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SIMPLIFICATIONS OF THE OSTROWSKI UPPER BOUNDS FOR THE SPECTRAL RADIUS OF NONNEGATIVE MATRICES*

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Abstract. A.M. Ostrowski in 1951 gave two well-known upper bounds for the spectral radius of nonnegative matrices. However, the bounds are not of much practical use because they all involve a parameter α in the interval $[0, 1]$, and it is not easy to decide the optimum value of α. In this paper, their equivalent forms which can be computed with the entries of matrix and without having to minimize the expressions of the bounds over all possible values of $\alpha \in [0, 1]$, are given.

Key words. Spectral radius, Ostrowski, Nonnegative matrices.

AMS subject classifications. 15A18, 15A42, 15A48.

1. Introduction. A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called nonnegative if $a_{ij} \geq 0$ for any $i, j \in N = \{1, 2, \ldots, n\}$. The well-known Perron-Frobenius theorem [1, 6, 7, 15] states that the spectral radius $\rho(A)$ of a nonnegative matrix A is the eigenvalue of A with a corresponding nonnegative eigenvector. One important problem in nonnegative matrices is to estimate the spectral radius of a nonnegative matrix [2, 6, 8, 9, 10, 11, 12, 19].

In 1912, G. Frobenius [6] provided the following upper bound for the spectral radius of nonnegative matrices.

THEOREM 1.1. [6, 18] *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be nonnegative. Then*

$$\rho(A) \leq \max_{i\in N} R_i(A), \tag{1.1}$$

where $R_i(A) = \sum\limits_{j\in N} a_{ij}$.

Since a matrix A and its transpose A^T have the same eigenvalue [1], we have $\rho(A) = \rho(A^T)$. Hence, for the nonnegative matrix A, we get from Theorem 1.1 that

$$\rho(A) \leq \max_{i\in N} C_i(A), \tag{1.2}$$

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where $C_i(A) = R_i(A^T)$. Combining inequality (1.1) with inequality (1.2) gives

$$\rho(A) \leq \min\{\max_{i\in N} R_i(A), \max_{i\in N} C_i(A)\}. \tag{1.3}$$

Here, we call the bound in (1.3) the Frobenius upper bound for $\rho(A)$.

To estimate $\rho(A)$ more precisely, many researchers gave some upper bounds [8, 9, 10, 11, 12], which are smaller than the Frobenius upper bound. Particularly, in 1951 A.M. Ostrowski [13] gave the following well-known upper bound; also see [11].

THEOREM 1.2. [11, 13] *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be nonnegative. Then for any $\alpha \in [0,1]$,*

$$\rho(A) \leq \max_{i\in N}(R_i(A))^{\alpha}(C_i(A))^{1-\alpha},$$

that is,

$$\rho(A) \leq \min_{\alpha\in[0,1]} \max_{i\in N}(R_i(A))^{\alpha}(C_i(A))^{1-\alpha}. \tag{1.4}$$

Moreover, from the generalized arithmetic-geometric mean inequality [4]:

$$\alpha a + (1-\alpha)b \geq a^{\alpha}b^{1-\alpha},$$

where $a, b \geq 0$ and $0 \leq \alpha \leq 1$, another upper bound is obtained easily.

THEOREM 1.3. [13] *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be nonnegative. Then for any $\alpha \in [0,1]$,*

$$\rho(A) \leq \max_{i\in N}\{\alpha R_i(A) + (1-\alpha)C_i(A)\},$$

that is,

$$\rho(A) \leq \min_{\alpha\in[0,1]} \max_{i\in N}\{\alpha R_i(A) + (1-\alpha)C_i(A)\}. \tag{1.5}$$

Although Ostrowski gave many well-known results, such as the bounds in [14], we here call the bounds in (1.4) and (1.5) the Ostrowski upper bounds for $\rho(A)$. Note that when $\alpha = 0$,

$$\max_{i\in N}(R_i(A))^{\alpha}(C_i(A))^{1-\alpha} = \max_{i\in N}\{\alpha R_i(A) + (1-\alpha)C_i(A)\} = \max_{i\in N} C_i(A)\},$$

and $\alpha = 1$,

$$\max_{i\in N}(R_i(A))^{\alpha}(C_i(A))^{1-\alpha} = \max_{i\in N}\{\alpha R_i(A) + (1-\alpha)C_i(A)\} = \max_{i\in N} R_i(A).$$

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Therefore,

$$\begin{aligned}\min_{\alpha\in[0,1]}\max_{i\in N}(R_i(A))^{\alpha}(C_i(A))^{1-\alpha} &\le \min_{\alpha\in[0,1]}\max_{i\in N}\{\alpha R_i(A)+(1-\alpha)C_i(A)\}\\ &\le \min\{\max_{i\in N}R_i(A),\max_{i\in N}C_i(A)\},\end{aligned}$$

which implies that the Ostrowski upper bounds are smaller than the Frobenius upper bound. However, they are not of much practical use because they all involve a parameter α and it is not easy to decide the optimum value of α. Therefore, one often take some special α in practical, such as $\alpha=\frac{1}{2},\ \frac{1}{4},\ \frac{3}{4}$ and so on, but this leads to that the estimating is not good enough.

In this paper, we focus on the simplification problem of the Ostrowski upper bounds, and give their equivalent forms which do not include a minimization over all parameters α in the interval $[0,1]$. Numerical examples are also given to verify the corresponding results.

2. Main results. In this section, we give equivalent forms of the Ostrowski upper bounds which do not include a minimization over all parameters $\alpha\in[0,1]$. First, we consider the Ostrowski upper bound $\min\limits_{\alpha\in[0,1]}\max\limits_{i\in N}\{\alpha R_i(A)+(1-\alpha)C_i(A)\}$, and give a lemma as follows.

LEMMA 2.1. *Let $A=(a_{ij})\in\mathbb{R}^{n\times n}$ be nonnegative. Then there exists $\alpha\in[0,1]$ such that for all $i\in N$,*

$$\rho(A)>\alpha R_i(A)+(1-\alpha)C_i(A), \tag{2.1}$$

if and only if the following two conditions hold:

(i) for any $i\in N$,

$$\rho(A)>\min\{R_i(A),C_i(A)\}, \tag{2.2}$$

(ii) for any $i\in\Lambda$ and any $j\in\Delta$,

$$\frac{\rho(A)-C_i(A)}{R_i(A)-C_i(A)}>\frac{C_j(A)-\rho(A)}{C_j(A)-R_j(A)}, \tag{2.3}$$

where $\Lambda=\{i\in N: R_i(A)>C_i(A)\}$ and $\Delta=\{j\in N: C_j(A)>R_j(A)\}$.

Proof. Let $\Xi=\{i\in N: R_i(A)=C_i(A)\}$. Then $N=\Xi\bigcup\Lambda\bigcup\Delta$.

First, suppose that there exists $\alpha\in[0,1]$ such that inequality (2.1) holds for all $i\in N$, then for any $i\in\Lambda$,

$$\frac{\rho(A)-C_i(A)}{R_i(A)-C_i(A)}>\alpha,$$

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and for any $j \in \Delta$,

$$\alpha > \frac{C_j(A) - \rho(A)}{C_j(A) - R_j(A)}.$$

Therefore, for any $i \in \Lambda$ and any $j \in \Delta$, inequality (2.3) holds. Furthermore, from $\alpha \in [0,1]$ it is easy to get that inequality (2.2) holds for any $i \in N$.

Conversely, suppose that the conditions (i) and (ii) hold. Obviously, inequality (2.1) always holds for each $i \in \Xi$. Thus, it remains to prove that inequality (2.1) holds for all $i \in \Lambda$ and all $j \in \Delta$.

For each $i \in \Lambda$, we have $R_i(A) - C_i(A) > 0$ and $\rho(A) - C_i(A) > 0$. Therefore,

$$\frac{\rho(A) - C_i(A)}{R_i(A) - C_i(A)} > 0. \tag{2.4}$$

And for each $j \in \Delta$, we have $C_j(A) - R_j(A) > 0$, $\rho(A) - R_j(A) > 0$, and

$$C_j(A) - R_j(A) > C_j(A) - \rho(A),$$

which implies

$$\frac{C_j(A) - \rho(A)}{C_j(A) - R_j(A)} < 1. \tag{2.5}$$

Combining inequality (2.3), inequality (2.4) with inequality (2.5) gives that there exists $\alpha \in [0,1]$ such that for all $i \in \Lambda$ and all $j \in \Delta$,

$$\max\left\{0, \frac{C_j(A) - \rho(A)}{C_j(A) - R_j(A)}\right\} < \alpha < \min\left\{\frac{\rho(A) - C_i(A)}{R_i(A) - C_i(A)}, 1\right\}. \tag{2.6}$$

By inequality (2.6), we have that for any $i \in \Lambda$,

$$\alpha < \frac{\rho(A) - C_i(A)}{R_i(A) - C_i(A)},$$

that is, $\rho(A) > \alpha R_i(A) + (1-\alpha)C_i(A)$, and that for any $j \in \Delta$,

$$\frac{C_j(A) - \rho(A)}{C_j(A) - R_j(A)} < \alpha,$$

that is, $\rho(A) > \alpha R_j(A) + (1-\alpha)C_j(A)$. Therefore, there exists $\alpha \in [0,1]$ such that inequality (2.1) holds for all $i \in N$. The proof is completed. ∎

According to Lemma 2.1, we can obtain the equivalent form of the Ostrowski upper bound $\min\limits_{\alpha\in[0,1]} \max\limits_{i \in N}\{\alpha R_i(A) + (1-\alpha)C_i(A)\}$.

Theorem 2.2. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be nonnegative. Then*

$$\rho(A) \le \max_{i\in N}\{\min\{R_i(A), C_i(A)\}\},$$

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or

$$\rho(A) \le \max_{\substack{i\in\Lambda,\\ j\in\Delta}} \frac{R_i(A)C_j(A) - C_i(A)R_j(A)}{R_i(A) - C_i(A) + C_j(A) - R_j(A)}.$$

That is,

$$\rho(A) \le \max\left\{\max_{i\in N}\{\min\{R_i(A), C_i(A)\}\}, \max_{\substack{i\in\Lambda,\\ j\in\Delta}} \frac{R_i(A)C_j(A) - C_i(A)R_j(A)}{R_i(A) - C_i(A) + C_j(A) - R_j(A)}\right\}. \tag{2.7}$$

Proof. Suppose on the contrary that

$$\rho(A) > \max\left\{\max_{i\in N}\{\min\{R_i(A), C_i(A)\}\}, \max_{\substack{i\in\Lambda,\\ j\in\Delta}} \frac{R_i(A)C_j(A) - C_i(A)R_j(A)}{R_i(A) - C_i(A) + C_j(A) - R_j(A)}\right\},$$

or equivalently,

$$\rho(A) > \max_{i\in N}\{\min\{R_i(A), C_i(A)\}\}$$

and

$$\rho(A) > \max_{\substack{i\in\Lambda,\\ j\in\Delta}} \frac{R_i(A)C_j(A) - C_i(A)R_j(A)}{R_i(A) - C_i(A) + C_j(A) - R_j(A)}.$$

This implies respectively that for any $i \in N$,

$$\rho(A) > \min\{R_i(A), C_i(A)\},$$

and that for any $i \in \Lambda$ and any $j \in \Delta$,

$$\rho(A) > \frac{R_i(A)C_j(A) - C_i(A)R_j(A)}{R_i(A) - C_i(A) + C_j(A) - R_j(A)}.$$

Furthermore, by Lemma 2.1, there exists $\alpha \in [0, 1]$ such that for all $i \in N$,

$$\rho(A) > \alpha R_i(A) + (1-\alpha)C_i(A),$$

that is,

$$\rho(A) > \max_{i\in N}\{\alpha R_i(A) + (1-\alpha)C_i(A)\}$$

which contradicts to Theorem 1.3. The conclusions follows. ∎

Next, we prove that the bound in inequality (2.7) is equivalent to the Ostrowski upper bound $\min\limits_{\alpha\in[0,1]} \max\limits_{i\in N}\{\alpha R_i(A) + (1-\alpha)C_i(A)\}$.

Theorem 2.3. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be nonnegative. Then*

$$\min_{\alpha\in[0,1]} \max_{i\in N}\{\alpha R_i(A) + (1-\alpha)C_i(A)\}$$

$$= \max\left\{\max_{i\in N}\{\min\{R_i(A), C_i(A)\}\}, \max_{\substack{i\in\Lambda,\\ j\in\Delta}} \frac{R_i(A)C_j(A) - C_i(A)R_j(A)}{R_i(A) - C_i(A) + C_j(A) - R_j(A)}\right\}.$$

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Proof. First we prove that if inequality (1.5) holds, then inequality (2.7) holds. This implies that

$$\min_{\alpha\in[0,1]}\max_{i\in N}\{\alpha R_i(A)+(1-\alpha)C_i(A)\} \le \max\left\{\max_{i\in N}\{\min\{R_i(A),C_i(A)\}\},\max_{\substack{i\in\Lambda,\\ j\in\Delta}}\frac{R_i(A)C_j(A)-C_i(A)R_j(A)}{R_i(A)-C_i(A)+C_j(A)-R_j(A)}\right\}. \tag{2.8}$$

In fact, if

$$\rho(A)>\max\left\{\max_{i\in N}\{\min\{R_i(A),C_i(A)\}\},\max_{\substack{i\in\Lambda,\\ j\in\Delta}}\frac{R_i(A)C_j(A)-C_i(A)R_j(A)}{R_i(A)-C_i(A)+C_j(A)-R_j(A)}\right\},$$

then by the proof of Theorem 2.2, there exists $\alpha\in[0,1]$ such that for all $i\in N$,

$$\rho(A)>\max_{i\in N}\{\alpha R_i(A)+(1-\alpha)C_i(A)\}.$$

This gives

$$\rho(A)>\min_{\alpha\in[0,1]}\max_{i\in N}\{\alpha R_i(A)+(1-\alpha)C_i(A)\}.$$

This is a contradiction to inequality (1.5). Hence, inequality (2.8) holds.

We now prove that if inequality (2.7) holds, then inequality (1.5) holds, which implies that

$$\max\left\{\max_{i\in N}\{\min\{R_i(A),C_i(A)\}\},\max_{\substack{i\in\Lambda,\\ j\in\Delta}}\frac{R_i(A)C_j(A)-C_i(A)R_j(A)}{R_i(A)-C_i(A)+C_j(A)-R_j(A)}\right\} \le \min_{\alpha\in[0,1]}\max_{i\in N}\{\alpha R_i(A)+(1-\alpha)C_i(A)\}. \tag{2.9}$$

In fact, if

$$\rho(A)>\min_{\alpha\in[0,1]}\max_{i\in N}\{\alpha R_i(A)+(1-\alpha)C_i(A)\},$$

that is, there exists $\alpha\in[0,1]$ such that for all $i\in N$,

$$\rho(A)>\max_{i\in N}\{\alpha R_i(A)+(1-\alpha)C_i(A)\}\ge\alpha R_i(A)+(1-\alpha)C_i(A),$$

then by Lemma 2.1 and the proof of Theorem 2.2, we have

$$\rho(A)>\max\left\{\max_{i\in N}\{\min\{R_i(A),C_i(A)\}\},\max_{\substack{i\in\Lambda,\\ j\in\Delta}}\frac{R_i(A)C_j(A)-C_i(A)R_j(A)}{R_i(A)-C_i(A)+C_j(A)-R_j(A)}\right\}.$$

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This is a contradiction to inequality (2.7). Hence, inequality (2.9) holds. The conclusion follows from inequality (2.8) and inequality (2.9). □

REMARK 2.4. From Theorem 2.3, we know that Theorem 2.2 provides an equivalent form of the Ostrowski upper bound $\min\limits_{\alpha\in[0,1]}\max\limits_{i\in N}\{\alpha R_i(A)+(1-\alpha)C_i(A)\}$. Obviously, this form only relates to the entries of A and has nothing to do with α, and hence, it is much easier to estimate the spectral radius of nonnegative matrices.

Similarly, we can obtain easily the equivalent form of the Ostrowski upper bound $\min\limits_{\alpha\in[0,1]}\max\limits_{i\in N}(R_i(A))^\alpha(C_i(A))^{1-\alpha}$.

LEMMA 2.5. *Let $A=(a_{ij})\in\mathbb{R}^{n\times n}$ be nonnegative. Then there exists $\alpha\in[0,1]$ such that for all $i\in N$,*

$$\rho(A)>(R_i(A))^\alpha(C_i(A))^{1-\alpha},$$

if and only if the following two conditions hold:

(i) for any $i\in N$,

$$\rho(A)>\min\{R_i(A),C_i(A)\},$$

(ii) for any $i\in\Lambda$, $C_i(A)\neq 0$ and any $j\in\Delta$, $R_j(A)\neq 0$,

$$log_{\frac{R_i(A)}{C_i(A)}}\frac{\rho(A)}{C_i(A)}>log_{\frac{C_j(A)}{R_j(A)}}\frac{C_j(A)}{\rho(A)}. \tag{2.10}$$

Proof. Similar to the proof of Lemma 2.1, the conclusion follows easily. □

LEMMA 2.6. *Let $A=(a_{ij})\in\mathbb{R}^{n\times n}$ be nonnegative. Then for any $i\in\Lambda$, $C_i(A)\neq 0$ and any $j\in\Delta$, $R_j(A)\neq 0$, inequality (2.10) holds if and only if*

$$\rho(A)>\left(C_j(A)*\left(\frac{C_j(A)}{R_j(A)}\right)^{log_{\frac{R_i(A)}{C_i(A)}}C_i(A)}\right)^{\frac{1}{1+log_{\frac{R_i(A)}{C_i(A)}}\frac{C_j(A)}{R_j(A)}}}.$$

Proof. Inequality (2.10) is equivalent to

$$log_{\frac{R_i(A)}{C_i(A)}}\rho(A)-log_{\frac{R_i(A)}{C_i(A)}}C_i(A)>log_{\frac{C_j(A)}{R_j(A)}}C_j(A)-log_{\frac{C_j(A)}{R_j(A)}}\rho(A),$$

that is,

$$log_{\frac{R_i(A)}{C_i(A)}}\rho(A)+log_{\frac{C_j(A)}{R_j(A)}}\rho(A)>log_{\frac{R_i(A)}{C_i(A)}}C_i(A)+log_{\frac{C_j(A)}{R_j(A)}}C_j(A). \tag{2.11}$$

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Note that $i \in \Lambda$ and $j \in \Delta$, then $\frac{R_i(A)}{C_i(A)} > 1$, $\frac{C_j(A)}{R_j(A)} > 1$ and $log_{\frac{R_i(A)}{C_i(A)}} \frac{C_j(A)}{R_j(A)} > 0$. Therefore, inequality (2.11) holds if and only if

$$log_{\frac{R_i(A)}{C_i(A)}} \rho(A) + \frac{log_{\frac{R_i(A)}{C_i(A)}} \rho(A)}{log_{\frac{R_i(A)}{C_i(A)}} \frac{C_j(A)}{R_j(A)}} > log_{\frac{R_i(A)}{C_i(A)}} C_i(A) + \frac{log_{\frac{R_i(A)}{C_i(A)}} C_j(A)}{log_{\frac{R_i(A)}{C_i(A)}} \frac{C_j(A)}{R_j(A)}},$$

or equivalently,

$$\begin{aligned} log_{\frac{R_i(A)}{C_i(A)}} \rho(A) \left(1 + log_{\frac{R_i(A)}{C_i(A)}} \frac{C_j(A)}{R_j(A)}\right) &> log_{\frac{R_i(A)}{C_i(A)}} C_i(A) * log_{\frac{R_i(A)}{C_i(A)}} \frac{C_j(A)}{R_j(A)} + log_{\frac{R_i(A)}{C_i(A)}} C_j(A) \\ &= log_{\frac{R_i(A)}{C_i(A)}} \left(\frac{C_j(A)}{R_j(A)}\right)^{log_{\frac{R_i(A)}{C_i(A)}} C_i(A)} + log_{\frac{R_i(A)}{C_i(A)}} C_j(A) \\ &= log_{\frac{R_i(A)}{C_i(A)}} \left(C_j(A) * \left(\frac{C_j(A)}{R_j(A)}\right)^{log_{\frac{R_i(A)}{C_i(A)}} C_i(A)}\right), \end{aligned}$$

that is,

$$\begin{aligned} log_{\frac{R_i(A)}{C_i(A)}} \rho(A) &> \frac{log_{\frac{R_i(A)}{C_i(A)}} \left(C_j(A) * \left(\frac{C_j(A)}{R_j(A)}\right)^{log_{\frac{R_i(A)}{C_i(A)}} C_i(A)}\right)}{\left(1 + log_{\frac{R_i(A)}{C_i(A)}} \frac{C_j(A)}{R_j(A)}\right)} \\ &= log_{\frac{R_i(A)}{C_i(A)}} \left(C_j(A) * \left(\frac{C_j(A)}{R_j(A)}\right)^{log_{\frac{R_i(A)}{C_i(A)}} C_i(A)}\right)^{\left(1 + log_{\frac{R_i(A)}{C_i(A)}} \frac{C_j(A)}{R_j(A)}\right)^{-1}}. \end{aligned} \tag{2.12}$$

Since $\frac{R_i(A)}{C_i(A)} > 1$, Then inequality (2.12) is equivalent to

$$\rho(A) > \left(C_j(A) * \left(\frac{C_j(A)}{R_j(A)}\right)^{log_{\frac{R_i(A)}{C_i(A)}} C_i(A)}\right)^{\left(1 + log_{\frac{R_i(A)}{C_i(A)}} \frac{C_j(A)}{R_j(A)}\right)^{-1}}.$$

The proof is completed. □

Similar to the proof of Theorems 2.2 and 2.3, we can obtain easily the following theorems from Lemmas 2.5 and 2.6.

THEOREM 2.7. *Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be nonnegative. Then*

$$\rho(A) \leq \max_{i \in N} \{\min\{R_i(A), C_i(A)\}\},$$

or

$$\rho(A) \leq \rho_1 = \max_{\substack{i \in \Lambda, C_i(A) \neq 0, \\ j \in \Delta, R_j(A) \neq 0}} \left(C_j(A) * \left(\frac{C_j(A)}{R_j(A)}\right)^{log_{\frac{R_i(A)}{C_i(A)}} C_i(A)}\right)^{\left(1 + log_{\frac{R_i(A)}{C_i(A)}} \frac{C_j(A)}{R_j(A)}\right)^{-1}}.$$

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That is,

$$\rho(A) \le \max\left\{\max_{i\in N}\{\min\{R_i(A), C_i(A)\}\}, \rho_1\right\}. \tag{2.13}$$

Theorem 2.8. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be nonnegative. Then*

$$\min_{\alpha\in[0,1]}\max_{i\in N}(R_i(A))^{\alpha}(C_i(A))^{1-\alpha} = \max\left\{\max_{i\in N}\{\min\{R_i(A), C_i(A)\}\}, \rho_1\right\},$$

where ρ_1 is defined as in Theorem 2.7.

Remark 2.9. Theorem 2.7 provides an equivalent form of the Ostrowski upper bound $\min_{\alpha\in[0,1]}\max_{i\in N}(R_i(A))^{\alpha}(C_i(A))^{1-\alpha}$. However, it is determined with more difficultly than that in Theorem 2.2 because of computing $log_{\frac{R_i(A)}{C_i(A)}} C_i(A)$ or $log_{\frac{R_i(A)}{C_i(A)}} \frac{C_j(A)}{R_j(A)}$ difficultly. So in general we estimate the spectral radius of nonnegative matrices by Theorem 2.2.

3. Numerical comparisons. Besides the Frobenius bound and the Ostrowski bounds, there are another results on upper bounds for the spectral radius of nonnegative matrices [2, 3, 5, 8, 11, 12, 16, 17]. We now list some of the well-known bounds, and compare with the bound in Theorem 2.2. In 1964, Derzko and Pfeffer [5] provided an upper bound for the spectral radius of complex matrices, which is also used to estimate the spectral radius of a nonnegative matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$.

$$\rho(A) \le \left(\epsilon(A)^2 - \left(\max_{i\in N}|\bar{R}_i(A) - \bar{C}_i(A)|\right)^2\right)^{\frac{1}{2}}, \tag{3.1}$$

where $\epsilon(A) = \left(\sum_{i,j\in N} a_{ij}^2\right)^{\frac{1}{2}}$, $\bar{R}_i(A) = \left(\left(\sum_{j\in N} a_{ij}^2\right) - a_{ii}^2\right)^{\frac{1}{2}}$ and $\bar{C}_i(A) = \bar{R}_i(A^T)$.

In 1974, Brauer and Gentry [2] derived the following bound:

$$\rho(A) \le \frac{1}{2}\max_{j\ne i}\left(a_{ii} + a_{jj} + \left((a_{ii} - a_{jj})^2 + 4R_i'(A)R_j'(A)\right)^{\frac{1}{2}}\right), \tag{3.2}$$

where $R_i'(A) = R_i(A) - a_{ii}$.

In 1994, Rojo and Jiménez [16] obtained the following decreasing sequence of upper bounds:

$$\rho(A) \le v_k(A) \le \cdots \le v_2(A) \le v_1(A), \tag{3.3}$$

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where $v_p(A) = \frac{trace(A)}{n} + \gamma_p(A)$,

$$\gamma_p(A) = \left(\frac{(n-1)^{2p-1}}{(n-1)^{2p-1}+1}\ trace \left(M(A) - \frac{trace(A)}{n}\, I \right)^{2p} \right)^{\frac{1}{2p}}$$

and $M(A) = \frac{A+A^T}{2}$.

In 1998, Taşçi and Kirkland [17] obtained another sequence of upper bounds based on an arithmetic symmetrization of powers of A:

$$\rho(A) \le \sigma_k \le \cdots \le \sigma_2 \le \sigma_1, \tag{3.4}$$

where $\sigma_k = \left(\rho\left(M(A^{2^k})\right)\right)^{2^{-k}}$.

In 2006, Kolotilina [11] provided the following bound:

$$\rho(A) \le \max_{i,j:a_{ij}\neq 0} \left\{ \left(R_i(A)^\alpha R_j(A)^{1-\alpha}\right)^\beta \left(C_i(A)^\alpha C_j(A)^{1-\alpha}\right)^{1-\beta} \right\}, \tag{3.5}$$

where $0 \le \alpha, \beta \le 1$.

In 2012, Melman [12] derived an upper bound for the spectral radius, that is,

$$\rho(A) \le \frac{1}{2} \max_{i\in N} \min_{j\neq i} \left\{ a_{ii} + a_{jj} + R''_{ij}(A) + ((a_{ii} - a_{jj} + R''_{ij}(A))^2 + 4a_{ij}R'_j(A))^{\frac{1}{2}} \right\}, \tag{3.6}$$

where $R''_{ij}(A) = R'_i(A) - a_{ij} = R_i(A) - a_{ii} - a_{ij}$.

Very recently, Butler and Siegel [3] obtained the following upper bound for the spectral radius of nonnegative matrices with nonzero row sums.

$$\rho(A) \le \max_{i,j\in N} \left\{ \left(\frac{R_i(A^{K+P})R_j(A^{K+Q})}{R_i(A^K)R_j(A^K)} \right)^{\frac{1}{P+Q}} : a_{ij}^{(P)} > 0 \right\}, \tag{3.7}$$

where $a_{ij}^{(P)}$ is the (i,j) entry of A^P, $P > 0$, $Q \ge 0$ and $K \ge 0$.

We now give some numerical examples to compare the Ostrowski upper bound $\min\limits_{\alpha\in[0,1]} \max\limits_{i\in N}\{\alpha R_i(A) + (1-\alpha)C_i(A)\}$, or equivalently, the bound in Theorem 2.2 with the listed bounds.

EXAMPLE 3.1. Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$with $n \ge 2$, where

$$a_{ij} = \frac{i+j}{n-i+j}.$$

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Obviously, A is nonnegative. We compute by Matlab 7.0 the Frobenius upper bound, the bounds in (3.1), (3.2), (3.3), (3.4), (3.5), (3.6), (3.7), and the Ostrowski upper bound $\min_{\alpha\in[0,1]} \max_{i\in N}\{\alpha R_i(A)+(1-\alpha)C_i(A)\}$, i.e., the bound in (2.7), which are showed in Table 1.

n	20	50	100	200
the Frobenius upper bound	59.1503	183.9587	429.1125	987.3623
the bound in (3.1)	35.3673	96.8358	207.5647	443.3573
the bound in (3.2)	50.7089	160.4870	380.0033	886.6063
the bound in (3.3)	35.0224	92.2129	192.3634	401.0254
the bound in (3.4)	27.3319	67.0738	133.3430	265.9199
the bound in (3.5)	30.6872	82.1932	174.0878	368.2669
the bound in (3.6)	47.5346	151.0051	358.3530	837.9700
the bound in (3.7)	34.9409	88.6291	178.2934	357.7011
the Ostrowski upper bound	43.4386	128.1360	288.8136	645.2955
$\rho(A)$	25.4223	62.1346	123.3112	245.6604

Table 1. Comparison of the bounds for the nonnegative matrix in Example 3.1.

Example 3.2. Let

$$A=\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 4 & 1 & 3 \end{bmatrix}.$$

The bounds are showed in Table 2.

the Frobenius upper bound	5
the bound in (3.1)	4.8214
the bound in (3.2)	4.3723
the bound in (3.3)	4.8868
the bound in (3.4)	4.4001
the bound in (3.5)	4.4267
the bound in (3.6)	4.2361
the bound in (3.7)	4.6104
the Ostrowski upper bound	4.5714
$\rho(A)$	4.0946

Table 2. Comparison of the bounds for the nonnegative matrix in Example 3.2.

Example 3.3. Let

$$A=\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$

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By computation, the Ostrowski upper bound is 5, and the bounds in (3.4) and (3.5) are 5.1938 and 5.4772, respectively. In fact, $\rho(A) = 5$.

REMARK 3.4. (I) In Examples 3.1, 3.2 and 3.3, the bounds in (3.3), (3.4) are given by $v_1(A)$ and $\sigma_1(A)$, respectively. And the bound in (3.5) is given for $\alpha = \beta = 1$. For the bound in (3.7), we compute its value for $K = P = Q = 1$ in Example 3.1, and for $P = 2$, $K = Q = 1$ in Example 3.2.

(II) From Examples 3.1 and 3.2, we have that the Ostrowski upper bound is smaller than the Frobenius upper bound, smaller than the bounds in (3.1), (3.2), (3.3), (3.6), and (3.7) in some cases.

(III) Example 3.3 shows that the Ostrowski upper bound is smaller than the bounds in (3.4) and (3.5), and that the Ostrowski upper bound is sharp.

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