

SUBMATRICES OF HADAMARD MATRICES: COMPLEMENTATION RESULTS*

TEO BANICA[†], ION NECHITA[‡], AND JEAN-MARC SCHLENKER[§]

Abstract. Two submatrices A, D of a Hadamard matrix H are called *complementary* if, up to a permutation of rows and columns, $H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. In this paper, an explicit formula for the polar decomposition of D is found. As an application, it is shown that under suitable smallness assumptions on the size of A , the complementary matrix D is an *almost Hadamard sign pattern*, i.e., its rescaled polar part is an almost Hadamard matrix.

Key words. Hadamard matrix, Almost Hadamard matrix.

AMS subject classifications. 15B34.

0. Introduction. A Hadamard matrix is a square matrix $H \in M_N(\pm 1)$, whose rows are pairwise orthogonal. The basic example is the Walsh matrix, having size $N = 2^n$:

$$W_N = \begin{bmatrix} + & + \\ + & - \end{bmatrix}^{\otimes n}.$$

In general, the Hadamard matrices can be regarded as “generalizations” of the Walsh matrices. Their applications, mostly to coding theory and to various engineering questions, parallel the applications of the Walsh functions and matrices.

Mathematically speaking, $W_N = W_2^{\otimes n}$ is the matrix of the Fourier transform over the group $G = \mathbb{Z}_2^n$, and so the whole field can be regarded as a “non-standard” branch of discrete Fourier analysis. Of particular interest here is the Hadamard conjecture: for any $N \in 4\mathbb{N}$, there exists a Hadamard matrix $H \in M_N(\pm 1)$. See [10], [11].

We are interested here in square submatrices of such matrices. Up to a permutation of rows and columns we can assume that our submatrix appears at top left or bottom right:

*Received by the editors on November 13, 2013. Accepted for publication on January 25, 2014.
Handling Editor: Shmuel Friedland.

[†]Department of Mathematics, Cergy-Pontoise University, 95000 Cergy-Pontoise, France (teo.banica@gmail.com).

[‡]CNRS, Laboratoire de Physique Théorique, IRSAMC, Université de Toulouse, UPS, 31062 Toulouse, France (nechita@irsamc.ups-tlse.fr).

[§]University of Luxembourg, Campus Kirchberg Mathematics Research Unit, BLG 6, rue Richard Coudenhove-Kalergi L-1359 Luxembourg (jean-marc.schlenker@uni.lu).

SETUP. We consider Hadamard matrices $H \in M_N(\pm 1)$ written as

$$H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

with $A \in M_r(\pm 1)$ and $D \in M_d(\pm 1)$, where $N = r + d$.

As a first observation, one can show that any ± 1 matrix appears as a submatrix of a certain large Walsh matrix, so nothing special can be said about A, D . That is, when regarded individually, A, D are just some “random” ± 1 matrices.

The meaning of the word “random” here is in fact quite tricky. In general, the random Bernoulli matrices $D \in M_d(\pm 1)$ are known to obey to the Tao-Vu rules [13], [14], and their refinements, and basically to nothing more, in the $d \rightarrow \infty$ limit.

For submatrices of Hadamard matrices, however, the situation is much more complicated, and what “random” should really mean is not clear at all. All this is of course related to the Hadamard Conjecture. See de Launey and Levin [8], [9].

Now back to our matrices A, D , the point is to consider them “together”. As a first remark here, the unitarity of $U = \frac{H}{\sqrt{N}}$ gives, as noted by Szöllősi in [12]:

FACT. If $A \in M_r(\pm 1), D \in M_d(\pm 1)$ are as above then the singular values of $\frac{A}{\sqrt{N}}, \frac{D}{\sqrt{N}}$ are identical, up to $|r - d|$ values of 1. In particular, $|\det \frac{A}{\sqrt{N}}| = |\det \frac{D}{\sqrt{N}}|$.

This simple fact brings a whole new perspective on the problem: we should call A, D “complementary”, and see if there are further formulae relating them.

Let us recall now a few findings from [1], [2], [3], [4]. As noted in [1], by Cauchy-Schwarz an orthogonal matrix $U \in O(N)$ satisfies $\|U\|_1 \leq N\sqrt{N}$, with equality if and only if $H = \sqrt{N}U$ is Hadamard. This is quite nice, and leads to:

DEFINITION. A square matrix $H \in M_N(\mathbb{R})$ is called *almost Hadamard* (AHM) if the following equivalent conditions are satisfied:

1. $U = H/\sqrt{N}$ is orthogonal, and locally maximizes the 1-norm on $O(N)$.
2. $U_{ij} \neq 0$ for any i, j , and $US^t > 0$, where $S_{ij} = \text{sgn}(U_{ij})$.

In this case, we say that $S \in M_N(\pm 1)$ is an almost Hadamard sign pattern (AHP).

In this definition, the equivalence (1) $\iff$ (2) follows from a differential geometry computation, performed in [1]. For results on these matrices, see [1], [2], [3], [4].

For the purposes of this paper, observe that we have a bijection as follows, implemented by $S_{ij} = \text{sgn}(H_{ij})$ in one direction, and by $H = \sqrt{N}\text{Pol}(S)$ in the other:

$$AHM \longleftrightarrow AHP.$$

With these notions in hand, let us go back to the matrices $A \in M_r(\pm 1)$, $D \in M_d(\pm 1)$ above. It was already pointed out in [4], or rather visible from the design computations performed there, that when $r = 1$, the matrix D must be AHP. We will show here that this kind of phenomenon holds under much more general assumptions. First, we have:

LEMMA. Let $H = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_N(\pm 1)$ be a Hadamard matrix, such that $A \in M_r(\pm 1)$ is invertible. Then, the polar decomposition $D = UT$ is given by

$$U = \frac{1}{\sqrt{N}}(D - E), \quad T = \sqrt{N}I_d - S,$$

where $E = C(\sqrt{N}I_r + \sqrt{A^t A})^{-1} \text{Pol}(A)^t B$ and $S = B^t(\sqrt{N}I_r + \sqrt{A A^t})^{-1} B$.

These formulae, which partly extend the work in [6], [7], [12], will allow us to estimate the quantity $\|E\|_\infty$, and then to prove the following result:

THEOREM. Given a Hadamard matrix $H = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_N(\pm 1)$ with $A \in M_r(\pm 1)$, D is an almost Hadamard sign pattern (AHP) if:

1. A is invertible, and $r = 1, 2, 3$.
2. A is Hadamard, and $N > r(r-1)^2$.
3. A is invertible, and $N > \frac{r^2}{4}(r + \sqrt{r^2 + 8})^2$.

The paper is organized as follows: Sections 1–2 are preliminary sections. In Section 3, we state and prove our main results, and in Sections 4–5, we discuss examples and present some further results.

1. Almost Hadamard matrices. A Hadamard matrix is a square matrix $H \in M_N(\pm 1)$, whose rows are pairwise orthogonal. By looking at the first 3 rows, we see that the size of such a matrix satisfies $N \in \{1, 2\} \cup 4\mathbb{N}$. In what follows we assume $N \geq 4$, so that $N \in 4\mathbb{N}$.

We are interested here in the submatrices of such matrices. In this section, we recall some needed preliminary material, namely: (1) the polar decomposition, and (2) the almost Hadamard matrices. Let us begin with the polar decomposition [5]:

PROPOSITION 1.1. Any matrix $D \in M_N(\mathbb{R})$ can be written as $D = UT$, with positive semi-definite $T = \sqrt{D^t D}$, and with orthogonal $U \in O(N)$. If D is invertible, then U is uniquely determined and we write $U = \text{Pol}(D)$.

The polar decomposition can be deduced from the singular value decomposition (again, see [5] for details):

PROPOSITION 1.2. If $D = V\Delta W^t$ with V, W orthogonal and Δ diagonal is the

singular values decomposition of D , then $\text{Pol}(D) = VW^t$.

Let us discuss now the notion of almost Hadamard matrix, from [1], [2], [3], [4]. Recall first that the coordinate 1-norm of a matrix $M \in M_N(\mathbb{C})$ is given by:

$$\|M\|_1 = \sum_{ij} |M_{ij}|.$$

The importance of the coordinate 1-norm in relation with Hadamard matrices was realized in [1], where the following observation was made:

PROPOSITION 1.3. For $U \in O(N)$ we have $\|U\|_1 \leq N\sqrt{N}$, with equality if and only if the rescaled matrix $H = \sqrt{N}U$ is Hadamard.

Proof. This follows from $\|U\|_2 = \sqrt{N}$ by Cauchy-Schwarz, see [1]. $\square$

Since the global maximum of the 1-norm over the orthogonal group is quite difficult to find, in [3], [4] the *local maxima* of the 1-norm were introduced:

DEFINITION 1.4. An *almost Hadamard matrix* (AHM) is a square matrix $H \in M_N(\mathbb{R})$ having the property that $U = H/\sqrt{N}$ is a local maximum of the 1-norm on $O(N)$.

According to Proposition 1.3, these matrices can be thought of as being generalizations of the Hadamard matrices. Here is a basic example, which works for any $N \geq 3$:

$$K_N = \frac{1}{\sqrt{N}} \begin{bmatrix} 2-N & 2 & \dots & 2 \\ 2 & 2-N & \dots & 2 \\ \dots & \dots & \dots & \dots \\ 2 & 2 & \dots & 2-N \end{bmatrix}.$$

We have the following characterization of rescaled AHM, from [3]:

THEOREM 1.5. A matrix $U \in O(N)$ locally maximizes the 1-norm on $O(N)$ if and only if $U_{ij} \neq 0$, and $U^t S \geq 0$, where $S_{ij} = \text{sgn}(U_{ij})$.

Proof. This follows from basic differential geometry, with $U_{ij} \neq 0$ coming from a rotation trick, and $U^t S \geq 0$ being the Hessian of the 1-norm around U . See [3]. $\square$

The above proof shows that U is a strict maximizer of the 1-norm, in the sense that we have $\|U\|_1 > \|U_\varepsilon\|_1$ for $U_\varepsilon \neq U$ close to U , when $U^t S > 0$. This is important for us, because in what follows we will sometimes need S to be invertible.

In what follows, we will be precisely interested in the sign matrices S :

DEFINITION 1.6. A matrix $S \in M_N(\pm 1)$ is called an *almost Hadamard sign*

pattern (AHP) if there exists an almost Hadamard matrix $H \in M_N(\mathbb{R})$ such that $S_{ij} = \text{sgn}(H_{ij})$.

Here, “P” comes at the same time from “pattern” and “phase”.

Note that if a sign matrix S is an AHP, then there exists a *unique* almost Hadamard matrix H such that $S_{ij} = \text{sgn}(H_{ij})$, namely, $H = \sqrt{N} \text{Pol}(S)$. Since the polar part is not uniquely defined for singular sign matrices, in what follows, we shall mostly be concerned with invertible AHPs and AHMs.

2. Submatrices of Hadamard matrices. In this section, we start analyzing square *submatrices* of Hadamard matrices. By permuting rows and columns, we can always reduce to the following situation:

DEFINITION 2.1. $D \in M_d(\pm 1)$ is called a *submatrix* of $H \in M_N(\pm 1)$ if we have

$$H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

up to a permutation of the rows and columns of H . We set $r = \text{size}(A) = N - d$.

We recall that the n -th Walsh matrix is $W_N = \begin{bmatrix} + & + \\ + & - \end{bmatrix}^{\otimes n}$, with $N = 2^n$. Here, and in what follows, we use the tensor product convention $(H \otimes K)_{ia,jb} = H_{ij}K_{ab}$, with the lexicographic order on the double indices. Here are the first 3 such matrices:

$$W_2 = \begin{bmatrix} + & + \\ + & - \end{bmatrix}, W_4 = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}, W_8 = \begin{bmatrix} + & + & + & + & + & + & + & + \\ + & - & + & - & + & - & + & - \\ + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + \\ + & + & + & + & - & - & - & - \\ + & - & + & - & - & + & - & + \\ + & + & - & - & - & - & + & + \\ + & - & - & + & - & + & + & - \end{bmatrix}.$$

Observe that any $D \in M_2(\pm 1)$ having distinct columns appears as a submatrix of W_4 , and that any $D \in M_2(\pm 1)$ appears as a submatrix of W_8 . In fact, we have:

PROPOSITION 2.2. *Let $D \in M_d(\pm 1)$ be an arbitrary sign matrix.*

1. *If D has distinct columns, then D appears as a submatrix of W_N , with $N = 2^d$.*
2. *In general, D appears as a submatrix of W_M , with $M = 2^{d + \lceil \log_2 d \rceil}$.*

Proof. (1) Set $N = 2^d$. If we use length d bit strings $x, y \in \{0, 1\}^d$ as indices, then:

$$(W_N)_{xy} = (-1)^{\sum x_i y_i}.$$

Let $\widetilde{W}_N \in M_{d \times N}(\pm 1)$ be the submatrix of W_N having as row indices the strings of type $x_i = (\underbrace{0 \dots 0}_i 1 \underbrace{0 \dots 0}_{N-i-1})$. Then for $i \in \{1, \dots, d\}$ and $y \in \{0, 1\}^d$, we have:

$$(\widetilde{W}_N)_{iy} = (-1)^{y_i}.$$

Thus, the columns of $\widetilde{W}_N$ are the N elements of $\{\pm 1\}^d$, which gives the result.

(2) Set $R = 2^{\lceil \log_2 d \rceil} \geq d$. Since the first row of W_R contains only 1s, $W_R \otimes W_N$ contains as a submatrix R copies of $\widetilde{W}_N$, in which D can be embedded, finishing the proof. $\square$

Let us go back now to Definition 2.1, and try to relate the matrices A, D appearing there. The following result, due to Szöllősi [12], is a first one in this direction:

THEOREM 2.3. *If $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is unitary, with $A \in M_r(\mathbb{C})$, $D \in M_d(\mathbb{C})$, then:*

1. *The singular values of A, D are identical, up to $|r - d|$ values of 1.*
2. *$\det A = \det U \cdot \overline{\det D}$, so in particular, $|\det A| = |\det D|$.*

Proof. Here is a simplified proof. From the unitarity of U , we get:

$$\begin{aligned} A^*A + C^*C &= I_r, \\ CC^* + DD^* &= I_d, \\ AC^* + BD^* &= 0_{r \times d}. \end{aligned}$$

(1) This follows from the first two equations, and from the well-known fact that the matrices CC^*, C^*C have the same eigenvalues, up to $|r - d|$ values of 0.

(2) By using the above unitarity equations, we have:

$$\begin{bmatrix} A & 0 \\ C & I \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & C^* \\ 0 & D^* \end{bmatrix}.$$

The result follows by taking determinants. $\square$

3. Polar parts, norm estimates. In this section, we state and prove our main results. Our first goal is to find a formula for the polar decomposition of D . Let us introduce:

DEFINITION 3.1. Associated to any $A \in M_r(\pm 1)$ are the matrices

$$\begin{aligned} X_A &= (\sqrt{N}I_r + \sqrt{A^t A})^{-1} \text{Pol}(A)^t \\ Y_A &= (\sqrt{N}I_r + \sqrt{A A^t})^{-1} \end{aligned}$$

depending on a parameter N .

Observe that, in terms of the polar decomposition $A = VP$, we have:

$$\begin{aligned} X_A &= (\sqrt{N} + P)^{-1} V^t \\ Y_A &= V(\sqrt{N} + P)^{-1} V^t. \end{aligned}$$

The idea now is that, under the general assumptions of Theorem 2.3, the polar parts of A, D are related by a simple formula, with the passage $Pol(A) \rightarrow Pol(D)$ involving the above matrices X_A, Y_A . In what follows we will focus on the case that we are interested in, namely, with $U \in U(N)$ replaced by $U = \sqrt{N}H$ with $H \in M_N(\pm 1)$ Hadamard.

In the non-singular case, we have the following lemma:

LEMMA 3.2. *If $H = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_N(\pm 1)$ is Hadamard, with $A \in M_r(\pm 1)$ invertible, $D \in M_d(\pm 1)$, and $\|A\| < \sqrt{N}$, the polar decomposition $D = UT$ is given by*

$$U = \frac{1}{\sqrt{N}}(D - E), \quad T = \sqrt{N}I_d - S,$$

with $E = CX_AB$ and $S = B^tY_AB$.

Proof. Since H is Hadamard, we can use the formulae coming from:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} A^t & C^t \\ B^t & D^t \end{bmatrix} = \begin{bmatrix} A^t & C^t \\ B^t & D^t \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix}.$$

We start from the singular value decomposition of A :

$$A = V \text{diag}(s_i) X^t.$$

Here, $V, X \in O(r)$, $s_i \in (0, \|A\|]$. From $AA^t + BB^t = NI_r$ we get $BB^t = V \text{diag}(N - s_i^2) V^t$, so the singular value decomposition of B is as follows, with $Y \in O(d)$:

$$B = V \begin{bmatrix} \text{diag}(\sqrt{N - s_i^2}) & 0_{r \times (d-r)} \end{bmatrix} Y^t.$$

Similarly, from $A^tA + C^tC = I_r$, we infer the singular value decomposition for C , the result being that there exists an orthogonal matrix $\tilde{Z} \in O(d)$ such that:

$$C = -\tilde{Z} \begin{bmatrix} \text{diag}(\sqrt{N - s_i^2}) \\ 0_{(d-r) \times r} \end{bmatrix} X^t.$$

From $B^t B + D^t D = NI_d$, we obtain:

$$D^t D = Y(\text{diag}(s_i^2) \oplus NI_{(d-r)})Y^t.$$

Thus, the polar decomposition of D reads:

$$D = UY(\text{diag}(s_i) \oplus \sqrt{N}I_{(d-r)})Y^t.$$

Let $Z = UY$ and use the orthogonality relation $CA^t + DB^t = 0_{d \times r}$ to obtain:

$$\tilde{Z} \begin{bmatrix} \text{diag}(s_i \sqrt{N - s_i^2}) \\ 0_{(d-r) \times r} \end{bmatrix} = Z \begin{bmatrix} \text{diag}(s_i \sqrt{N - s_i^2}) \\ 0_{(d-r) \times r} \end{bmatrix}.$$

From the hypothesis, we have $s_i \sqrt{N - s_i^2} > 0$ and thus $Z^t \tilde{Z} = I_r \oplus Q$, for some orthogonal matrix $Q \in O(d)$. Plugging $\tilde{Z} = Z(I_r \oplus Q)$ in the singular value decomposition formula for C , we obtain:

$$C = -Z(I_r \oplus Q) \begin{bmatrix} \text{diag}(\sqrt{N - s_i^2}) \\ 0_{(d-r) \times r} \end{bmatrix} X^t = -Z \begin{bmatrix} \text{diag}(\sqrt{N - s_i^2}) \\ 0_{(d-r) \times r} \end{bmatrix} X^t.$$

To summarize, we have found $V, X \in O(r)$ and $Y, Z \in O(d)$ such that:

$$\begin{aligned} A &= V \text{diag}(s_i) X^t, \\ B &= V \begin{bmatrix} \text{diag}(\sqrt{N - s_i^2}) & 0_{r \times (d-r)} \end{bmatrix} Y^t, \\ C &= -Z \begin{bmatrix} \text{diag}(\sqrt{N - s_i^2}) \\ 0_{(d-r) \times r} \end{bmatrix} X^t, \\ D &= Z(\text{diag}(s_i) \oplus \sqrt{N}I_{(d-r)})Y^t. \end{aligned}$$

Now with U, T, E, S defined as in the statement, we obtain:

$$\begin{aligned} U &= ZY^t, \\ E &= Z(\text{diag}(\sqrt{N} - s_i) \oplus 0_{d-r})Y^t, \\ \sqrt{A^t A} &= X \text{diag}(s_i) X^t, \\ (\sqrt{N}I_r + \sqrt{A^t A})^{-1} &= X \text{diag}(1/(\sqrt{N} + s_i)) X^t, \\ X_A &= X \text{diag}(1/(\sqrt{N} + s_i)) V^t, \\ CX_A B &= Z(\text{diag}(\sqrt{N} - s_i) \oplus 0_{d-r})Y^t. \end{aligned}$$

Thus we have $E = CX_AB$, as claimed. Also, we have:

$$\begin{aligned} T &= Y(\text{diag}(s_i) \oplus \sqrt{N}I_{d-r})Y^t, \\ S &= Y(\text{diag}(\sqrt{N} - s_i) \oplus 0_{d-r})Y^t, \\ \sqrt{AA^t} &= V\text{diag}(s_i)V^t, \\ Y_A &= V\text{diag}(1/(\sqrt{N} + s_i))V^t, \\ B^tY_AB &= Y(\text{diag}(\sqrt{N} - s_i) \oplus 0_{d-r})Y^t. \end{aligned}$$

Hence, $S = B^tY_AB$, as claimed, and we are done. $\square$

Note that, in the above statement, when $r < \sqrt{N}$, the condition $\|A\| < \sqrt{N}$ is automatically satisfied.

As a first application, let us try to find out when D is AHP. For this purpose, we must estimate the quantity $\|E\|_\infty = \max_{ij} |E_{ij}|$:

LEMMA 3.3. Let $H = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_N(\pm 1)$ be a Hadamard matrix, with $A \in M_r(\pm 1)$, $D \in M_d(\pm 1)$ and $r \leq d$. Then, $\text{Pol}(D) = \frac{1}{\sqrt{N}}(D - E)$, with E satisfying:

1. $\|E\|_\infty \leq \frac{r\sqrt{r}}{\sqrt{r} + \sqrt{N}}$ when A is Hadamard.
2. $\|E\|_\infty \leq \frac{r^2 c \sqrt{N}}{N - r^2}$ if $r^2 < N$, with $c = \|\text{Pol}(A) - \frac{A}{\sqrt{N}}\|_\infty$.
3. $\|E\|_\infty \leq \frac{r^2(1 + \sqrt{N})}{N - r^2}$ if $r^2 < N$.

Proof. We use the basic fact that for two matrices $X \in M_{p \times r}(\mathbb{C})$, $Y \in M_{r \times q}(\mathbb{C})$ we have $\|XY\|_\infty \leq r\|X\|_\infty\|Y\|_\infty$. Thus, according to Lemma 3.2, we have:

$$\|E\|_\infty = \|CX_AB\|_\infty \leq r^2\|C\|_\infty\|X_A\|_\infty\|B\|_\infty = r^2\|X_A\|_\infty.$$

(1) If A is Hadamard, $AA^t = rI_r$, $\text{Pol}(A) = A/\sqrt{r}$ and thus:

$$X_A = (\sqrt{N}I_r + \sqrt{r}I_r)^{-1} \frac{A^t}{\sqrt{r}} = \frac{A^t}{r + \sqrt{rN}}.$$

Thus $\|X_A\|_\infty = \frac{1}{r + \sqrt{rN}}$, which gives the result.

(2) According to the definition of X_A , we have:

$$\begin{aligned} X_A &= (\sqrt{N}I_r + \sqrt{A^tA})^{-1} \text{Pol}(A)^t \\ &= (NI_r - A^tA)^{-1} (\sqrt{N}I_r - \sqrt{A^tA}) \text{Pol}(A)^t \\ &= (NI_r - A^tA)^{-1} (\sqrt{N} \text{Pol}(A) - A)^t. \end{aligned}$$

We therefore obtain:

$$\|X_A\|_\infty \leq r \|(NI_r - A^t A)^{-1}\|_\infty \|\sqrt{N} \text{Pol}(A) - A\|_\infty = \frac{rc}{\sqrt{N}} \left\| \left(I_r - \frac{A^t A}{N} \right)^{-1} \right\|_\infty.$$

Now by using $\|A^t A\|_\infty \leq r$, we obtain:

$$\left\| \left(I_r - \frac{A^t A}{N} \right)^{-1} \right\|_\infty \leq \sum_{k=0}^{\infty} \frac{\|(A^t A)^k\|_\infty}{N^k} \leq \sum_{k=0}^{\infty} \frac{r^{2k-1}}{N^k} = \frac{1}{r} \cdot \frac{1}{1 - r^2/N} = \frac{N}{rN - r^3}.$$

Thus, $\|X_A\|_\infty \leq \frac{rc}{\sqrt{N}} \cdot \frac{N}{rN - r^3} = \frac{c\sqrt{N}}{N - r^2}$, which gives the result.

(3) This follows from (2), because $c \leq \|\text{Pol}(A)\|_\infty + \|A/\sqrt{N}\|_\infty \leq 1 + \frac{1}{\sqrt{N}}$. $\square$

We can now state and prove our main result in this paper:

THEOREM 3.4. *Let $H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be Hadamard, with $A \in M_r(\pm 1)$, $H \in M_N(\pm 1)$.*

1. *If A is Hadamard, and $N > r(r-1)^2$, then D is AHP.*
2. *If $N > \frac{r^2}{4}(x + \sqrt{x^2 + 4})^2$, where $x = r\|\text{Pol}(A) - \frac{A}{\sqrt{N}}\|_\infty$, then D is AHP.*
3. *If $N > \frac{r^2}{4}(r + \sqrt{r^2 + 8})^2$, then D is AHP.*

Proof. (1) This follows from Lemma 3.3 (1), because:

$$\frac{r\sqrt{r}}{\sqrt{r} + \sqrt{N}} < 1 \iff r < 1 + \sqrt{N/r} \iff r(r-1)^2 < N.$$

(2) This follows from Lemma 3.3 (2), because:

$$\frac{r^2 c \sqrt{N}}{N - r^2} < 1 \iff N - r^2 c \sqrt{N} > r^2 \iff (2\sqrt{N} - r^2 c)^2 > r^4 c^2 + 4r^2.$$

Indeed, this is equivalent to $2\sqrt{N} > r^2 c + r\sqrt{r^2 c^2 + 4} = r(x + \sqrt{x^2 + 4})$, for $x = rc = r\|\text{Pol}(A) - \frac{A}{\sqrt{N}}\|_\infty$.

(3) This follows from Lemma 3.3 (3), because:

$$\frac{r^2(1 + \sqrt{N})}{N - r^2} < 1 \iff N - r^2 \sqrt{N} > 2r^2 \iff (2\sqrt{N} - r^2)^2 > r^4 + 8r^2.$$

Indeed, this is equivalent to $2\sqrt{N} > r^2 + r\sqrt{r^2 + 8}$, which gives the result. $\square$

As a technical comment, for $A \in M_r(\pm 1)$ Hadamard, Lemma 3.3 (2) gives:

$$\|E\|_\infty \leq \frac{r^2 \sqrt{N}}{N - r^2} \left(\frac{1}{\sqrt{r}} - \frac{1}{\sqrt{N}} \right) = \frac{r\sqrt{r}N - r^2}{N - r^2}.$$

Thus, $\|E\|_\infty < 1$ for $N > r^3$, which is slightly weaker than Theorem 3.4 (1).

4. Complements of small sign patterns. In view of the results above, it is convenient to make the following convention:

DEFINITION 4.1. We denote by $\{x\}_{m \times n} \in M_{m \times n}(\mathbb{R})$ the all- x , $m \times n$ matrix, and by

$$\begin{pmatrix} x_{11} & \dots & x_{1l} \\ \dots & \dots & \dots \\ x_{k1} & \dots & x_{kl} \end{pmatrix}_{(m_1, \dots, m_k) \times (n_1, \dots, n_l)}$$

the matrix having all- x_{ij} rectangular blocks $X_{ij} = \{x_{ij}\}_{m_i \times n_j} \in M_{m_i \times n_j}(\mathbb{R})$, of prescribed size. In the case of square diagonal blocks, we simply write $\{x\}_n = \{x\}_{n \times n}$ and

$$\begin{pmatrix} x_{11} & \dots & x_{1k} \\ \dots & \dots & \dots \\ x_{kk} & \dots & x_{kk} \end{pmatrix}_{n_1, \dots, n_k} = \begin{pmatrix} x_{11} & \dots & x_{1k} \\ \dots & \dots & \dots \\ x_{k1} & \dots & x_{kk} \end{pmatrix}_{(n_1, \dots, n_k) \times (n_1, \dots, n_k)}.$$

Modulo equivalence, the ± 1 matrices of size $r = 1, 2$ are as follows:

$$[+]_{(1)}, \quad \begin{bmatrix} + & + \\ + & - \end{bmatrix}_{(2)}, \quad \begin{bmatrix} + & + \\ + & + \end{bmatrix}_{(2')}.$$

In the cases (1) and (2) above, where the matrix A is invertible, the spectral properties of their complementary matrices are as follows:

THEOREM 4.2. For the $N \times N$ Hadamard matrices of type

$$\begin{bmatrix} + & + \\ + & D \end{bmatrix}_{(1)}, \quad \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & D_{00} & D_{01} \\ + & - & D_{10} & D_{11} \end{bmatrix}_{(2)}$$

the polar decomposition $D = UT$ with $U = \frac{1}{\sqrt{N}}(D - E)$, $T = \sqrt{N}I - S$ is given by:

$$E_{(1)} = \left\{ \frac{1}{1+\sqrt{N}} \right\}_{N-1}, \quad E_{(2)} = \frac{2}{2+\sqrt{2N}} \begin{Bmatrix} 1 & 1 \\ 1 & -1 \end{Bmatrix}_{N/2-1, N/2-1},$$

$$S_{(1)} = \left\{ \frac{1}{1+\sqrt{N}} \right\}_{N-1}, \quad S_{(2)} = \frac{2}{\sqrt{2}+\sqrt{N}} \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix}_{N/2-1, N/2-1}.$$

In particular, all the matrices D above are AHP.

Proof. For $A \in M_r(\pm 1)$ Hadamard, the quantities in Definition 3.1 are:

$$X_A = \frac{A^t}{r + \sqrt{rN}}, \quad Y_A = \frac{I_r}{\sqrt{r} + \sqrt{N}}.$$

These formulae follow indeed from $AA^t = A^tA = rI_r$ and $Pol(A) = A/\sqrt{r}$.

(1) Using the notation introduced in Definition 3.1, we have here $B_{(1)} = \{1\}_{1 \times N-1}$ and $C_{(1)} = B_{(1)}^t$. Since $A_{(1)} = [1]$ is Hadamard we have $X_{A_{(1)}} = Y_{A_{(1)}} = \frac{1}{1+\sqrt{N}}$, and so:

$$\begin{aligned} E_{(1)} &= \frac{1}{1 + \sqrt{N}} \{1\}_{N-1 \times 1} [1] \{1\}_{1 \times N-1} = \frac{1}{1 + \sqrt{N}} \{1\}_{N-1}, \\ S_{(1)} &= \frac{1}{1 + \sqrt{N}} \{1\}_{N-1 \times 1} \{1\}_{1 \times N-1} = \frac{1}{1 + \sqrt{N}} \{1\}_{N-1}. \end{aligned}$$

(2) Using the orthogonality of the first two lines in $H_{(2)}$, we find that the matrices D_{00} and D_{11} have size $N/2 - 1$. Since $A_{(2)} = \begin{bmatrix} + & + \\ + & - \end{bmatrix}$ is Hadamard we have $X_{A_{(2)}} = \frac{A}{2+\sqrt{2N}}$, $Y_{A_{(2)}} = \frac{I_2}{\sqrt{2}+\sqrt{N}}$, and so:

$$\begin{aligned} E_{(2)} &= \frac{1}{2 + \sqrt{2N}} \begin{Bmatrix} 1 & 1 \\ 1 & -1 \end{Bmatrix}_{(N/2-1, N/2-1) \times (1,1)} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}_{(1,1) \times (N/2-1, N/2-1)} \begin{Bmatrix} 1 & 1 \\ 1 & -1 \end{Bmatrix} \\ &= \frac{2}{2 + \sqrt{2N}} \begin{Bmatrix} 1 & 1 \\ 1 & -1 \end{Bmatrix}_{N/2-1, N/2-1} \\ S_{(2)} &= \frac{1}{\sqrt{2} + \sqrt{N}} \begin{Bmatrix} 1 & 1 \\ 1 & -1 \end{Bmatrix}_{(N/2-1, N/2-1) \times (1,1)} \begin{Bmatrix} 1 & 1 \\ 1 & -1 \end{Bmatrix}_{(1,1) \times (N/2-1, N/2-1)} \\ &= \frac{2}{\sqrt{2} + \sqrt{N}} \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix}_{N/2-1, N/2-1} \quad \square \end{aligned}$$

As an illustration for the above computations, let us first work out the case $r = 1, N = 2$, with $H = \begin{bmatrix} + & + \\ + & - \end{bmatrix}$ being the first Walsh matrix. Here we have:

$$E = S = \frac{1}{1 + \sqrt{2}} \implies U = -1, T = 1.$$

At $r = 2, N = 4$, consider the second Walsh matrix, written as in Theorem 4.2:

$$W'_4 = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & + & - & + \end{bmatrix}.$$

We obtain the polar decomposition $D = UT$ of the corner $D = \begin{bmatrix} - & - \\ - & + \end{bmatrix}$:

$$E = \frac{1}{1 + \sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, S = \frac{2}{2 + \sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \implies U = -\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, T = \sqrt{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let us record as well the following consequence of Theorem 4.2:

COROLLARY 4.3. *We have the formulae*

$$\begin{aligned} \det(\lambda - T_{(1)}) &= (\lambda - 1)(\lambda - \sqrt{N})^{4N-2}, \\ \det(\lambda - T_{(2)}) &= (\lambda - \sqrt{2})^2(\lambda - \sqrt{N})^{N-4}, \end{aligned}$$

so in particular $|\det D_{(1)}| = N^{N/2-1}$, $|\det D_{(2)}| = 2N^{N/2-2}$.

Proof. As already noted in [12], these formulae from [6], [7] follow from Theorem 2.3. They are even more clear from the above formulae of $S_{(1)}$ and $S_{(2)}$, using the fact that the non-zero eigenvalues of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{a,b}$ are a and b . $\square$

Modulo equivalence, the ± 1 matrices of size $r = 3$ are as follows:

$$\begin{bmatrix} + & + & + \\ + & - & + \\ + & + & - \end{bmatrix}_{(3)}, \quad \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & - \end{bmatrix}_{(3')}, \quad \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & + \end{bmatrix}_{(3'')}.$$

Among those, only (3) is invertible, here is the result:

PROPOSITION 4.4. *For the $N \times N$ Hadamard matrices of type*

$$\begin{bmatrix} + & + & + & + & + & + & + \\ + & - & + & + & + & - & - \\ + & + & - & + & - & + & - \\ + & + & + & D_{00} & D_{01} & D_{02} & D_{03} \\ + & + & - & D_{10} & D_{11} & D_{12} & D_{13} \\ + & - & + & D_{20} & D_{21} & D_{22} & D_{23} \\ + & - & - & D_{30} & D_{31} & D_{32} & D_{33} \end{bmatrix}_{(3)}$$

the polar decomposition $D = UT$ with $U = \frac{1}{\sqrt{N}}(D - E)$, $T = \sqrt{N}I - S$ is given by:

$$E_{(3)} = \frac{1}{\sqrt{N} + 1} \begin{bmatrix} x & y & y & 1 \\ y & -y & x & -1 \\ y & x & -y & -1 \\ 1 & -1 & -1 & -3 \end{bmatrix}_{N/4-1, N/4-1, N/4-1, N/4},$$

$$S_{(3)} = \frac{1}{\sqrt{N}+1} \begin{pmatrix} z & t & t & -1 \\ t & z & -t & 1 \\ t & -t & z & 1 \\ -1 & 1 & 1 & 3 \end{pmatrix}_{N/4-1, N/4-1, N/4-1, N/4},$$

where

$$x = \frac{7\sqrt{N}+6}{3\sqrt{N}+6}, \quad y = \frac{5\sqrt{N}+6}{3\sqrt{N}+6}, \quad z = \frac{9\sqrt{N}+10}{3\sqrt{N}+6}, \quad t = \frac{3\sqrt{N}+2}{3\sqrt{N}+6}.$$

In particular, if $N > \|A\|^2 = 4$, then D is an AHP.

Proof. By direct computation, we have

$$X_A = \frac{1}{3(\sqrt{N}+1)(\sqrt{N}+2)} \begin{bmatrix} \sqrt{N} & 2\sqrt{N}+3 & 2\sqrt{N}+3 \\ 2\sqrt{N}+3 & -(2\sqrt{N}+3) & \sqrt{N} \\ 2\sqrt{N}+3 & \sqrt{N} & -(2\sqrt{N}+3) \end{bmatrix}.$$

From the orthogonality condition for the first three lines of $H_{(3)}$, we find that the dimensions of the matrices D_{00}, D_{11}, D_{22} and D_{33} are, respectively, $N/4 - 1, N/4 - 1, N/4 - 1$ and $N/4$. Thus, we have

$$B_{(3)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}_{(1,1,1) \times (N/4-1, N/4-1, N/4-1, N/4)}$$

and $C_{(3)} = B_{(3)}^t$. The formulas for $E_{(3)}$ and $S_{(3)}$ follow by direct computation.

Since

$$3 > \frac{7\sqrt{N}+6}{3\sqrt{N}+6} > \frac{5\sqrt{N}+6}{3\sqrt{N}+6} > 1,$$

we have $\|E_{(3)}\|_\infty = 3/(\sqrt{N}+1)$ and the conclusion about D being AHP follows. $\square$

Note that in the case $N = 4$, there is a unique way to complement the matrix $A_{(3)}$ above into a 4×4 Hadamard matrix. Since the complement in this case is simply $D = [1]$, we conclude that, for all $N \geq 4$, the complement of $A_{(3)}$ inside a $N \times N$ Hadamard matrix is AHP.

In this case, using Theorem 2.3, we get $\det(\lambda - T) = (\lambda+1)(\lambda-2)^2(\lambda-1/\sqrt{N})^{N-3}$, and so $|\det D| = 2N^{(N-3)/2}$.

5. Examples of non-AHP sign patterns. In the previous section, we have shown that for $r = 1, 2, 3$, all invertible $r \times r$ sign patterns are complemented by AHP matrices inside Hadamard matrices. In the following proposition we show that this is

not the case for larger values of r . Recall that for a matrix D to be AHP, it must be invertible, its polar part $U = \text{Pol}(D)$ must have non-zero entries, and $D = \text{sgn}(U)$ must hold.

PROPOSITION 5.1. *Consider the Walsh matrix W_8 , and the Paley matrix H_{12} .*

1. W_8 has a 4×4 submatrix which is not AHP, due to a $U_{ij} = 0$ reason.
2. H_{12} has a 7×7 submatrix which is not AHP, due to a $D_{ij} = -1, U_{ij} > 0$ reason.

Proof. (1) Let A be the submatrix of W_8 (see Section 2) having the rows and columns with indices 1, 2, 3, 5. Then A, D and $U = \text{Pol}(D)$ are as follows:

$$A = \begin{bmatrix} + & + & + & + \\ + & - & + & + \\ + & + & - & + \\ + & + & + & - \end{bmatrix}, \quad D = \begin{bmatrix} + & - & - & + \\ - & + & - & + \\ - & - & + & + \\ + & + & + & - \end{bmatrix}, \quad U = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{\sqrt{3}} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{1}{\sqrt{3}} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix}.$$

- (2) Consider indeed the unique 12×12 Hadamard matrix, written as:

$$H_{12} = \begin{bmatrix} + & - & - & - & - & - & - & - & - & - & - & - \\ + & + & - & + & - & - & - & + & + & + & - & + \\ + & + & + & - & + & - & - & - & + & + & + & - \\ + & - & + & + & - & + & - & - & - & + & + & + \\ + & + & - & + & + & - & + & - & - & - & + & + \\ + & + & + & + & - & + & + & - & + & - & - & + \\ + & - & + & + & + & - & + & + & - & + & - & - \\ + & - & - & + & + & + & - & + & + & - & + & - \\ + & - & - & - & + & + & + & - & + & + & - & + \\ + & + & - & - & - & + & + & + & - & + & + & - \\ + & - & + & - & - & - & + & + & + & - & + & + \end{bmatrix}.$$

Let A be the submatrix having the rows and columns with indices 1, 2, 3, 5, 6:

$$A = \begin{bmatrix} + & - & - & - & - \\ + & + & - & - & - \\ + & + & + & + & - \\ + & + & - & + & - \\ + & + & + & + & + \end{bmatrix}, \quad D = \begin{bmatrix} + & - & - & - & + & + & + \\ + & + & - & + & - & - & - \\ + & + & + & - & + & - & - \\ + & - & + & + & - & + & - \\ - & + & - & + & + & - & + \\ - & + & + & - & + & + & - \\ - & + & + & + & - & + & + \end{bmatrix}.$$

Then D is invertible, and its polar part is given by:

$$U \approx \begin{bmatrix} 0.51 & -0.07 & -0.37 & -0.22 & 0.35 & 0.51 & 0.37 \\ 0.37 & 0.51 & -0.51 & 0.22 & -0.35 & -0.07 & -0.37 \\ 0.51 & 0.37 & 0.51 & -0.22 & 0.35 & -0.37 & -0.07 \\ 0.35 & -0.35 & 0.35 & 0.61 & 0.03 & 0.35 & -0.35 \\ -0.22 & 0.22 & -0.22 & 0.61 & 0.61 & -0.22 & 0.22 \\ -0.37 & 0.37 & 0.07 & -0.22 & 0.35 & 0.51 & -0.51 \\ -0.07 & 0.51 & 0.37 & 0.22 & -0.35 & 0.37 & 0.51 \end{bmatrix}.$$

Now since $D_{45} = -1$ and $U_{45} \approx 0.03 > 0$, this gives the result. $\square$

Acknowledgment. The work of I.N. was supported by the ANR grants “OS-QPI” 2011 BS01 008 01 and “RMTQIT” ANR-12-IS01-0001-01, and by the PEPS-ICQ CNRS project “Cogit”.

REFERENCES

- [1] T. Banica, B. Collins, and J.-M. Schlenker. On orthogonal matrices maximizing the 1-norm. *Indiana Univ. Math. J.*, 59:839–856, 2010.
- [2] T. Banica and I. Nechita. Almost Hadamard matrices: the case of arbitrary exponents. *Discrete Appl. Math.*, 161:2367–2379, 2013.
- [3] T. Banica, I. Nechita, and J.-M. Schlenker. Analytic aspects of the circulant Hadamard conjecture. Preprint, arXiv:1212.3589.
- [4] T. Banica, I. Nechita, and K. Życzkowski. Almost Hadamard matrices: General theory and examples. *Open Syst. Inf. Dyn.*, 19:1–26, 2012.
- [5] R.A. Horn and C.R. Johnson. *Matrix Analysis*. Cambridge University Press, 1985.
- [6] C. Koukouvinos, M. Mitrouli, and J. Seberry. An algorithm to find formulae and values of minors for Hadamard matrices. *Linear Algebra Appl.*, 330:129–147, 2001.
- [7] C. Koukouvinos, M. Mitrouli, and J. Seberry. An algorithm to find formulae and values of minors for Hadamard matrices. II. *Linear Algebra Appl.*, 371:111–124, 2003.
- [8] W. de Launey and D.A. Levin. $(1, -1)$ -matrices with near-extremal properties. *SIAM J. Discrete Math.*, 23:1422–1440, 2009.
- [9] W. de Launey and D.A. Levin. A Fourier-analytic approach to counting partial Hadamard matrices. *Cryptogr. Commun.*, 2:307–334, 2010.
- [10] J. Seberry and M. Yamada. *Hadamard Matrices, Sequences, and Block Designs*. Wiley, 1992.
- [11] F.R. Sharpe. The maximum value of a determinant. *Bull. Amer. Math. Soc.*, 14:121–123, 1907.
- [12] F. Szöllősi. Exotic complex Hadamard matrices and their equivalence. *Cryptogr. Commun.*, 2:187–198, 2010.
- [13] T. Tao and V. Vu. On random ± 1 matrices: singularity and determinant. *Random Structures Algorithms*, 28:1–23, 2006.
- [14] T. Tao and V. Vu. On the singularity probability of random Bernoulli matrices. *J. Amer. Math. Soc.*, 20:603–628, 2007.