

MAXIMA OF THE Q -INDEX: DEGENERATE GRAPHS*

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Abstract. Let G be a k -degenerate graph of order n . It is well-known that G has no more edges than $S_{n,k}$, the join of a complete graph of order k and an independent set of order $n - k$. In this note, it is shown that $S_{n,k}$ is extremal for some spectral parameters of G as well. More precisely, letting $\mu(H)$ and $q(H)$ denote the largest eigenvalues of the adjacency matrix and the signless Laplacian of a graph H , the inequalities

$$\mu(G) < \mu(S_{n,k}) \quad \text{and} \quad q(G) < q(S_{n,k})$$

hold, unless $G = S_{n,k}$.

The latter inequality is deduced from the following general bound, which improves some previous bounds on $q(G)$:

If G is a graph of order n , with m edges, with maximum degree Δ and minimum degree δ , then

$$q(G) \leq \min \left\{ 2\Delta, \frac{1}{2} \left(\Delta + 2\delta - 1 + \sqrt{(\Delta + 2\delta - 1)^2 + 16m - 8(n - 1 + \Delta)\delta} \right) \right\}.$$

Equality holds if and only if G is regular or G has a component of order $\Delta + 1$ in which every vertex is of degree δ or Δ , and all other components are δ -regular.

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1. Introduction and main results. A graph G is said to be k -degenerate if every subgraph of G contains a vertex of degree at most k . Clearly, k -degenerate graphs constitute a monotone graph family, as every subgraph of a k -degenerate graph is k -degenerate itself. In fact, many much studied classes of graphs consist of degenerate graphs; e.g., forests are 1-degenerate, planar graphs are 5-degenerate, graphs of tree-width k are k -degenerate, etc.

By induction on n one can show that if $n \geq k$ and G is a k -degenerate graph of order n , then the number of edges $e(G)$ of G satisfies

$$e(G) \leq kn - (k^2 + k)/2.$$

Equality holds for numerous graphs, in particular if $G = S_{n,k}$, where $S_{n,k}$ is the join of a complete graph of order k and an independent set of order $n - k$.

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It turns out that the graph $S_{n,k}$ is extremal for some spectral parameters of k -degenerate graphs as well. A notable early contribution is due to Hong, who showed in [9] that if G has tree-width k , then the largest eigenvalue $\mu(G)$ of the adjacency matrix of G satisfies $\mu(G) < \mu(S_{n,k})$, unless $G = S_{n,k}$. Here we prove the following stronger theorem.

THEOREM 1.1. *If G is a k -degenerate graph of order $n \geq k$, then $\mu(G) < \mu(S_{n,k})$, unless $G = S_{n,k}$.*

We shall show that Theorem 1.1 readily follows from a well-known bound on $\mu(G)$, stated below as Theorem A. The main effort of this note, however, is to prove a similar bound on the largest eigenvalue $q(G)$ of the signless Laplacian of a k -degenerate graph G .

THEOREM 1.2. *If G is a k -degenerate graph of order $n \geq k$, then $q(G) < q(S_{n,k})$, unless $G = S_{n,k}$.*

The simple proof strategy for Theorem 1.1 did not apply to Theorem 1.2 because until recently there was no appropriate bound on $q(G)$ similar to Theorem A. To fill in this void we came up with the following theorem, which is of separate interest and may have other applications as well.

THEOREM 1.3. *If G is a graph of order n , with m edges, with maximum degree Δ and minimum degree δ , then*

$$q(G) \leq \min \left\{ 2\Delta, \frac{\Delta + 2\delta - 1 + \sqrt{(\Delta + 2\delta - 1)^2 + 16m - 8(n - 1 + \Delta)\delta}}{2} \right\}. \quad (1.1)$$

Equality holds if and only if G is regular or G has a component of order $\Delta + 1$ in which every vertex is of degree δ or Δ , and all other components are δ -regular.

Inequality (1.1) seems quite unwieldy at first glance, so some remarks are due here. First, obviously the right side is nondecreasing in m ; a simple calculation shows that it is equal to 2Δ whenever

$$2m \geq \Delta^2 + \Delta + (n - 1 - \Delta)\delta.$$

Furthermore, the dependence of the right side of (1.1) on δ and Δ is clarified by the following proposition whose simple proof is omitted.

PROPOSITION 1.4. *Let $n \geq 1$ and $0 \leq m \leq n(n - 1)/2$. If $x \leq 2m/n$ and $y \geq 2m/n$, then the functions*

$$f(x, y) = y + 2x - 1 + \sqrt{(y + 2x - 1)^2 + 16m - 8(n - 1 + y)x}$$

and

$$g(x, y) = \min \left\{ 2y, \frac{1}{2} \left(y + 2x - 1 + \sqrt{(y + 2x - 1)^2 + 16m - 8(n - 1 + y)x} \right) \right\}$$

are decreasing in x and increasing in y .

Now, using Proposition 1.4 and the fact that $\Delta \leq n - 1$, we obtain the following simplified version of Theorem 1.3.

COROLLARY 1.5. *If G is a graph of order n , with m edges and with minimum degree δ , then*

$$q(G) \leq \frac{1}{2} \left(n + 2\delta - 2 + \sqrt{(n + 2\delta - 2)^2 + 16m - 16(n - 1)\delta} \right).$$

Equality holds if and only if each degree of G is equal to δ or $n - 1$.

Likewise, using that $\delta \geq 0$, we obtain another bound.

COROLLARY 1.6. *If G is a graph of order n , with m edges and with maximum degree Δ , then*

$$q(G) \leq \min \left\{ 2\Delta, \frac{1}{2} \left(\Delta - 1 + \sqrt{(\Delta - 1)^2 + 16m} \right) \right\}.$$

Equality holds if and only if G is Δ -regular with possibly some isolated vertices.

Similar bounds on $q(G)$ have been known in the literature, and it seems that Theorem 1.3 compares favorably with most of them. For example, Theorem 1.3 improves the bound

$$q(G) \leq \frac{1}{2} \left(\delta - 1 + \sqrt{(\delta - 1)^2 + 16m + 8\Delta^2 - 8(n - 1)\delta} \right),$$

obtained in [5], and the bound

$$q(G) \leq \frac{1}{2} \left(\Delta + \delta - 1 + \sqrt{(\Delta + \delta - 1)^2 + 16m - 8(n - 1)\delta} \right),$$

obtained in [6].

Very likely Theorem 1.3 can be used for other extremal problems about $q(G)$, just like Theorem A below has been used for many extremal problems about $\mu(G)$.

The rest of the paper is dedicated to the proofs of Theorems 1.1, 1.2 and 1.3.

2. Proofs. For graph notation and concepts undefined here, we refer the reader to [1]. For introductory material on the signless Laplacian see the survey of Cvetković [2] and its references. In particular, for a graph G we write:

- $V(G)$ and $E(G)$ for the sets of vertices and edges of G ;
- $v(G)$ for $|V(G)|$ and $e(G)$ for $|E(G)|$;
- d_u for the degree of a vertex u .

In the proof of Theorem 1.1, we shall use the following inequality proved in [7], with condition for equality proved in [10].

THEOREM A. *If G is a graph of order n , with m edges, and with minimum degree δ , then.*

$$q(G) \leq \frac{\delta - 1}{2} + \sqrt{2m - n\delta + \frac{(\delta + 1)^2}{4}}. \quad (2.1)$$

Equality holds if and only if G is regular or G has a component of order $\Delta + 1$ in which every vertex is of degree δ or Δ , and all other components are δ -regular.

For connected graphs inequality (2.1) has been proved independently by Hong, Shu and Fang in [8], however, here the unnecessary requirement for connectedness blurs the situation, as equality holds in (2.1) for a much smaller set of graphs.

We shall also need the following characterization of k -degenerate graphs:

LEMMA B. *A graph G is k -degenerate if and only if its vertices can be ordered so that every vertex has at most k preceding neighbors.*

Finally, note the following facts about $S_{n,k}$:

$$\mu(S_{n,k}) = \frac{k-1}{2} + \sqrt{kn - \frac{3k^2 + 2k - 1}{4}}; \quad (2.2)$$

$$q(S_{n,k}) = \frac{1}{2} \left(n + 2k - 2 + \sqrt{(n + 2k - 2)^2 - 8(k^2 - k)} \right). \quad (2.3)$$

Proof of Theorem 1.1. Our first goal is to prove that $\mu(G) \leq \mu(S_{n,k})$. Without loss of generality we may assume that G is edge maximal, that is to say, if H is obtained by adding a new edge to G , then H is not k -degenerate. Order the vertices of G as follows: Choose a vertex $v \in V(G)$ with $d(v) = \delta(G)$ and set $v_n = v$; further, having chosen v_n, \dots, v_{j+1} , let H be the graph induced by the set $V(G) \setminus \{v_n, \dots, v_{j+1}\}$, choose $v \in V(H)$ with $d_H(v) = \delta(H)$ and set $v_j = v$.

Since G is k -degenerate, in the ordering v_n, \dots, v_1 every vertex has at most k preceding neighbors. Now, if $\delta(G) < k$, then we can add some edge incident to v_n , and in the ordering v_n, \dots, v_1 , every vertex will still have at most k preceding

neighbors. Hence, Lemma B implies that the resulting graph is k -degenerate. This contradicts that G is edge maximal, so we conclude that $\delta(G) = k$.

Next, as G is k -degenerate, we have $e(G) \leq kn - (k^2 + k)/2$, and Theorem A implies that

$$\begin{aligned} \mu(G) &\leq \frac{k-1}{2} + \sqrt{2m - nk + \frac{(k+1)^2}{4}} \\ &\leq \frac{k-1}{2} + \sqrt{2kn - (k^2 + k) - nk + \frac{(k+1)^2}{4}} \\ &= \mu(S_{n,k}). \end{aligned}$$

Note that if $\mu(G) = \mu(S_{n,k})$, then $e(G) = kn - (k^2 + k)/2$.

It remains to prove that if $\mu(G) = \mu(S_{n,k})$, then $G = S_{n,k}$. Assume that G' is an edge maximal k -degenerate graph of order n containing G . We have

$$\mu(S_{n,k}) = \mu(G) \leq \mu(G') \leq \mu(S_{n,k}),$$

and so

$$\mu(G) = \mu(G') = \mu(S_{n,k}),$$

implying in turn that

$$e(G) = kn - (k^2 + k)/2 = e(G'),$$

and so $G = G'$.

Arguing as above, we see that G is connected and $\delta(G) = k$. Now, the condition for equality in Theorem A implies that each degree of G is equal to k or to $n-1$. Let x be the number of vertices of degree $n-1$ in G . We have

$$x(n-1) + (n-x)k = 2e(G) = 2kn - k^2 - k,$$

and so $x = k$. Clearly, this implies that $G = S_{n,k}$, completing the proof of Theorem 1.1. \square

Proof of Theorem 1.3. Our proof is based on well-known ideas, used in the same context first in [4]. Write $r_k(B)$ for the k 'th rowsum of a matrix B . Write d_1, \dots, d_n for the degrees of G , and let Q denote the signless Laplacian of G . Define the matrix M as

$$M = Q^2 - (\Delta + 2\delta - 1)Q.$$

We shall show that the rowsums of M do not exceed $4m - 4(n - 1)\delta$. To begin with, recall the well-known fact (see, e.g., [3])

$$r_k(Q^2) = 2d_k^2 + 2 \sum_{\{i,k\} \in E(G)} d_i.$$

Now, clearly

$$\sum_{\{i,k\} \in E(G)} d_i = \sum_{i \in V(G)} d_i - \sum_{\{i,k\} \notin E(G)} d_i \leq 2m - d_k - (n - 1 - d_k)\delta,$$

and so

$$\begin{aligned} r_k(M) &= 2d_k^2 + 2 \sum_{i \sim k} d_i - 2(\Delta + 2\delta - 1)d_k \\ &\leq 4m - 2d_k - 2(n - 1 - d_k)\delta + 2d_k^2 - 2(\Delta + 2\delta - 1)d_k \\ &= 4m - 2(n - 1)\delta + 2d_k^2 - 2d_k + 2\delta d_k - 2\Delta d_k - 4\delta d_k + 2d_k \\ &= 4m - 2(n - 1)\delta + 2d_k^2 - 2(\Delta + \delta)d_k. \end{aligned}$$

To eliminate the term $2d_k^2$ we shall show that for every $k \in V(G)$,

$$d_k^2 - (\Delta + \delta)d_k \leq -\Delta\delta.$$

Indeed, the function $f(x) = x^2 - (\Delta + \delta)x$ is convex and, in view of $\delta \leq d_k \leq \Delta$, we find that

$$\begin{aligned} d_k^2 - (\Delta + \delta)d_k &= f(d_k) \leq \max\{f(\delta), f(\Delta)\} \\ &= \max\{\delta^2 - (\Delta + \delta)\delta, \Delta^2 - (\Delta + \delta)\Delta\} \\ &= -\Delta\delta. \end{aligned}$$

Hence,

$$2d_k^2 \leq 2(\Delta + \delta)d_k - 2\Delta\delta,$$

and, returning back to $r_k(M)$, we get

$$\begin{aligned} r_k(M) &\leq 4m - 2(n - 1)\delta + 2(\Delta + \delta)d_k - 2\Delta\delta - 2(\Delta + \delta)d_k \\ &= 4m - 2(n - 1 + \Delta)\delta. \end{aligned}$$

Now, writing $\mu(M)$ for the largest eigenvalue of M , we see that

$$\mu(M) \leq \max_{k \in V(G)} r_k(M) \leq 4m - 2(n - 1 + \Delta)\delta.$$

On the other hand, letting $q = q(G)$, it follows that $q^2 - (\Delta + 2\delta - 1)q$ is an eigenvalue of M and so,

$$q^2 - (\Delta + 2\delta - 1)q - 4m + 2(n - 1 + \Delta)\delta \leq 0. \quad (2.4)$$

Solving this inequality, and using the well-known $q(G) \leq 2\Delta$, we obtain (1.1).

To prove the condition for equality, first we assume that G is connected. If equality holds in (1.1) then Lemma 2.1 of [4] implies that all rowsums of M are equal and so

$$\sum_{\{i,k\} \notin E(G) \text{ and } i \neq k} d_i = (n-1-d_k)\delta$$

for every $k \in V(G)$. Therefore, if a vertex is not of degree $n-1$, then it must be of degree δ , implying that G is regular or every vertex is of degree either δ or $n-1$.

Now suppose that G is not connected and let equality hold in (2.4). Assume that $q(G) = q(G_1)$ for a component G_1 of G and set $n_1 = v(G_1)$, $m_1 = e(G_1)$, $\Delta_1 = \Delta(G_1)$, and $\delta_1 = \delta(G_1)$. In view of $\delta_1 \geq \delta$ and $\Delta_1 \leq \Delta$, inequality (1.1) and Proposition 1.4 imply that

$$\begin{aligned} q(G) &\leq \frac{1}{2} \left(\Delta_1 + 2\delta_1 - 1 + \sqrt{(\Delta_1 + 2\delta_1 - 1)^2 + 16m_1 - 8(n_1 - 1 + \Delta_1)\delta_1} \right) \\ &\leq \frac{1}{2} \left(\Delta + 2\delta - 1 + \sqrt{(\Delta + 2\delta - 1)^2 + 16m_1 - 8(n_1 - 1 + \Delta)\delta} \right) \\ &\leq \frac{1}{2} \left(\Delta + 2\delta - 1 + \sqrt{(\Delta + 2\delta - 1)^2 + 16m - 8(n - 1 + \Delta)\delta} \right). \end{aligned}$$

In the above derivation, we also use that $16m_1 - 8n_1\delta \leq 16m - 8n\delta$. Since the first and the last term of the above chain of inequalities are equal, it follows that all inequalities are equalities. In particular, all vertices which do not belong to G_1 have degree precisely δ . Also, either G_1 is δ -regular or $n_1 = \Delta + 1$ and every vertex of G_1 is of degree δ or Δ . This completes the proof of the necessity of the condition for equality in (1.1). The sufficiency of the condition is simple and we omit the calculations. The proof of Theorem 1.3 is completed. \square

Proof of Theorem 1.2. Our first goal is to prove that $q(G) \leq q(S_{n,k})$. As in the proof of Theorem 1.1, we can assume that $\delta = k$. Using Corollary 1.5 and the fact the $2m \leq 2kn - k^2 - k$, we find that

$$\begin{aligned} q(G) &\leq \frac{1}{2} \left(n + 2k - 2 + \sqrt{(n + 2k - 2)^2 + 16m - 16(n-1)k} \right) \\ &\leq \frac{1}{2} \left(n + 2k - 2 + \sqrt{(n + 2k - 2)^2 + 16kn - 8k^2 - 8k - 16(n-1)k} \right) \\ &= \frac{1}{2} \left(n + 2k - 2 + \sqrt{(n + 2k - 2)^2 - 8(k^2 - k)} \right) \\ &= q(S_{n,k}). \end{aligned}$$

Note that if $q(G) = q(S_{n,k})$, then $e(G) = kn - (k^2 + k)/2$.

It remains to prove that if $q(G) = q(S_{n,k})$, then $G = S_{n,k}$. Assume that G' is an edge maximal k -degenerate graph of order n containing G . We have

$$q(S_{n,k}) = q(G) \leq q(G') \leq q(S_{n,k}),$$

and so

$$q(G) = q(G') = q(S_{n,k}).$$

Since

$$e(G) = kn - (k^2 + k)/2 = e(G'),$$

it turns out that $G = G'$. Arguing as in the proof of Theorem 1.1, we see that $\delta(G) = k$. Now, the condition for equality of Corollary 1.5 implies that each degree of G is equal to k or to $n - 1$. From here the proof of Theorem 1.2 is completed exactly as the proof of Theorem 1.1. \square

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