREM 2.6. *Let k, n, d be positive integers with $k \le n$. Then there exists a symmetric $(0,1)$-matrix A of term rank k with exactly d 1's if and only if $k \le d \le \Phi(n,k)$.*

Proof. Suppose that A is a symmetric $(0,1)$-matrix of term rank k with exactly d 1's. By Theorem 2.5, we have $k \le d \le \Phi(n,k)$. Hence, we need only show the "if" part.

We only prove that if $k \le d \le nk - \frac{k^2}{4}$, where $2 \le k \le \frac{4n}{5}$ and k is even, then there exists $A' \in \Delta(n,k)$ with d ones. The proofs of the other two cases are similar.

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In this case, $\Phi(n,k) = nk - \frac{k^2}{4}$. Furthermore, the maximum is attained if and only if A is permutation similar to $A_{\frac{k}{2}} := \begin{bmatrix} J_{\frac{k}{2}} & J_{\frac{k}{2},\, n-\frac{k}{2}} \\ J_{n-\frac{k}{2},\, \frac{k}{2}} & O_{n-\frac{k}{2}} \end{bmatrix}$. Setting

$$A_1 = \begin{bmatrix} O_{\frac{k}{2}} & O & C \\ O & O_{n-k} & O \\ C & O & O_{\frac{k}{2}} \end{bmatrix} \in \Delta(n,k)$$

and constructing the following matrices recursively,

$$A_m = A_{m-1} + E_{m,\, m} + \sum_{m<j\le n,\, m+j\neq n+1} (E_{m,\, j} + E_{j,\, m}), \quad m = 2, \ldots, \frac{k}{2},$$

where C is the matrix of order $\frac{k}{2}$ with each entry on the cross diagonal being 1 and all other entries being 0. Note that for any $m \in \{1, 2, \ldots, \frac{k}{2} - 1\}$, we have

$$A_m \le A_{m+1},\ \sharp(A_m) = k + (2n - m)m - 2m.$$

It is obvious that $\sharp(A_m)$ is a monotonically increasing function in m and

$$A_1 \le A_2 \le \cdots \le A_{\frac{k}{2}}.$$

By Lemma 2.2, we have $A_i \in \Delta(n,k)$, for $i \in \{1, 2, \ldots, \frac{k}{2}\}$. It is clear that

$$k = \sharp(A_1) < \sharp(A_2) < \cdots < \sharp\left(A_{\frac{k}{2}}\right) = nk - \frac{k^2}{4}.$$

Thus, for any given positive integer d, there exists some $s \in \{1, 2, \ldots, \frac{k}{2} - 1\}$ such that $d \in [\sharp(A_s), \sharp(A_{s+1})]$. For any given $w \in \{1, 2, \ldots, \frac{k}{2} - 1\}$, we need only show that if d is a positive integer with $\sharp(A_w) < d < \sharp(A_{w+1})$, then there exists some $A^{'} \in \Delta(n,k)$ with $\sharp(A^{'}) = d$.

It is easily seen that $\sharp(A_{w+1}) = \sharp(A_w) + 1 + 2(n - 2 - w)$. Thus,

$$1 \le d - \sharp(A_w) \le 1 + 2(n - 2 - w).$$

Let $p = d - \sharp(A_w)$ and denote $\Psi_{w+1} := \{(m+1, j) \mid m + 1 < j \le n, j + w = n + 1\}$. We distinguish two cases by considering the parity of p.

If p is even, then $\frac{p}{2} \le |\Psi_{w+1}|$. Hence, there exists a subset Ψ_0 of Ψ_{w+1} with $|\Psi_0| = \frac{p}{2}$ such that

$$A^{'} = A_w + \sum_{(i,j)\in\Psi_0} (E_{i,j} + E_{j,i}).$$

Thus, $A_w \le A^{'} \le A_{w+1}$ and $\sharp(A^{'}) = d$. By Lemma 2.2, we have $A^{'} \in \Delta(n,k)$.

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Similarly, if p is odd, then $\frac{p-1}{2} \le |\Psi_{w+1}|$ and there exists a subset Ψ_1 of Ψ_{w+1} with $|\Psi_1| = \frac{p-1}{2}$ such that

$$A^{'} = A_w + \sum_{(i,j)\in\Psi_1} (E_{i,j} + E_{j,i}) + E_{m+1,m+1}.$$

Thus, $A_w \le A^{'} \le A_{w+1}$ and $\sharp(A^{'}) = d$. By Lemma 2.2, we have $A^{'} \in \Delta(n,k)$. $\square$

Let G be a simple undirected graph. Note that the adjacency matrix of G is a symmetric $(0,1)$-matrix with 0's on the main diagonal. Clearly, the adjacency matrix is just another way of specifying the graph. Finally, we consider the class of symmetric $(0,1)$-matrices with 0's on the main diagonal. This class is also of interest from the point of view of graph theory.

Denote by $\Gamma(n)$ the set of all $n\times n$ symmetric $(0,1)$-matrices with each main diagonal entry being 0 and

$$\Delta_0(n,k) = \{A \in \Gamma(n) | \tau(A) = k\}.$$

It is clear that if $A \in \Gamma(n)$, then $\tau(A) \ne 1$. Thus, we may assume that $k > 1$.

Let

$$\Phi_0(n,k) = \max\{\sharp(A) | A \in \Delta_0(n,k)\}.$$

We will determine $\Phi_0(n,k)$ and the matrices in $\Delta_0(n,k)$ that attain it. Denote $J_t^{'} = J_t - I_t$.

Theorem 2.7.

$$\Phi_0(n,k) = \begin{cases} nk - (\frac{k}{2})^2 - \frac{k}{2}, & \text{if } 2 \le k \le \frac{4n+2}{5} \text{ and } k \text{ is even}, \\ n(k-3) - (\frac{k-3}{2})^2 - \frac{k-3}{2} + 6, & \text{if } 2 \le k \le \frac{4n-7}{5} \text{ and } k \text{ is odd}, \\ k^2 - k, & \text{otherwise}. \end{cases}$$

Furthermore, a matrix $A \in \Delta_0(n,k)$ has exactly $\Phi_0(n,k)$ 1's if and only if A is permutation similar to one of the following forms:

(4)

$$\begin{bmatrix} J_{\frac{k}{2}}^{'} & J_{\frac{k}{2},\, n-\frac{k}{2}} \\ J_{n-\frac{k}{2},\, \frac{k}{2}} & O \end{bmatrix},$$

where k is even;

(5)

$$\begin{bmatrix} J_{\frac{k-3}{2}}^{'} & J_{3,\,\frac{k-3}{2}}^{T} & J_{n-\frac{k+3}{2},\,\frac{k-3}{2}}^{T} \\ J_{3,\,\frac{k-3}{2}} & J_3^{'} & O \\ J_{n-\frac{k+3}{2},\,\frac{k-3}{2}} & O & O \end{bmatrix},$$

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where k is odd;
(6)

$$\begin{bmatrix} J_k^{'} & O_{k,\ n-k} \\ O_{n-k,\ k} & O \end{bmatrix}.$$

Proof. Use the notation and argument in the proof of Theorem 5. We distinguish the following two cases.

Case 1. $N_1 \bigcap N_2 = \emptyset$. We consider two subcases.

Subcase 1.1. $N_1 = \emptyset$ or $N_2 = \emptyset$. Therefore, A is permutation similar to a matrix of the form

$$\begin{bmatrix} A_0 & O \\ O & O \end{bmatrix},$$

where $A_0 \in \Gamma(k)$. It is obvious that $\sharp(A) = \sharp(A_0) \leq k^2 - k$. The equality holds with $A_0 = J_k^{'}$.

Subcase 1.2. $N_1 \neq \emptyset$ and $N_2 \neq \emptyset$. Suppose $\mid N_1 \mid= t$. Then $\mid N_2 \mid= k - t$ with $1 \leq t \leq k - 1$. Therefore, A is permutation similar to a matrix of the form

$$\begin{bmatrix} A_1 & O & O \\ O & A_2 & O \\ O & O & O \end{bmatrix},$$

where $A_1 \in \Gamma(t)$ and $A_2 \in \Gamma(k - t)$. We have $\sharp(A) = \sharp(A_1) + \sharp(A_2)$ and

$$\sharp(A) < k^2 - k.$$

Case 2. $N_1 \bigcap N_2 \neq \emptyset$. We also consider two subcases.

Subcase 2.1. $N_1 \bigcap N_2 = N_1$, or $N_1 \bigcap N_2 = N_2$. Suppose $\mid N_1 \mid= t$. Then $\mid N_2 \mid= k - t$ with $1 \leq t \leq \frac{k}{2}$. Therefore, A is permutation similar to a matrix of the form

$$\begin{bmatrix} A_1 & U^T & V^T \\ U & A_2 & O \\ V & O & O \end{bmatrix},$$

where $A_1 \in \Gamma(t)$ and $A_2 \in \Gamma(k-2t)$. Therefore, $\sharp(A) = \sharp(A_1)+\sharp(A_2)+2(\sharp(U)+\sharp(V))$ and

$$\sharp(A) \leq g(t) := k^2 - k + 3t^2 + (2n - 4k + 1)t.$$

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First assume that k is odd. Then the order of A_2 is odd with $k-2t \geq 3$ and so $1 \leq t \leq \frac{k-3}{2}$.

(a) Suppose $2n-4k+1 \geq 0$, i.e., $2 \leq k \leq \frac{n}{2}+\frac{1}{4}$. We have $g(t) \geq k^2-k$ and

$$\max_{1\leq t\leq \frac{k-3}{2}} g(t) = g\left(\frac{k-3}{2}\right) = n(k-3) - \left(\frac{k-3}{2}\right)^2 - \frac{k-3}{2} + 6.$$

(b) Suppose $2n-4k+1 < 0$, i.e., $\frac{n}{2}+\frac{1}{4} < k \leq n$. If $g(t) \geq k^2-k$, then by the monotonicity of the quadratic polynomial $g(t)$, we have

$$\frac{4k-2n-1}{3} \leq \frac{k-3}{2}, \quad \text{i.e.,} \quad k \leq \frac{4n}{5} - \frac{7}{5}.$$

and

$$\max_{1\leq t\leq \frac{k-3}{2}} g(t) = g\left(\frac{k-3}{2}\right) = n(k-3) - \left(\frac{k-3}{2}\right)^2 - \frac{k-3}{2} + 6.$$

Combining (a) and (b), it follows that if $2 \leq k \leq \frac{4n-7}{5}$ and k is odd, then

$$\sharp(A) \leq \max_{1\leq t\leq \frac{k}{2}} g(t) = g\left(\frac{k-3}{2}\right) = n(k-3) - \left(\frac{k-3}{2}\right)^2 - \frac{k-3}{2} + 6.$$

Moreover, A attains the largest number of ones with

$$\sharp(A) = g\left(\frac{k}{2}\right) = n(k-3) - \left(\frac{k-3}{2}\right)^2 - \frac{k-3}{2} + 6$$

if and only if $N_1 \subseteq N_2, \mid N_2 \backslash N_1 \mid = 3$. If $\frac{4n-7}{5} < k \leq n$ and k is odd, then

$$\sharp(A) < k^2 - k.$$

Now assume that k is even.

(c) Suppose $2n-4k+1 \geq 0$, i.e., $2 \leq k \leq \frac{n}{2}+\frac{1}{4}$. We have $g(t) \geq k^2-k$ and

$$\max_{1\leq t\leq \frac{k}{2}} g(t) = g\left(\frac{k}{2}\right) = nk - \frac{k^2}{4} - \frac{k}{2}.$$

(d) Suppose $2n-4k+1 < 0$, i.e., $\frac{n}{2}+\frac{1}{4} < k \leq n$. If $g(t) \geq k^2-k$, then by the monotonicity of the quadratic polynomial $g(t)$, we have

$$\frac{4k-2n-1}{3} \leq \frac{k}{2}, \quad \text{i.e.,} \quad k \leq \frac{4n}{5} + \frac{2}{5}.$$

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and

$$\max_{1\le t\le \frac{k}{2}} g(t)=g\left(\frac{k}{2}\right)=nk-\frac{k^2}{4}-\frac{k}{2}.$$

Combining (c) and (d), it follows that if $2\le k\le \frac{4n+2}{5}$ and k is even, then

$$\sharp(A)\le \max_{1\le t\le \frac{k}{2}} g(t)=g\left(\frac{k}{2}\right)=nk-\frac{k^2}{4}.$$

Moreover, A attains the largest number of ones with $\sharp(A)=g\left(\frac{k}{2}\right)=nk-\frac{k^2}{4}$ if and only if $N_1=N_2$. If $\frac{4n+2}{5}<k\le n$ and k is even, then

$$\sharp(A)<k^2-k.$$

In Subcase 2.1, we have

$$\sharp(A)\le \max_{1\le t\le \frac{k}{2}} g(t)=\begin{cases} nk-\frac{k^2}{4}-\frac{k}{2}, & \text{if } 2\le k\le \frac{4n+2}{5} \text{ and } k \text{ is even},\\ n(k-3)-\left(\frac{k-3}{2}\right)^2-\frac{k-3}{2}+6, & \text{if } 2\le k\le \frac{4n-7}{5} \text{ and } k \text{ is odd},\\ <k^2-k, & \text{otherwise .}\end{cases}$$

Subcase 2.2. $N_1\bigcap N_2\ne N_1$ and $N_1\bigcap N_2\ne N_2$. Suppose $\mid N_1\mid\ =t$ and $\mid N_1\bigcap N_2\mid\ =m$. Then

$$\mid N_1\backslash(N_1\bigcap N_2)\mid\ =t-m>0,\ \ \mid N_2\backslash(N_1\bigcap N_2)\mid\ =k-t-m>0.$$

Now A is permutation similar to a matrix of the form

$$\begin{bmatrix} J_m^{'} & J_{m,t-m} & J_{m,\ k-t-m}\\ J_{t-m,\ m} & A_1 & O\\ J_{k-t-m,\ m} & O & A_2\end{bmatrix},$$

where $A_1\in\Gamma(t-m)$ and $A_2\in\Gamma(k-t-m)$.

Then we can show that $\sharp(A)<g(m)=k^2+3m^2+(2n-4k)m$ with $1\le m\le \frac{k}{2}-1$. Comparing this $g(m)$ with $g(t)$ in Subcase 2.1, we have

$$\sharp(A)<\begin{cases} nk-\frac{k^2}{4}-\frac{k}{2}, & \text{if } 2\le k\le \frac{4n+2}{5} \text{ and } k \text{ is even},\\ n(k-3)-\left(\frac{k-3}{2}\right)^2-\frac{k-3}{2}+6, & \text{if } 2\le k\le \frac{4n-7}{5} \text{ and } k \text{ is odd},\\ k^2-k, & \text{otherwise .}\end{cases}$$

Now from the above proof, we can conclude that

$$\Phi_0(n,k)\ =\begin{cases} nk-\frac{k^2}{4}-\frac{k}{2}, & \text{if } 2\le k\le \frac{4n+2}{5} \text{ and } k \text{ is even},\\ n(k-3)-(\frac{k-3}{2})^2-\frac{k-3}{2}+6, & \text{if } 2\le k\le \frac{4n-7}{5} \text{ and } k \text{ is odd},\\ k^2-k, & \text{otherwise .}\end{cases}$$

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First, observe from Subcase 2.1 that if $\sharp(A) = \Phi_0(n,k)$ and k is even with $2 \leq k \leq \frac{4n+2}{5}$, then there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} J'_{\frac{k}{2}} & J_{\frac{k}{2},\ n-\frac{k}{2}} \\ J_{n-\frac{k}{2},\ \frac{k}{2}} & O \end{bmatrix}.$$

Second, if $\sharp(A) = \Phi_0(n,k)$ and k is odd with $2 \leq k \leq \frac{4n-7}{5}$, then there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} J'_{\frac{k-3}{2}} & J^T_{3,\ \frac{k-3}{2}} & J^T_{n-\frac{k+3}{2},\ \frac{k-3}{2}} \\ J_{3,\ \frac{k-3}{2}} & J'_3 & O \\ J_{n-\frac{k+3}{2},\ \frac{k-3}{2}} & O & O \end{bmatrix}.$$

Otherwise, if $\sharp(A) = \Phi_0(n,k) = k^2 - k$, then there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} J'_k & 0_{k,\ n-k} \\ 0_{n-k,k} & O \end{bmatrix}. \quad \square$$

Next, we will determine the possible numbers of nonzero entries of matrices in $\Delta_0(n,k)$. To this end, we need determine the minimal number of nonzero entries of matrices in $\Delta_0(n,k)$. Denote

$$\phi_0(n,k) = \min\{\sharp(A) \mid A \in \Delta_0(n,k)\}.$$

Lemma 2.8.

$$\phi_0(n,k) = \begin{cases} k, & \textit{if } k \textit{ is even}, \\ k+3, & \textit{if } k \textit{ is odd}. \end{cases}$$

Proof. First note that if $A \in \Delta_0(n,k)$, then $\sharp(A) \geq k$ and $\sharp(A)$ is even. We consider the following two cases according to the parity of k.

If k is even, we need only show that there exists a matrix $A \in \Delta_0(n,k)$ such that $\sharp(A) = k$. Let $A = (\oplus_{i=1}^{k/2})J'_2 \oplus O_{n-k}$, then $A \in \Delta_0(n,k)$ and $\sharp(A) = k$. Thus, in this case, $\phi_0(n,k) = k$.

If k is odd and $A \in \Delta_0(n,k)$, then A has exactly k nonzero entries with no two of these entries on a line. Since $\sharp(A)$ is even, A has at least $k+1$ nonzero entries. If $\sharp(A) = k+1$, then there are $\frac{k+1}{2}$ pairs of nonzero entries which are pairwise symmetric. Without loss of generality, let $a_1, a_2, \ldots, a_k, a_{k+1}$ be the $k+1$ nonzero entries of A

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and a_i, a_{i+1} be symmetric in A, $i = 1, 3, \ldots, k$. We may assume that $a_1, a_2, \ldots, a_k$ are k nonzero entries with no two on a line. Since $\tau(A) = k$, there exists a positive integer i, $1 \le i \le k-1$, such that a_i and a_{k+1} are on a line. This can not be true since a_k and a_i are not on a line. Hence, if k is odd and $A \in \Delta_0(n,k)$, then $\sharp(A) > k+1$, i.e., $\sharp(A) \ge k+3$. Moreover, let $A = (\oplus_{i=1}^{\frac{k-3}{2}} J_2') \oplus J_3' \oplus O_{n-k}$. Then, $A \in \Delta_0(n,k)$ and $\sharp(A) = k+3$. Therefore, if k is odd, then $\phi_0(n,k) = k+3$. $\square$

THEOREM 2.9. *Let k, n, d be positive integers with $k \le n$. Then there exists $A \in \Delta_0(n,k)$ with exactly d 1's if and only if $\phi_0(n,k) \le d \le \Phi_0(n,k)$ and d is even.*

Proof. Suppose $A \in \Delta_0(n,k)$ has exactly d 1's. By Theorem 2.7 and Lemma 2.8, $\phi_0(n,k) \le d \le \Phi_0(n,k)$ and d is even. Hence, we need only show the "if" part.

According to the parity of k, we consider the following two cases: k is even; k is odd. Next, we only show the case: k is even, while the left case is similar.

Let k be even. If $2 \le k \le \frac{4n+2}{5}$, then $\Phi_0(n,k) = nk - \frac{k^2}{4} - \frac{k}{2}$. Without loss of generality, we can set

$$A_1 = \begin{bmatrix} J'_{\frac{k}{2}} & J \\ J & O_{n-\frac{k}{2}} \end{bmatrix} = \begin{bmatrix} J'_{\frac{k}{2}} & J & J \\ J & O_{n-k} & O \\ J & O & O_{\frac{k}{2}} \end{bmatrix}.$$

Taking

$$A_0 = \begin{bmatrix} O_{\frac{k}{2}} & O & C \\ O & O_{n-k} & O \\ C & O & O_{\frac{k}{2}} \end{bmatrix},$$

where C is the matrix of order $\frac{k}{2}$ with each entry on the cross diagonal being 1 and all other entries being 0. Then $A_0 \in \Delta_0(n,k)$ and $A_0 \le A_1$. By Lemma 2.2, for any symmetric matrix B with $A_0 \le B \le A_1$ we have $B \in \Delta_0(n,k)$. Hence, for any given even number d with $k \le d \le \Phi_0(n,k)$, there exists $B_0 \in \Delta_0(n,k)$ such that $\sharp(B_0) = d$.

Otherwise, $\Phi_0(n,k) = k^2 - k$. Without loss of generality, we can set

$$A_3 = \begin{bmatrix} J'_k & O \\ O & O_{n-k} \end{bmatrix}.$$

Taking

$$A_2 = \begin{bmatrix} C_k & O \\ O & O_{n-k} \end{bmatrix},$$

where C_k is the matrix of order k with each entry on the cross diagonal being 1 and all other entries being 0. Then $A_2 \in \Delta_0(n,k)$ and $A_2 \le A_3$, By Lemma 2.2, for any

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symmetric matrix B with $A_2 \leq B \leq A_3$ we have $B \in \Delta_0(n,k)$. Thus, for any given even number d with $k \leq d \leq \Phi_0(n,k)$, there exists $B_0 \in \Delta_0(n,k)$ such that $\sharp(B_0) = d$. This completes the proof. $\square$

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