Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 124-145, February 2014

ELA

REAL AND COMPLEX INVARIANT SUBSPACES FOR MATRICES WHICH ARE H-POSITIVE REAL IN AN INDEFINITE INNER PRODUCT SPACE*

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Abstract. In this paper, the equivalence of the existence of unique real and complex A-invariant semidefinite subspaces for real H-positive real matrices are shown.

Key words. H-Positive real matrices, Invariant maximal semidefinite subspaces.

AMS subject classifications. 15A18, 15A57.

1. Introduction. In this article, we investigate invariant maximal semidefinite subspaces for real matrices A which are H-positive real in the indefinite inner product given by the invertible real symmetric matrix H. This means that $HA + A^T H \geq 0$, that is, it is positive semidefinite. We can view this matrix as acting on $\mathbb{R}^n$ as well as on $\mathbb{C}^n$, both equipped with the indefinite inner product given by H. We shall switch back and forth between these two points of view. Recall that for a complex matrix A, we say that A is H-positive real if $HA + A^* H \geq 0$.

As mentioned, our main interest is the study of A-invariant subspaces $\mathcal{M}$ which are maximal H-nonnegative, respectively, maximal H-nonpositive, and which have the additional property that the spectrum $\sigma(A|_{\mathcal{M}})$ is contained in the closed right half plane $\overline{\mathbb{C}}_{right}$, respectively, the closed left half plane $\overline{\mathbb{C}}_{left}$. Recall that a vector $x \in \mathbb{C}^n$ or in $\mathbb{R}^n$ is called H-nonnegative if $\langle Hx, x\rangle \geq 0$, H-nonpositive if $\langle Hx, x\rangle \leq 0$, and H-neutral if $\langle Hx, x\rangle = 0$. A subspace $\mathcal{M}$ is called H-nonnegative, respectively, H-nonpositive, H-neutral if every vector in $\mathcal{M}$ is H-nonnegative, respectively, H-nonpositve, H-neutral. A subspace $\mathcal{M}$ is called H-nondegenerate if $(H\mathcal{M})^\perp \cap \mathcal{M} = \{0\}$.

*Received by the editors on August 18, 2013. Accepted for publication on January 25, 2014. Handling Editor: Christian Mehl. This work was partially funded by the National Research Foundation (NRF) of South Africa.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 124-145, February 2014

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Before continuing, let us introduce some notation. We shall denote by $[x, y] = \langle Hx, y\rangle$ the (H-)indefinite inner product of the vectors x and y. The adjoint of a matrix A with respect to this indefinite inner product is denoted by $A^{[*]}$. One easily checks that $A^{[*]} = H^{-1}A^*H$. If $\mathcal{M}$ and $\mathcal{N}$ are two subspaces, then the notation $\mathcal{X} = \mathcal{M}[\dot{+}]\mathcal{N}$ means that $\mathcal{X}$ is the H-orthogonal direct sum of $\mathcal{M}$ and $\mathcal{N}$. In particular this means that the subspaces are H-orthogonal to each other, and have intersection consisting only of the zero vector.

From the main theorem, Theorem 5.1 in Section 5 of this manuscript, it follows that the following statements are equivalent for a given real H-positive real matrix A:

(a) There exists a unique complex A-invariant maximal H-nonnegative subspace $\mathcal{M}$, such that $\sigma(A|_{\mathcal{M}}) \subseteq \overline{\mathbb{C}}_{right}$.
(b) There exists a unique real A-invariant maximal H-nonnegative subspace $\mathcal{M}$, such that $\sigma(A|_{\mathcal{M}}) \subseteq \overline{\mathbb{C}}_{right}$.
(c) There exists a unique complex A-invariant maximal H-nonpositive subspace $\mathcal{M}$, such that $\sigma(A|_{\mathcal{M}}) \subseteq \overline{\mathbb{C}}_{left}$.
(d) There exists a unique real A-invariant maximal H-nonpositive subspace $\mathcal{M}$, such that $\sigma(A|_{\mathcal{M}}) \subseteq \overline{\mathbb{C}}_{left}$.

The proof of Theorem 5.1 will make use of the fact that the class of matrices we are studying is closely related to H-dissipative matrices. Recall that a complex matrix A is H-dissipative if $\frac{1}{i}(HA - A^*H) \geq 0$. It is easily seen that A is H-positive real if and only if iA is H-dissipative, and hence, it follows that $-iA$ is H-positive real if and only if A is H-dissipative.

It should be observed that an A-invariant maximal H-nonnegative subspace $\mathcal{M}$ such that $\sigma(A|_{\mathcal{M}}) \subseteq \overline{\mathbb{C}}_{right}$ always exists. Indeed, the usual proof of this fact runs as follows (compare [1], where the dissipative case was done this way). Let $\varepsilon > 0$ and consider $A(\varepsilon) = A + \varepsilon H$. Then $HA(\varepsilon) + A(\varepsilon)^T H = HA + A^T H + 2\varepsilon H^2$, and hence, this is strictly positive definite. By the well-known inertia theorem (see e.g., [6], Chapter 13), $A + \varepsilon H$ has no eigenvalues on the imaginary line, and the spectral subspace $\mathcal{M}_+(\varepsilon)$, respectively $\mathcal{M}_-(\varepsilon)$ of $A + \varepsilon H$ corresponding to its eigenvalues in the open right, respectively left, half plane is H-nonnegative, respectively H-nonpositive. By counting the dimensions, we see that these subspaces are maximal H-nonnegative, respectively, maximal H-nonpositive. Now let $\varepsilon \downarrow 0$. Then, in the gap metric on the set of subspaces, the subspaces $\mathcal{M}_\pm(\varepsilon)$ converge to A-invariant maximal H-nonnegative, respectively, H-nonpositive, subspaces $\mathcal{M}_\pm$. Since A is real, the subspaces $\mathcal{M}_\pm(\varepsilon)$ have a real basis as well, and hence, also their limits have a real basis. This shows that existence of the subspaces mentioned in the theorem above is not an issue.

The construction outlined in the previous paragraph is obviously far from an

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 124-145, February 2014

explicit construction. For that reason, explicit constructions were carried out in [10] for the dissipative case, and in [4] for the complex and real case. The construction in [4] was taken a bit further in [5]. We shall give a brief outline of the construction in Section 4. The construction is based on reduction of the pair (A, H) to a so-called simple form, that is, a basis transformation such that with respect to this basis A is in (real or complex) Jordan canonical form, and H is in a simple form. Section 2.2 of [5] follows the line of argument of [4] and [10] for the case of complex H-positive real matrices.

Uniqueness of invariant maximal H-nonnegative subspaces with a spectral constraint has been discussed for H-dissipative matrices in [10], see also [8, 9]. It turns out that this is equivalent to a condition involving only the pair (A, H), which may be read off from the simple form constructed in [10]. This condition is called the numerical range condition. It only concerns the simple form as far as it pertains to the eigenvalues with zero real parts of the matrix A. Section 2.4 of [5] discusses the numerical range condition for complex matrices A which are H-positive real. Using this we discuss here the uniqueness and stability of invariant maximal semidefinite subspaces. Finally, in Section 5, the real case is discussed; see also Theorem 3.1 of [4] and Example 2.1.

2. Preliminaries. The paper [4] considers both the complex and real case. In the real case, $H = H^T$ is an invertible real symmetric matrix, and A is a real matrix satisfying $HA + A^T H \geq 0$, i.e., A is H-positive real. Compared to the complex case the additional difficulty is that the eigenvalues now appear in complex conjugate pairs, and that we shall be interested in real A-invariant maximal H-nonnegative subspaces. The real canonical form is used instead of the complex Jordan normal form. Recall that the real canonical form of a real matrix A is given by $A = SJS^{-1}$, where S is a real invertible matrix and $J = \operatorname{diag}(J_1, \ldots, J_N)$, where J_i is either a standard Jordan block with a real eigenvalue λ, given by

$$J_i = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix},$$

or a so-called real Jordan block corresponding to a pair of complex conjugate eigenvalues $a \pm bi$, given by

$$J_i = \begin{bmatrix} \alpha & I & & \\ & \ddots & \ddots & \\ & & \ddots & I \\ & & & \alpha \end{bmatrix},$$

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A publication of the International Linear Algebra Society
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where $\alpha = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, and I denotes the (2×2) identity matrix.

We will refer to Theorem 3.1 of [4] for the simple form of the matrix H in the special case where the spectrum of A, $\sigma(A)$, is just a pair of complex conjugate eigenvalues with zero real part and where $A = J_1 \oplus J_2 \oplus \cdots \oplus J_N$ is H-positive real. As was already mentioned in the Introduction, and as we shall see later on, the numerical range condition only uses the simple form for eigenvalues with zero real parts.

The proof of Theorem 5.1 below depends very heavily on the simple form of the pair (A, H), which was developed in [4]. As an illustration of how this simple form is obtained we shall present a small example below, and in doing so, also make an additional observation that is not in [4] (compare the proof of Theorem 3.1 in [4]). Incidentally, this observation also played a role in [3], where eigenvalues of rank one perturbations $B(t) = A + tuu^T H$ $(t > 0)$ were studied.

Example 2.1. Let $A = \begin{bmatrix} \gamma H_0 & I_2 \\ 0 & \gamma H_0 \end{bmatrix}$, and $H = \begin{bmatrix} 0 & H_{12} \\ H_{12}^T & H_{22} \end{bmatrix}$, where $H_{12} = aI_2 + bH_0$, for some $a, b \in \mathbb{R}$, and $H_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Assume that $HA + A^T H \geq 0$. A direct computation gives

$$HA + A^T H = \begin{bmatrix} 0 & 0 \\ 0 & H_{12} + H_{12}^T + \gamma H_{22} H_0 - \gamma H_0 H_{22} \end{bmatrix}.$$

Now $H_{22} = \begin{bmatrix} c & d \\ d & e \end{bmatrix}$ for some real numbers c, d and e. Further, $H_{12} + H_{12}^T = 2aI_2$. Computing the $(2, 2)$ block entry of $HA + A^T H$, which we denote by D, gives

$$D = \begin{bmatrix} 2a - 2\gamma d & \gamma(c - e) \\ \gamma(c - e) & 2a + 2\gamma d \end{bmatrix}.$$

For A to be H-positive real, the matrix D needs to be positive semidefinite. In particular, both diagonal entries of D need to be greater than or equal to zero. So $2a \geq 2\gamma d$ and $2a \geq -2\gamma d$. It follows that $2a \geq |2\gamma d| \geq 0$. In particular we conclude that $a \geq 0$.

Something similar holds for all blocks of even size with eigenvalues with zero real parts: let us assume that A is of size $2n$, and n is even, say $n = 2k$, and that A

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A publication of the International Linear Algebra Society
Volume 27, pp. 124-145, February 2014

consists of one real Jordan block with eigenvalues $\pm\gamma i$. Thus,

$$A = \begin{bmatrix} \gamma H_0 & I_2 & & \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ & & & \gamma H_0 \end{bmatrix}.$$

Write $H = \left[H_{i,j}\right]_{i,j=1}^{n}$, where each $H_{i,j}$ is a two by two matrix. Then we know from [4, 5] that $H_{i,j} = 0$ when $i+j < n+1$. Moreover, $H_{1,n}$ is of the form $H_{1,n} = aI_2 + bH_0$. It can be shown that in this case the following must hold: $(-1)^{k-1}a \geq 0$.

The simple forms for H in general, are given in Theorem 3.9 of [4].

3. Numerical range condition. In this section, we introduce the notion of the numerical range condition for the pair (A, H). Thereafter, we shall apply it to study the stability of invariant maximal nonnegative subspaces for an H-positive real matrix. Throughout this section, we shall view the real matrix A as acting on the complex vector space $\mathbb{C}^n$. Thus, we shall work with the standard Jordan normal form, and not with the real Jordan form. The following is taken from the article [8]. See also [9] and [10], Chapter 3.

Let A be an H-positive real matrix and suppose $\lambda \in i\mathbb{R}$ is an eigenvalue of A. Then iA is H-dissipative and $i\lambda \in \mathbb{R}$. Let κ denote the number of negative eigenvalues of H. By Corollary 2.2.4 in [5], the maximum length of a Jordan chain of A corresponding to λ does not exceed $2\kappa + 1$. Consider a Jordan basis for $\mathcal{R}(A, \{\lambda\})$, where

$$\mathcal{R}(A, \{\lambda\}) = \{x \in \mathbb{C}^n \mid (A - \lambda I)^s x = 0, \text{ for some positive integer } s\}.$$

The Jordan basis for $\mathcal{R}(A, \{\lambda\})$ splits into the sets $J(\lambda, j)$. Here $J(\lambda, j)$ consists of the basis vectors belonging to Jordan chains of length j. Denote by $n_{\lambda,j}$ the number of chains of length j and the basis vectors in $J(\lambda, j)$ that are not in the set $\mathrm{Ker}(A-\lambda)^{j-1}$ by $\{x_{j;1}, \ldots, x_{j;n_{\lambda,j}}\}$. These are all the end vectors of Jordan chains of length j. Let m_j be $\frac{j+1}{2}$ in case j is odd, and $\frac{j}{2}$ in case j is even. Furthermore, let

$$y_{j;k} = (A - \lambda)^{m_j - 1} x_{j;k},$$

which are the vectors in the middle of the chains.

In [10], there is a discussion of the numerical range condition for a pair (B, K) of matrices with B a K-dissipative matrix. We call this condition the "numerical range condition for dissipative matrices". Modelled on the discussion in [10], we now introduce a numerical range condition for a pair (A, H) of matrices, whereby H is a

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A publication of the International Linear Algebra Society
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symmetric matrix and A is a H-positive real matrix. We will call this condition the "numerical range condition for H-positive real matrices". For Jordan chains of odd length, let

$$CM_j = \begin{bmatrix} \langle Hy_{j;1}, y_{j;1}\rangle & \cdots & \langle Hy_{j;n_{\lambda,j}}, y_{j;1}\rangle \\ \vdots & & \vdots \\ \langle Hy_{j;1}, y_{j;n_{\lambda,j}}\rangle & \cdots & \langle Hy_{j;n_{\lambda,j}}, y_{j;n_{\lambda,j}}\rangle \end{bmatrix},$$

for $j = 1, 3, \ldots, 2\kappa + 1$. (Here CM stands for the *characteristic matrix*). Define

$$CM_{odd}(A, \lambda) = \operatorname{diag}\{CM_1, CM_3, \ldots, CM_{2\kappa+1}\}.$$

Let $n_{odd} = n_{\lambda,1} + n_{\lambda,3} + \cdots + n_{\lambda,2\kappa+1}$ and put

$$NR_{odd}(A, \lambda) = \{\langle CM_{odd}(A, \lambda)x, x\rangle \mid x \in \mathbb{C}^{n_{odd}}, x \neq 0\}.$$

(Here NR stands for the *numerical range*).

The matrix $CM_{odd}(A, \lambda)$ is the same as the matrix $CM_{odd}(iA, i\lambda)$ in [10], where iA is H-dissipative and $i\lambda \in \mathbb{R}$. Similarly, $NR_{odd}(A, \lambda)$ corresponds to $NR_{odd}(iA, i\lambda)$ in [10]. Recall from [10], Section 3.1.1, that $CM_{odd}(iA, i\lambda)$ is Hermitian and invertible and $NR_{odd}(iA, i\lambda)$ is independent of the choice of the Jordan basis. Hence, the same properties hold for $CM_{odd}(A, \lambda)$ and $NR_{odd}(A, \lambda)$. We may therefore define the *odd numerical range condition* for (A, H) as follows:

Definition 3.1. Let A be H-positive real, then the pair (A, H) is said to satisfy the odd numerical range condition if $0 \notin NR_{odd}(A, \lambda)$ for all $\lambda \in i\mathbb{R} \cap \sigma(A)$. (Thus, (A, H) has the odd numerical range condition if and only if (iA, H) has the odd numerical range condition for the dissipative matrices in [10]).

For even length chains, let $j, k \in \{2, 4, \ldots, 2\kappa\}$ and put

$$CM_{j,k} = \begin{bmatrix} \langle H(A-\lambda)y_{j;1}, y_{k;1}\rangle & \cdots & \langle H(A-\lambda)y_{j;n_{\lambda,j}}, y_{k;1}\rangle \\ \vdots & & \vdots \\ \langle H(A-\lambda)y_{j;1}, y_{k;n_{\lambda,k}}\rangle & \cdots & \langle H(A-\lambda)y_{j;n_{\lambda,j}}, y_{k;n_{\lambda,k}}\rangle \end{bmatrix},$$

where the vectors $y_{j,1}, y_{j,2}, \ldots, y_{j,n_{\lambda,j}}$ and the vectors $y_{k,1}, y_{k,2}, \ldots, y_{k,n_{\lambda,k}}$ are defined as before. Let

$$CM_{even}(A, \lambda) = \begin{bmatrix} CM_{2,2} & CM_{4,2} & \cdots & CM_{2\kappa,2} \\ \vdots & & & \vdots \\ CM_{2,2\kappa} & CM_{4,2\kappa} & \cdots & CM_{2\kappa,2\kappa} \end{bmatrix}.$$

From equation (2.5) in [10], it follows that $CM_{even}(A, \lambda)$ is a block upper triangular matrix and is invertible. Let $n_{even} = n_{\lambda,2} + n_{\lambda,4} + \cdots + n_{\lambda,2\kappa}$. The numerical

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 124-145, February 2014

range

$$NR_{even}(A,\lambda) = \{\langle CM_{even}(A,\lambda)x, x\rangle \mid x \in \mathbb{C}^{n_{even}}, x \neq 0\}$$

of $CM_{even}(A,\lambda)$ is called the *even numerical range* of A at λ and is independent of the choice of the Jordan basis one starts with.

Definition 3.2. Let A be H-positive real, then the pair (A,H) is said to satisfy the even numerical range condition if $0 \notin NR_{even}(A,\lambda)$ for all $\lambda \in i\mathbb{R} \cap \sigma(A)$.

Definition 3.3. We say that a pair (A,H) satisfies the numerical range condition if it satisfies both the odd numerical range condition and the even numerical range condition.

Note that our definition of the numerical range condition for H-positive real matrices resembles the definition of the numerical range condition for H-dissipative matrices in the following sense:

Lemma 3.4. (Lemma 2.4.2 in [5]) *The pair (A,H) has the numerical range condition for H-positive real matrices if and only if the pair (iA,H) has the numerical range condition for H-dissipative matrices.*

Although the result in the above lemma is to be expected, it follows by a non-trivial technical argument, which we will not discuss here. The reader is referred to [5] for a proof of this fact.

We will explain the numerical range condition by means of a simple example. The example illustrates the case where the numerical range condition is not satisfied.

Example 3.5. Let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad H = I_2 \oplus I_2 \oplus -I_2.$$

Then A has eigenvalues $\pm i$, such that there are three Jordan blocks of size one associated with eigenvalue i, respectively, $-i$. The matrix CM_{odd} corresponding to both i and $-i$ is in this case given by $I_2 \oplus (-1)$, so for neither of these eigenvalues is the numerical range condition satisfied.

Example 3.6. Let us examine how the numerical range condition behaves under perturbations. We consider rank one perturbed matrices of the form $B(u) = A + uu^T H$ for a vector $u \in \mathbb{R}^6$, where the pair (A,H) is as in the previous example. It may be seen from some experimentation with the help of Matlab that there are vectors u_1 and u_2 such that the pair $(B(u_1),H)$ does satisfy the numerical range condition, while the pair $(B(u_2),H)$ does not satisfy the numerical range condition.

We introduce the following notation and definitions of $\mathbb{A}$-stable and $\mathbb{D}$-stable subspaces. Let $\mathcal{M}$ and $\mathcal{N}$ be subspaces of $\mathbb{C}^n$ and let $\theta(\mathcal{M},\mathcal{N}) = \|P_{\mathcal{M}} - P_{\mathcal{N}}\|$ where

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$P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ are the unique orthogonal projections of $\mathcal{M}$ and $\mathcal{N}$. Then it is clear that θ is a metric on the set $S(\mathbb{C}^n)$ of all subspaces of $\mathbb{C}^n$.

Definition 3.7. Let $A \in \mathbb{A}$, where $\mathbb{A} \subset \mathbb{R}^{n\times n}$ denotes the class of all real $n\times n$ H-positive real matrices. We call an A-invariant maximal H-nonnegative (respectively, H-nonpositive) subspace $\mathcal{M}$, $\mathbb{A}$-*stable* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every $B \in \mathbb{A}$ with $\|A - B\| < \delta$ there exists a H-nonnegative (respectively, H-nonpositive) B-invariant subspace $\mathcal{N}$, such that

$$\theta(\mathcal{M}, \mathcal{N}) < \epsilon.$$

Definition 3.8. Let $A \in \mathbb{D}$, where $\mathbb{D} \subset \mathbb{C}^{n\times n}$ denotes the class of all complex $n \times n$ H-dissipative matrices. We call an A-invariant maximal H-nonnegative (respectively, H-nonpositive) subspace $\mathcal{M}$, $\mathbb{D}$-*stable* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every $B \in \mathbb{D}$ with $\|A - B\| < \delta$ there exists a H-nonnegative (respectively, H-nonpositive) B-invariant subspace $\mathcal{N}$, such that

$$\theta(\mathcal{M}, \mathcal{N}) < \epsilon.$$

Now we state Theorem 2.1 in [8], (see also [5] and [10]), which describes uniqueness and stability of A-invariant maximal H-nonnegative subspaces.

Theorem 3.9. *The following statements are equivalent for given H-dissipative matrix A:*

(i) there exists a $\mathbb{D}$-stable A-invariant maximal H-nonnegative subspace,
(ii) there exists a $\mathbb{D}$-stable A-invariant maximal H-nonpositive subspace,
(iii) the numerical range condition, in the sense of [10]*, holds for the pair (A, H).*
(iv) there is a unique A-invariant maximal H-nonnegative subspace $\mathcal{M}$, with $\sigma(A|_{\mathcal{M}})$ contained in the closed upper half plane,
(v) there is a unique A-invariant maximal H-nonpositive subspace $\mathcal{M}$, with $\sigma(A|_{\mathcal{M}})$ contained in the closed lower half plane.

In that case, there is a unique stable A-invariant maximal H-nonnegative subspace, being the one with $\sigma(A|_{\mathcal{M}})$ contained in the closed upper half plane, and there is a unique stable A-invariant maximal H-nonpositive subspace, being the one with $\sigma(A|_{\mathcal{M}})$ contained in the closed lower half plane.

Next, recall Theorem 2.5.4 in [5], which states the equivalence between the numerical range condition and the existence of $\mathbb{A}$-stable A-invariant maximal H-nonpositive and A-invariant maximal H-nonnegative subspaces.

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A publication of the International Linear Algebra Society
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THEOREM 3.10. *The following statements are equivalent for a given H-positive real matrix A:*

(i) there exists an $\mathbb{A}$-stable A-invariant maximal H-nonnegative subspace, say $\mathcal{M}_+$,
(ii) there exists an $\mathbb{A}$-stable A-invariant maximal H-nonpositive subspace, say $\mathcal{M}_-$,
(iii) the numerical range condition holds for the pair (A, H),
(iv) there is a unique A-invariant maximal H-nonnegative subspace $\mathcal{M}_+$, with $\sigma(A|_{\mathcal{M}_+})$ contained in the closed right half plane,
(v) there is a unique A-invariant maximal H-nonpositive subspace $\mathcal{M}_-$, with $\sigma(A|_{\mathcal{M}_-})$ contained in the closed left half plane.

Proof. First, note that the pair (A, H) where $A \in \mathbb{A}$ satisfies the numerical range condition of Definition 3.3 if and only if the pair (iA, H) satisfies the numerical range condition in the sense of [10].

We first prove (iii) $\Leftrightarrow$ (iv): Assume (iv) holds. There exists a unique (complex) A-invariant maximal H-nonnegative subspace $\mathcal{M}_+$, with $\sigma(A|_{\mathcal{M}_+})$ contained in the closed right half plane. Take note that, $A \in \mathbb{A}$ implies that iA is H-dissipative and that $\mathcal{M}_+$ is also iA-invariant. Then $\mathcal{M}_+$ is the unique iA-invariant maximal H-nonnegative subspace with $\sigma(iA|_{\mathcal{M}_+})$ contained in the closed upper half plane. Thus, by Theorem 3.9 (iii), the numerical range condition holds for (iA, H) in the sense of [10], and hence, (iii) holds for the pair (A, H).

Conversely, let the pair (A, H) satisfy the numerical range condition for H-positive real matrices, equivalently, the pair (iA, H) satisfies the numerical range condition for H-dissipative matrices, (cf. Lemma 3.4). Take note, iA is H-dissipative and $\mathcal{M}_+$ is iA-invariant. From Theorem 3.9 (iv) it follows that there exists a unique iA-invariant maximal H- nonnegative subspace $\mathcal{M}_+$, with $\sigma(iA|_{\mathcal{M}_+})$ contained in the closed upper half plane. Furthermore, $\mathcal{M}_+$ is also A-invariant and $\sigma(iA|_{\mathcal{M}_+}) \subseteq$ "closed upper half plane" implies $\sigma(A|_{\mathcal{M}_+}) \subseteq$ "closed right half plane". Thus, there is a unique A-invariant maximal H-nonnegative subspace $\mathcal{M}_+$, with $\sigma(A|_{\mathcal{M}_+})$ contained in the closed right half plane.

The equivalence of (iii) and (v) follows similarly.

Next, we show (i) $\Leftrightarrow$ (iii). Suppose that (iii) is not satisfied. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be different A-invariant maximal H-nonnegative subspaces with $\sigma(A|_{\mathcal{M}_i})$, $i = 1, 2$ contained in closed right half plane. Choose a real subspace $\mathcal{N}$ which is maximal and strictly H-negative, then $\mathbb{C}^n = \mathcal{M}_1 \dot{+} \mathcal{N}$. With respect to this decomposition we may

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write

$$\tilde{A} = S^{-1}AS = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad \tilde{H} = S^T HS = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{bmatrix},$$

where $H_{11} \geq 0$ and $H_{22} < 0$. So without loss of generality we may assume A and H are already in this form (compare [8]). For $\epsilon > 0$, let

$$A_\epsilon = \begin{bmatrix} A_{11} + \epsilon I & A_{12} \\ 0 & A_{22} - \epsilon I \end{bmatrix}.$$

Then

$$HA_\epsilon + A_\epsilon^T H = HA + A^T H + \begin{bmatrix} 2\epsilon H_{11} & 0 \\ 0 & -2\epsilon H_{22} \end{bmatrix}.$$

Since $H_{11} \geq 0$ and $H_{22} < 0$, we see that A_ϵ is H-positive real. Moreover, $\sigma(A_\epsilon) = \{\sigma(A_{11} + \epsilon I)\} \cup \{\sigma(A_{22} - \epsilon I)\}$. So, $\sigma(A_\epsilon) \cap i\mathbb{R} = \emptyset$ for ϵ small enough. Thus, there is a unique A_ϵ-invariant maximal H-nonnegative subspace which must be $\mathcal{R}(A_\epsilon, \overline{\mathbb{C}}_{right}) = \mathcal{M}_1$.

Indeed, suppose this is not the case. Then, according to Theorem 4.9 in [9] there exists an A_ϵ-invariant H-neutral subspace $\mathcal{N}$ with $\sigma(A_\epsilon|_{\mathcal{N}}) \subset \mathbb{C}_{left}$. In particular, let $z = \begin{bmatrix} x \\ y \end{bmatrix}$ be an eigenvector of A_ϵ in $\mathcal{N}$, and let λ be the corresponding eigenvalue. Then $\operatorname{Re}\lambda < 0$. Consider

$$\langle (HA_\epsilon + A_\epsilon^T H)z, z\rangle = (\lambda + \overline{\lambda})\langle Hz, z\rangle = 0,$$

since $\mathcal{N}$ is a H-neutral subspace. On the other hand, we have that this is equal to

$$\langle (HA + A^T H)z, z\rangle + \epsilon(\langle H_{11}x, x\rangle - \langle H_{22}y, y\rangle) \geq 0.$$

As either term in this expression is larger than or equal to zero, we derive that in particular $\langle H_{22}y, y\rangle = 0$. But, as $H_{22} < 0$ it follows that $y = 0$. Then $z = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$ which cannot be an eigenvector of A_ϵ corresponding to an eigenvalue in the left half plane. Thus, such an A_ϵ-invariant H-neutral subspace cannot exist.

Now let ϵ tend to zero. We see that $\mathcal{R}(A_\epsilon, \overline{\mathbb{C}}_{right})$ converges to $\mathcal{M}_1$. So, $\mathcal{M}_1$ is the only possibility for a stable invariant maximal nonnegative subspace. But, replacing $\mathcal{M}_1$ with $\mathcal{M}_2$ throughout the previous argument we see that there is no stable invariant maximal nonnegative subspace if the numerical range condition is not satisfied, i.e., (i) is not satisfied.

Conversely, assume that (iii) holds. Then, equivalently by Lemma 3.4, the pair (iA, H) satisfies the numerical range condition in the sense of [10]. By Theorem

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A publication of the International Linear Algebra Society
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3.9 (i) there exists a $\mathbb{D}$-stable iA-invariant maximal H-nonnegative subspace $\mathcal{M}_+$. The subspace $\mathcal{M}_+$ is clearly also A-invariant. We now verify that $\mathcal{M}_+$ is also $\mathbb{A}$-stable: Let $\epsilon > 0$ be given. Then there exists a $\delta > 0$ such that for every $B \in \mathbb{D}$ with $\|iA - B\| < \delta$, there exists a nonnegative B-invariant subspace $\mathcal{N}_+$ such that $\theta(\mathcal{M}_+, \mathcal{N}_+) < \epsilon$ (cf. Definition 3.8). Now, if we let $B_0 \in \mathbb{A}$ with $\|A - B_0\| < \delta$, then $iB_0 \in \mathbb{D}$ and also

$$\|iA - iB_0\| = \|A - B_0\| < \delta.$$

Therefore, there exists a iB_0-invariant H-nonnegative subspace $\mathcal{N}_{+0}$ such that $\theta(\mathcal{M}_+, \mathcal{N}_{+0}) < \epsilon$. Since $\mathcal{N}_{+0}$ is also B_0-invariant, it follows that $\mathcal{M}_+$ is $\mathbb{A}$-stable with respect to the pair (A, H). This proves (i) $\Leftrightarrow$ (iii).

The equivalence of (ii) and (iii) follows similarly. $\square$

4. The construction of maximal invariant subspaces. Before giving the main theorem of this article, regarding the existence of the complex A-invariant maximal semidefinite subspaces and their real counterpart, we first give the construction of the invariant subspaces, as it plays an important role in the proof of the theorem. The construction, which is based on results in [10], is reproduced from [5]. In this section, we shall view the real matrix A as acting on $\mathbb{C}^n$.

Let A be an H-positive real matrix with corresponding eigenvalue $\lambda \in i\mathbb{R}$. Let $J(\lambda) = \{J(\lambda, 1), \ldots, J(\lambda, 2\kappa+1)\}$ be a Jordan basis of $\mathcal{R}(A, \{\lambda\})$, where $J(\lambda, j)$ is the set of all elements of the Jordan basis that belong to some Jordan chain of length j. Let $n_{\lambda,j}$ be the number of Jordan chains in the set $J(\lambda, j)$ and let $x_{j,1}, x_{j,2}, \ldots, x_{j,n_{\lambda,j}}$ be the vectors in $J(\lambda, j) \backslash \mathrm{Ker}(A - \lambda)^{j-1}$. Thus, each $x_{j,k}$ is the last element in one of the $n_{\lambda,j}$ chains in $J(\lambda, j)$. Then

$$J(\lambda, j) = \{(A - \lambda)^k x_{j,l} \ \mid \ k = 0, 1, \ldots, j-1 \ \text{ and } \ l = 1, 2, \ldots, n_{\lambda,j}\}.$$

Put $m_j = [\frac{j+1}{2}]$. Let $y_{j,k} = (A - \lambda)^{m_j - 1} x_{j,k}$ for $k = 1, 2, \ldots, n_{\lambda,j}$. If $\mathrm{Sp}\{y_{j,1}, \ldots, y_{j,n_{\lambda,j}}\}$ is nondegenerate, then from Proposition 1.0.3 in [10] it follows that $\mathrm{Sp}\{y_{j,1}, \ldots, y_{j,n_{\lambda,j}}\} = \mathcal{M}_-(\lambda, j)[\dot{+}]\mathcal{M}_+(\lambda, j)$ where $\mathcal{M}_-(\lambda, j)$ and $\mathcal{M}_+(\lambda, j)$ are nonpositive and nonnegative subspaces, respectively. It also follows from the construction that an element, say u, of $\mathcal{M}_-(\lambda, j)$ (similarly for an element of $\mathcal{M}_+(\lambda, j)$) can be written as $u = \sum_{s=1}^{n_{\lambda,j}} g_s y_{j,s}$, for some choice of g_s.

We use the subspaces $\mathcal{M}_-(\lambda, j)$ and $\mathcal{M}_+(\lambda, j)$ to construct invariant maximal nonnegative and nonpositive subspaces as follows:

(i) First observe that

$$[\mathrm{Ker}(A - \lambda)^{m_j - 1} \cap \mathrm{Sp}J(\lambda, j)] \cap \mathcal{M}_-(\lambda, j) = \{0\}.$$

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Indeed, if $u \in (\mathrm{Ker}(A-\lambda)^{m_j-1} \cap \mathrm{Sp}J(\lambda,j)) \cap \mathcal{M}_-(\lambda,j)$, then $u = \sum_{s=1}^{n_{\lambda,j}} g_s y_{j,s}$. Therefore, $(A-\lambda)^{m_j-1}u = 0$, and thus, $\sum_{s=1}^{n_{\lambda,j}} g_s (A-\lambda)^{m_j-1} y_{j,s} = 0$. Since $\{(A-\lambda)^{m_j-1} y_{j,s} \mid s = 1,\ldots,n_{\lambda,j}\}$ is linearly independent, it follows that $g_s = 0$ for $s = 1,2,\ldots,n_{\lambda,j}$ and hence $u = 0$.

(ii) Let $x \in \mathrm{Ker}(A-\lambda)^{m_j-1} \cap \mathrm{Sp}J(\lambda,j)$. Then $x \in \mathrm{Sp}J(\lambda,j)$ means it can be written as a linear combination $x = \sum_{k=0}^{j-1}\sum_{l=1}^{n_{\lambda,j}} \beta_{k,l}(A-\lambda)^k x_{j,l}$ and also since $x \in \mathrm{Ker}(A-\lambda)^{m_j-1}$ it follows that

$$\sum_{k=0}^{j-1}\sum_{l=1}^{n_{\lambda,j}} \beta_{k,l}(A-\lambda)^k (A-\lambda)^{m_j-1} x_{j,l} = (A-\lambda)^{m_j-1}x = 0, \text{ i.e.}$$
$$\sum_{k=0}^{j-1}\sum_{l=1}^{n_{\lambda,j}} \beta_{k,l}(A-\lambda)^{k+m_j-1} x_{j,l} = 0.$$

However, $(A-\lambda)^{k+m_j-1}x_{j,l} = 0$ for $k+m_j-1 \geq j$, that is, for $k \geq j-m_j+1$. We may therefore write $0 = \sum_{k=0}^{j-m_j}\sum_{l=1}^{n_{\lambda,j}} \beta_{k,l}(A-\lambda)^{k+m_j-1}x_{j,l}$. It then follows from the linear independence of the vectors in the Jordan basis that $\beta_{k,l} = 0$ for all $k = 0,1,\ldots,j-m_j$ and $l = 1,2,\ldots,n_{\lambda,j}$. Thus,

$$x = \sum_{k=j-m_j+1}^{j-1}\sum_{l=1}^{n_{\lambda,j}} \beta_{k,l}(A-\lambda)^k x_{j,l}.$$

On the other hand, let $y \in \mathcal{M}_-(\lambda,j)$, then as before $y = \sum_{s=1}^{n_{\lambda,j}} g_s y_{j,s}$. Thus, we have,

$$\begin{aligned}[x,y] &= [\sum_{k=j-m_j+1}^{j-1}\sum_{l=1}^{n_{\lambda,j}} \beta_{k,l}(A-\lambda)^k x_{j,l}, \sum_{s=1}^{n_{\lambda,j}} g_s y_{j,s}] \\ &= \sum_{k=j-m_j+1}^{j-1}\sum_{l=1}^{n_{\lambda,j}}\sum_{s=1}^{n_{\lambda,j}} \beta_{k,l}\overline{g}_s[(A-\lambda)^k x_{j,l}, y_{j,s}].\end{aligned}$$

Here $y_{j,s} = (A-\lambda)^{m_j-1}x_{j,s}$ and $x_{j,s}$ is the last element of the sth Jordan chain, i.e., $y_{j,s}$ is the middle term of the sth chain. Since $x_{j,l}$ is the last element of the lth chain, $(A-\lambda)^k x_{j,l}$ precedes it in the same chain. It follows from Corollary 2.2.6 in [5] that $[(A-\lambda)^k x_{j,l}, y_{j,s}] = 0$. Thus, $[x,y] = 0$.
We may therefore form the orthogonal direct sum:

$$\mathcal{N}_-(\lambda,j) := (\mathrm{Ker}(A-\lambda)^{m_j-1} \cap \mathrm{Sp}J(\lambda,j))[\dot{+}]\mathcal{M}_-(\lambda,j). \tag{4.1}$$

Similarly, using the same arguments, we may form the following direct sum:

$$\mathcal{N}_+(\lambda,j) := (\mathrm{Ker}(A-\lambda)^{m_j-1} \cap \mathrm{Sp}J(\lambda,j))[\dot{+}]\mathcal{M}_+(\lambda,j). \tag{4.2}$$

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(iii) It remains to show that $\mathcal{N}_-(\lambda, j)$ and $\mathcal{N}_+(\lambda, j)$ are nonpositive and nonnegative subspaces respectively. We only prove the nonpositivity of $\mathcal{N}_-(\lambda, j)$; the nonnegativity of $\mathcal{N}_+(\lambda, j)$ follows similarly.

Let $x = u+v$ with $u \in \mathcal{M}_-(\lambda, j)$ and $v \in (\mathrm{Ker}(A-\lambda)^{m_j-1} \cap \mathrm{Sp}J(\lambda, j))$. Thus, $u = \sum_{s=1}^{n_{\lambda,j}} g_s y_{j,s}$ and $v = \sum_{k=0}^{j-1} \sum_{l=1}^{n_{\lambda,j}} \beta_{k,l}(A-\lambda)^k x_{j,l}$. Now, $[x, x] = [u + v, u + v] = [u, u] + [u, v] + [v, u] + [v, v]$. Clearly $[u, u] \leq 0$, since $u \in \mathcal{M}_-(\lambda, j)$ and $[u, v] = [v, u] = 0$ from the orthogonal direct sum (see (ii)). We now prove that $[v, v] = 0$. Since $v \in (\mathrm{Ker}(A - \lambda)^{m_j-1} \cap \mathrm{Sp}J(\lambda, j))$, we obtain as before

$$v = \sum_{k=j-m_j+1}^{j-1} \sum_{l=1}^{n_{\lambda,j}} \beta_{k,l}(A - \lambda)^k x_{j,l}.$$

The vector v is the linear combination of the first $m_j - 1$ elements of the Jordan chains of length j in the Jordan basis. Now from Lemma 2.2.11 (iii) and Corollary 2.2.6 in [5], it follows that $[v, v] = 0$. Hence, $\mathcal{N}_-(\lambda, j)$ is a nonpositive subspace.

(iv) We prove for $x \in \mathcal{N}_-(\lambda, j) \cup \mathcal{N}_+(\lambda, j)$ that $A^{[*]}x = -Ax$. Let $x \in \mathcal{N}_-(\lambda, j)$. As before, $x = u+v$ with $u \in \mathcal{M}_-(\lambda, j)$ and $v \in (\mathrm{Ker}(A-\lambda)^{m_j-1} \cap \mathrm{Sp}J(\lambda, j))$. Now, $A^{[*]}x = A^{[*]}(u + v) = A^{[*]}u + A^{[*]}v$ and from Lemma 2.2.2 in [5] we have that,

$$A^{[*]}u = \sum_{s=1}^{n_{\lambda,j}} g_s A^{[*]} y_{j,s} = -\sum_{s=1}^{n_{\lambda,j}} g_s A y_{j,s} = -Au$$

and

$$A^{[*]}v = \sum_{k=0}^{j-1} \sum_{l=1}^{n_{\lambda,j}} \beta_{k,l} A^{[*]}(A - \lambda)^k x_{j,l} = \sum_{k=0}^{j-1} \sum_{l=1}^{n_{\lambda,j}} \beta_{k,l}(-A(A - \lambda)^k x_{j,l}) = -Av.$$

Thus, $A^{[*]}x = -Au - Av = -A(u + v) = -Ax$. Similarly it can be shown that $A^{[*]}x = -Ax$ for all $x \in \mathcal{N}_+(\lambda, j)$. Thus, it holds for all $x \in \mathcal{N}_-(\lambda, j) \cup \mathcal{N}_+(\lambda, j)$.

THEOREM 4.1. *Let A be H-positive real and $\lambda \in i\mathbb{R}$ an eigenvalue of A. Then $\mathcal{R}(A, \{\lambda\})$ is nondegenerate. For any $j \in \{1, 3, \ldots, 2\kappa+1\}$ let $\mathcal{N}_-(\lambda, j)$ and $\mathcal{N}_+(\lambda, j)$ be as in* (4.1) *and* (4.2)*, respectively. Then the subspace $\mathcal{N}_-(\lambda, j)$ is A-invariant and nonpositive, the subspace $\mathcal{N}_+(\lambda, j)$ is A-invariant and nonnegative and* $\dim \mathcal{N}_-(\lambda, j) + \dim \mathcal{N}_+(\lambda, j) = \dim SpJ(\lambda, j)$.

Proof. The fact that $\mathcal{R}(A, \{\lambda\})$ is nondegenerate follows from [2], see also Corollary 2.2.10 in [5]. For λ a pure imaginary eigenvalue of the H-positive real matrix A,

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A publication of the International Linear Algebra Society
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we know that $i\lambda$ is a real eigenvalue of the H-dissipative matrix iA. We can therefore claim that $\mathcal{N}_-(\lambda, j)$ (and also $\mathcal{N}_+(\lambda, j)$) of the matrix A, equals $\mathcal{N}_-(i\lambda, j)$ (and $\mathcal{N}_+(i\lambda, j)$) of the matrix iA, respectively. This is true because one finds that for the matrix A (with eigenvalue $\lambda \in i\mathbb{R}$) and iA (with eigenvalue $i\lambda \in \mathbb{R}$) that

(i) $\mathrm{Sp}J(i\lambda, j) = \mathrm{Sp}J(\lambda, j)$
(ii) $\mathrm{Ker}(iA - i\lambda)^{m_j-1} = \mathrm{Ker}((i)^{m_j-1}(A - \lambda)^{m_j-1}) = \mathrm{Ker}(A - \lambda)^{m_j-1}$
(iii) $\mathcal{M}_-(\lambda, j) = \mathcal{M}_-(i\lambda, j)$, since

$$u \in \mathcal{M}_-(i\lambda, j) \Leftrightarrow u = \sum_{s=1}^{n_{\lambda,j}} g_s y'_{j,s} = \sum_{s=1}^{n_{\lambda,j}} (i)^{\frac{3(j-1)}{2}} g_s y_{j,s},$$

where $y_{j,s}$ is as before (see (i)) and $y'_{j,s} = (i)^{\frac{3(j-1)}{2}} y_{j,s}$ is as in the proof of Lemma 2.2.13 in [5].

Therefore, from Theorem 2.3.5 in [10], it follows that $\mathcal{N}_-(\lambda, j)$ and $\mathcal{N}_+(\lambda, j)$ are A-invariant and

$$\dim \mathcal{N}_-(\lambda, j) + \dim \mathcal{N}_+(\lambda, j) = \dim \mathrm{Sp}J(\lambda, j). \quad \square$$

Furthermore,

(i) $\dim \mathrm{Sp}J(\lambda, j) = jn_{\lambda,j}$;
(ii) $\dim \mathcal{N}_-(\lambda, j) = (m_j - 1)n_{\lambda,j} + \dim \mathcal{M}_-(\lambda, j)$;
(iii) $\dim \mathcal{N}_+(\lambda, j) = (m_j - 1)n_{\lambda,j} + \dim \mathcal{M}_+(\lambda, j)$;
(iv) $\dim \mathcal{M}_-(\lambda, j) + \dim \mathcal{M}_+(\lambda, j) = n_{\lambda,j}$.

For even j, we have $m_j = \frac{j}{2}$. For $j \in \{2, 4, \ldots, 2\kappa\}$, let

$$\mathcal{N}(\lambda, j) = \mathcal{N}_-(\lambda, j) = \mathcal{N}_+(\lambda, j) = \mathrm{Ker}(A - \lambda)^{m_j} \cap \mathrm{Sp}J(\lambda, j). \tag{4.3}$$

From the proof of Theorem 4.1 we observe that for an H-positive real matrix A with eigenvalue $\lambda \in i\mathbb{R}$ (and iA being H-dissipative with eigenvalue $i\lambda \in \mathbb{R}$) that

$$\begin{aligned} \mathrm{Ker}(A - \lambda)^{m_j} &= \mathrm{Ker}(iA - i\lambda)^{m_j}, \\ \mathrm{Sp}J(\lambda, j) &= \mathrm{Sp}J(i\lambda, j). \end{aligned}$$

Therefore, we can conclude that $\mathcal{N}(\lambda, j) = \mathcal{N}(i\lambda, j)$. Then according to Theorem 2.3.6 in [10], it follows that $\mathcal{N}(\lambda, j)$ is a neutral subspace. Indeed, we use Lemma 2.2.11 in [5] as follows: if $v \in \mathcal{N}(\lambda, j) = \mathrm{Ker}(A - \lambda)^{m_j} \cap \mathrm{Sp}J(\lambda, j)$, then since $v \in \mathrm{Sp}J(\lambda, j)$ it follows that

$$v = \sum_{k=0}^{j-1} \sum_{l=1}^{n_{\lambda,j}} \beta_{k,l} (A - \lambda)^k x_{j,l}$$

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A publication of the International Linear Algebra Society
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and because $v \in \operatorname{Ker}(A-\lambda)^{m_j}$, it follows that

$$\sum_{k=0}^{j-1}\sum_{l=1}^{n_{\lambda,j}} \beta_{k,l}(A-\lambda)^k(A-\lambda)^{m_j}x_{j,l} = 0, \text{ i.e.}$$
$$\sum_{k=0}^{j-1}\sum_{l=1}^{n_{\lambda,j}} \beta_{k,l}(A-\lambda)^{k+m_j}x_{j,l} = 0.$$

However, $(A-\lambda)^{k+m_j}x_{j,l} = 0$ for $k+m_j \geq j$, that is, for $k \geq j-m_j$. We may therefore write $0 = \sum_{k=0}^{j-m_j-1}\sum_{l=1}^{n_{\lambda,j}} \beta_{k,l}(A-\lambda)^{k+m_j}x_{j,l}$. It therefore follows from the linear independence of the vectors in the Jordan basis that $\beta_{k,l} = 0$ for all $k = 0, 1, \ldots, j-m_j-1$ and $l = 1, 2, \ldots, n_{\lambda,j}$. Thus,

$$v = \sum_{k=j-m_j}^{j-1}\sum_{l=1}^{n_{\lambda,j}} \beta_{k,l}(A-\lambda)^k x_{j,l}.$$

The vector v is the linear combination of the first m_j elements of the Jordan chains of length j in the Jordan basis. Now from Lemma 2.2.11 (iii) in [5] it follows that $[v, v] = 0$. Hence, $\mathcal{N}(\lambda, j)$ is a neutral subspace.

In particular, it also follows from Lemma 2.3.6 in [10] that $\mathcal{N}_-(\lambda, j)$ is an A-invariant (also iA-invariant) nonpositive subspace and $\mathcal{N}_+(\lambda, j)$ is an A-invariant nonnegative subspace. From (i) and (ii) in the proof of Theorem 4.1 it follows that

$$\dim \mathcal{N}_-(\lambda, j) + \dim \mathcal{N}_+(\lambda, j) = \dim \operatorname{Sp} J(\lambda, j).$$

Also, as before $A^{[*]}x = -Ax$ for all $x \in \mathcal{N}(\lambda, j)$.

For odd and even j, it follows from the proof of Theorem 4.1 that

$$\mathcal{N}_-(\lambda, j) = \mathcal{N}_-(i\lambda, j),$$
$$\mathcal{N}_+(\lambda, j) = \mathcal{N}_+(i\lambda, j).$$

With $\mathcal{N}_-(\lambda, j)$ and $\mathcal{N}_+(\lambda, j)$ as in (4.1) to (4.3), let

$$\mathcal{N}_-(\lambda) = \mathcal{N}_-(\lambda, 1)[\dot{+}]\mathcal{N}_-(\lambda, 2)[\dot{+}]\cdots[\dot{+}]\mathcal{N}_-(\lambda, 2\kappa+1), \tag{4.4}$$

$$\mathcal{N}_+(\lambda) = \mathcal{N}_+(\lambda, 1)[\dot{+}]\mathcal{N}_+(\lambda, 2)[\dot{+}]\cdots[\dot{+}]\mathcal{N}_+(\lambda, 2\kappa+1). \tag{4.5}$$

Thus, we get that

$$\mathcal{N}_-(\lambda) = \mathcal{N}_-(i\lambda), \qquad \mathcal{N}_+(\lambda) = \mathcal{N}_+(i\lambda).$$

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A publication of the International Linear Algebra Society
Volume 27, pp. 124-145, February 2014

According to Corollary 2.2.6 in [5], the direct sums are indeed orthogonal direct sums. The subspaces $\mathcal{N}_-(\lambda)$ and $\mathcal{N}_+(\lambda)$ are nonpositive, respectively, nonnegative subspaces of maximal dimension within $\mathcal{R}(A, \{\lambda\})$. From the preceding discussion and combining with Theorem 2.3.7 in [10] the following theorem holds:

THEOREM 4.2. *Let $\lambda \in i\mathbb{R}$ be an eigenvalue of an H-positive real matrix A. The subspace $\mathcal{N}_-(\lambda)$ from* (4.4) *is A-invariant nonpositive, the subspace $\mathcal{N}_+(\lambda)$ from* (4.5) *is A-invariant nonnegative and*

$$\dim \mathcal{N}_-(\lambda) + \dim \mathcal{N}_+(\lambda) = \dim \mathcal{R}(A, \{\lambda\}).$$

Moreover, $A^{[*]}x = -Ax$ for all $x \in \mathcal{N}_-(\lambda) \cup \mathcal{N}_+(\lambda)$.

For distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with zero real parts of an H-positive real matrix A, let

$$\mathcal{N}_- = \mathcal{N}_-(\lambda_1)[\dot{+}]\mathcal{N}_-(\lambda_2)[\dot{+}]\cdots[\dot{+}]\mathcal{N}_-(\lambda_k)$$

and

$$\mathcal{N}_+ = \mathcal{N}_+(\lambda_1)[\dot{+}]\mathcal{N}_+(\lambda_2)[\dot{+}]\cdots[\dot{+}]\mathcal{N}_+(\lambda_k).$$

The subspaces $\mathcal{N}_-(\lambda_i)$ and $\mathcal{N}_+(\lambda_i)$ for $i = 1, 2, \ldots, k$ are constructed as in (4.4) and (4.5). It follows from Corollary 2.2.8 in [5] that the direct sums are orthogonal direct sums. Hence, the subspaces $\mathcal{N}_-$ and $\mathcal{N}_+$ are the same as the corresponding subspaces $\mathcal{N}_-$ and $\mathcal{N}_+$ as in [10], p.37.

Now let,

$$\mathcal{M}_- = \mathcal{N}_-[\dot{+}]\mathcal{R}(A, \mathbb{C}_{left}), \tag{4.6}$$

$$\mathcal{M}_+ = \mathcal{N}_+[\dot{+}]\mathcal{R}(A, \mathbb{C}_{right}). \tag{4.7}$$

By Lemma 2.2.9 in [5], it means that

$$\mathcal{M}_- = \mathcal{N}_-[\dot{+}]\mathcal{R}(iA, \mathbb{C}_{low}),$$
$$\mathcal{M}_+ = \mathcal{N}_+[\dot{+}]\mathcal{R}(iA, \mathbb{C}_{upp}).$$

Thus, we get the same subspaces $\mathcal{M}_-$ and $\mathcal{M}_+$ as in [10] for real eigenvalues. Therefore, it follows that $\mathcal{M}_-$ from (4.6) and $\mathcal{M}_+$ from (4.7) are A-invariant maximal nonpositive and nonnegative subspaces, respectively.

We now turn to the viewpoint that the real matrix A is viewed as a map from $\mathbb{R}^n$ to itself. We consider the construction of a real maximal H-nonnegative A-invariant

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A publication of the International Linear Algebra Society
Volume 27, pp. 124-145, February 2014

subspace. A subspace $\mathcal{M}$ of $\mathbb{C}^n$ will be called real if $\mathcal{M} = \overline{\mathcal{M}}$. Observe that, since A is real, we have that for any complex subspace $\mathcal{M}$ of $\mathbb{C}^n$ $A\overline{\mathcal{M}} = \overline{A\mathcal{M}}$. In particular, $\mathcal{R}(A, \{\bar{\lambda}\}) = \overline{\mathcal{R}(A, \{\lambda\})}$, and thus, $\mathcal{R}(A, \mathbb{C}_{left})$ and $\mathcal{R}(A, \mathbb{C}_{right})$ are real subspaces. In addition, when $\lambda \in i\mathbb{R}$, then $\mathcal{R}(A, \{\lambda\}) \dot{+} \mathcal{R}(A, \{\bar{\lambda}\})$ is real, A-invariant and H-nondegenerate. The simple form developed in [4] for the real case, combined with the construction above, then shows that the subspaces $\mathcal{N}_{\pm}(\bar{\lambda}) = \overline{\mathcal{N}_{\pm}(\lambda)}$. It follows that the subspaces $\mathcal{N}_{\pm}(\lambda) \dot{+} \mathcal{N}_{\pm}(\bar{\lambda})$ are real. This implies that also $\mathcal{N}_{\pm}$ are real, and hence also $\mathcal{M}_{\pm}$. In particular, there exists a basis consisting of real vectors for these subspaces.

5. The main theorem. We state here the main theorem of this article. Note that the four equivalent statements mentioned in the introduction are extended here with an additional statement (e), which at this point will be clear to the reader.

THEOREM 5.1. *The following statements are equivalent for a given real H-positive real matrix A.*

(a) There exists a unique complex A-invariant maximal H-nonnegative subspace $\mathcal{M}$, such that $\sigma(A|_{\mathcal{M}}) \subseteq \overline{\mathbb{C}}_{right}$.
(b) There exists a unique real A-invariant maximal H-nonnegative subspace $\mathcal{M}$, such that $\sigma(A|_{\mathcal{M}}) \subseteq \overline{\mathbb{C}}_{right}$.
(c) There exists a unique complex A-invariant maximal H-nonpositive subspace $\mathcal{M}$, such that $\sigma(A|_{\mathcal{M}}) \subseteq \overline{\mathbb{C}}_{left}$.
(d) There exists a unique real A-invariant maximal H-nonpositive subspace $\mathcal{M}$, such that $\sigma(A|_{\mathcal{M}}) \subseteq \overline{\mathbb{C}}_{left}$.
(e) The numerical range condition is satisfied.

Proof. We consider A as a linear transformation acting on $\mathbb{C}^n$, i.e., $A : \mathbb{C}^n \to \mathbb{C}^n$, and we define an indefinite inner product on $\mathbb{C}^n$ by $[x, y] := \langle Hx, y\rangle$, where $x, y \in \mathbb{C}^n$, and $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{C}^n$.

(a)$\Rightarrow$(b): Assume (a), i.e., there exists an unique complex A-invariant maximal H-nonnegative subspace $\mathcal{M}$, with $\sigma(A|_M) \subseteq \overline{\mathbb{C}}_{right}$, such that $\langle Hx, x\rangle \geq 0$ for all $x \in \mathcal{M}$.

By the construction of the A invariant maximal H-nonnegative and H-nonpositive subspaces in the previous section, there is a real basis for $\mathcal{M}$. So, let $\{x_1, x_2, \ldots, x_m\}$ be a real basis in $\mathcal{M}$, and let $\mathcal{N}$ be the linear span over $\mathbb{R}$, i.e.,

$$\mathcal{N} = \{x = \sum_{j=1}^{m} \alpha_j x_j | \alpha_j \in \mathbb{R}\}.$$

Observe that $\mathcal{M} = \mathcal{N} + i\mathcal{N}$. For $x \in \mathcal{N}$ we have $x = \overline{x}$, so that $Ax = A\overline{x} = \overline{Ax} \in \mathcal{M}$, since $\mathcal{M}$ is A-invariant. But $x \in \mathcal{N}$ if and only if $x \in \mathcal{M}$ and $x = \overline{x}$, thus $Ax \in \mathcal{N}$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
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Hence, $\mathcal{N}$ is A-invariant. Also, since $\mathcal{N}$ is considered as a subspace over $\mathbb{R}$ and has the same basis as $\mathcal{M}$ over $\mathbb{C}$, it follows that $\dim \mathcal{N} = \dim \mathcal{M} = m$. Furthermore, $\mathcal{N}$ is H-nonnegative since $\mathcal{N} \subset \mathcal{M}$, thus if $x \in \mathcal{N} \subset \mathcal{M}$ it follows that $\langle Hx, x\rangle \geq 0$. Hence, we have that $\mathcal{N}$ is A-invariant and H-nonnegative. The subspace $\mathcal{N}$ satisfies $\sigma(A|_{\mathcal{N}}) = \sigma(A|_{\mathcal{M}}) \subset \overline{\mathbb{C}}_{right}$: Let $\lambda \in \sigma(A|_{\mathcal{M}})$, then there exist a $0 \neq x \in \mathcal{M}$, such that $Ax = \lambda x$. Since $\mathcal{M} = \mathcal{N} + i\mathcal{N}$, there exist $x_1, x_2 \in \mathcal{N}$ such that $x = x_1 + ix_2$ (where either $x_1 \neq 0$ or $x_2 \neq 0$ or both $x_1, x_2 \neq 0$). Now, $Ax = \lambda x \Leftrightarrow Ax_1 + iAx_2 = \lambda x_1 + i\lambda x_2$, i.e., $Ax_1 = \lambda x_1$ and $Ax_2 = \lambda x_2$. Thus, $\sigma(A|_{\mathcal{M}}) \subseteq \sigma(A|_{\mathcal{N}})$. The inclusion $\sigma(A|_{\mathcal{N}}) \subseteq \sigma(A|_{\mathcal{M}})$ is obvious.

We now prove the maximality of $\mathcal{N}$. To see this, let $\tilde{\mathcal{N}}$ be a real H-nonnegative subspace such that $\sigma(A|_{\tilde{\mathcal{N}}}) \subseteq \overline{\mathbb{C}}_{right}$ and $\mathcal{N} \subseteq \tilde{\mathcal{N}}$ and $A(\tilde{\mathcal{N}}) \subseteq \tilde{\mathcal{N}}$. Let $\tilde{\mathcal{M}} = \tilde{\mathcal{N}} + i\tilde{\mathcal{N}}$, then $\mathcal{M} \subseteq \tilde{\mathcal{M}}$. Now $A(\tilde{\mathcal{M}}) = A(\tilde{\mathcal{N}}) + iA(\tilde{\mathcal{N}}) \subseteq \tilde{\mathcal{N}} + i\tilde{\mathcal{N}} = \tilde{\mathcal{M}}$, i.e., $\tilde{\mathcal{M}}$ is A-invariant. Let $x \in \tilde{\mathcal{M}}$ then $x = x_1 + ix_2$ with $x_1, x_2 \in \tilde{\mathcal{N}}$. Then

$$\begin{aligned}\langle H(x_1 + ix_2), x_1 + ix_2\rangle &= \langle Hx_1, x_1\rangle + i\langle Hx_2, x_1\rangle - i\langle Hx_1, x_2\rangle + \langle Hx_2, x_2\rangle \\ &= \langle Hx_1, x_1\rangle + \langle Hx_2, x_2\rangle \\ &\geq 0,\end{aligned}$$

which proves $\tilde{\mathcal{M}}$ is H-nonnegative. As before, $\sigma(A|_{\tilde{\mathcal{M}}}) = \sigma(A|_{\tilde{\mathcal{N}}}) \subseteq \overline{\mathbb{C}}_{right}$. The maximality of $\mathcal{M}$ implies that $\mathcal{M} = \tilde{\mathcal{M}}$. Thus, $\mathcal{N} = \tilde{\mathcal{N}}$, and therefore, $\mathcal{N}$ is maximal.

To prove the uniqueness, let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two real A-invariant maximal H-nonnegative subspaces such that $\sigma(A|_{N_j}) \subseteq \overline{\mathbb{C}}_{right}$ for $j = 1, 2$. Set $\mathcal{M}_j = \{x + iy | x, y \in \mathcal{N}_j\}$. As before $\mathcal{M}_j$ is A-invariant and H-nonnegative such that $\sigma(A|_{\mathcal{M}_j}) \subseteq \overline{\mathbb{C}}_{right}$. We prove that $\mathcal{M}_j$ is maximal with these properties: Suppose not. Let $\tilde{\mathcal{M}}_j$ have the same properties where $\mathcal{M}_j \subset \tilde{\mathcal{M}}_j$. As before, let $\tilde{\mathcal{M}}_j = \tilde{\mathcal{N}}_j + i\tilde{\mathcal{N}}_j$ with $\tilde{\mathcal{N}}_j$ being a real A-invariant, maximal and H-nonnegative subspace. Then $\mathcal{N}_j \subset \tilde{\mathcal{N}}_j$ for $j = 1, 2$. The maximality of $\mathcal{N}_j$ with these properties then implies that $\mathcal{N}_j = \tilde{\mathcal{N}}_j$, i.e., $\mathcal{M}_j = \tilde{\mathcal{M}}_j$. By the uniqueness property it follows that $\mathcal{M}_1 = \mathcal{M}_2$. Thus, $\mathcal{N}_1 = \mathcal{N}_2$.

Therefore, there exists a unique real A-invariant maximal H-nonnegative subspace $\mathcal{M}$ such that $\sigma(A|_{\mathcal{M}}) \subseteq \overline{\mathbb{C}}_{right}$. Thus, we have proved that (a) implies (b).

We next show that (b) implies (e) and thereafter that (e) implies (a). The remaining relations are proved similarly, i.e., (c) implies (d), (d) implies (e) and again that (e) implies (c).

Suppose that the numerical range condition for H-positive real matrices is not satisfied. Then there exists a $\lambda \in i\mathbb{R}$ for which it fails to hold, and then it fails to hold at $\overline{\lambda}$ by the simple form of the pair (A, H), (see [5]). In the construction above, in Theorem 4.1, there exists an A-invariant subspace $\mathcal{N}_+$, different from $\mathcal{N}_+(\lambda, j)$ which is maximal H-nonnegative in $\mathcal{R}(A, \{\lambda\})$.

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Construct $\tilde{\mathcal{M}}_+(\lambda,\overline{\lambda}) = \mathcal{N}_+ \dot{+} \overline{\mathcal{N}_+}$ and define $\tilde{\mathcal{M}}_+$ by replacing in the construction in Section 4, the subspace $\mathcal{N}_+(\lambda)\dot{+}\mathcal{N}_+(\overline{\lambda})$ by $\tilde{\mathcal{M}}_+(\lambda,\overline{\lambda})$. Then, $\tilde{\mathcal{M}}_+$ is A-invariant, real and maximal H-nonnegative. So, there is not a unique real maximal H-nonnegative subspace. Therefore, (b) does not hold and we have shown (b) implies (e).

The implication (e) implies (a) is actually part of Theorem 3.10. □

6. Examples. We consider a few examples in this section.

Example 6.1. Let

$$A = \begin{bmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{bmatrix}, \qquad H = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}.$$

Then the numerical range condition does not hold (cf. Example 3.5). It is easy to verify that

$$\mathcal{M}(\alpha) = \mathrm{Sp}\left\{ \begin{bmatrix} 1 \\ 0 \\ \alpha \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \alpha \end{bmatrix} \right\}$$

is a real A-invariant subspace for all $\alpha \in \mathbb{R}$. Furthermore, we have

$$\left\langle H \begin{bmatrix} 1 \\ 0 \\ \alpha \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ \alpha \\ 0 \end{bmatrix} \right\rangle = 1-\alpha^2, \quad \left\langle H \begin{bmatrix} 0 \\ 1 \\ 0 \\ \alpha \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \alpha \end{bmatrix} \right\rangle = 1-\alpha^2, \quad \left\langle H \begin{bmatrix} 1 \\ 0 \\ \alpha \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \alpha \end{bmatrix} \right\rangle = 0.$$

Thus, $\mathcal{M}(\alpha)$ is maximal H-nonnegative if and only if $|\alpha| < 1$. So, there are infinitely many real A-invariant maximal H-nonnegative subspaces. This confirms Theorem 5.1.

Example 6.2. In this example, the matrix A we consider will always be

$$A = J_2(0) \oplus J_2(0)$$

but the matrix H determining the indefinite inner product will be different. This is connected to the following issue: from the construction of the simple form for H as given in [5], we know that it is possible that for the matrix A under consideration $\mathrm{rank}(HA + A^T H)$ takes three values: it is either 0, 1 or 2.

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A publication of the International Linear Algebra Society
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Case 1. Consider the case $H_0A + A^TH_0 = 0$. Then it is known (see, e.g, [7]) that H_0 has the following canonical form

$$H_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Case 2. In the case $\operatorname{rank}(H_1A + A^TH_1) = 1$, H_1 has the following canonical form (compare [3], Section 3):

$$H_1 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Case 3. If $\operatorname{rank}(H_2A + A^TH_2) = 2$, then H_2 has the following canonical form for some real number v (compare [3], Section 3):

$$H_2 = \begin{bmatrix} 0 & 1 & 0 & v \\ 1 & 0 & -v & 0 \\ 0 & -v & 0 & 1 \\ v & 0 & 1 & 0 \end{bmatrix}.$$

Observe that in both Cases 1 and 2 there is clearly more than one A-invariant H-nonnegative subpace, namely, both $\operatorname{Ker} A$ and $\operatorname{Sp}\{e_3, e_4\}$ are A-invariant and H-nonnegative (in fact H-neutral). Let us analyse what happens in Case 3. An A-invariant maximal H-nonnegative subspace has dimension 2. The 2-dimensional A-invariant subspaces $\mathcal{M}$ come in two types: either $\dim((\operatorname{Ker} A) \cap \mathcal{M}) = 1$, or $\dim((\operatorname{Ker} A) \cap \mathcal{M}) = 2$. In the latter case $\mathcal{M} = \operatorname{Ker} A$. In the former case, let

$$(\operatorname{Ker} A) \cap \mathcal{M} = \operatorname{Sp}\left\{ \begin{bmatrix} x_1 \\ 0 \\ x_2 \\ 0 \end{bmatrix} \right\},$$

with $x_1^2 + x_2^2 \neq 0$. Then, for some y_1 and y_2

$$\mathcal{M} = \operatorname{Sp}\left\{ \begin{bmatrix} x_1 \\ 0 \\ x_2 \\ 0 \end{bmatrix}, \begin{bmatrix} y_1 \\ x_1 \\ y_2 \\ x_2 \end{bmatrix} \right\}.$$

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A publication of the International Linear Algebra Society
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Let us consider the Gram matrix of this basis for $\mathcal{M}$ with respect to the indefinite inner product given by H_2. A straigthforward computation gives that this is equal to

$$\begin{bmatrix} 0 & x_1^2 + x_2^2 \\ x_1^2 + x_2^2 & * \end{bmatrix},$$

where $*$ denotes a number the value of which is immaterial to our purpose. Since $x_1^2 + x_2^2 \neq 0$, this Gram matrix is indefinite, and so $\mathcal{M}$ is H_2-indefinite.

We conclude that the following holds: if $A = J_2(0) \oplus J_2(0)$ is H-positive real, then there exists a unique A-invariant maximal H-nonnegative subspace if and only if the pair (A, H) is such that $\operatorname{rank}(HA + A^*H) = 2$, and in that case the unique A-invariant maximal H-nonnegative subspace is $\operatorname{Ker} A$.

Next, let us consider the numerical range condition. The relevant matrices are as follows:

Case 1. $CM_{even} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, with the set $\{\langle CM_{even}x, x\rangle \mid x \neq 0\}$ equal to $\{0\}$,

Case 2. $CM_{even} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, with the set $\{\langle CM_{even}x, x\rangle \mid x \neq 0\}$ containing 0 (take $x = e_2$),

Case 3. $CM_{even} = \begin{bmatrix} 1 & -v \\ v & 1 \end{bmatrix}$, with the set $\{\langle CM_{even}x, x\rangle \mid x \neq 0\}$ just being $\mathbb{R}_+$. Indeed, in this case, CM_{even} is of the form $I + K$ for a skew-symmetric K, and so $\langle CM_{even}x, x\rangle = \|x\|^2 > 0$.

So, only in the case where $\operatorname{rank}(HA + A^T H) = 2$ does the numerical range condition hold. Of course, this completely agrees with the fact that this is the only case in which there is uniqueness of the A-invariant maximal H-nonnegative subspace.

Acknowledgement. The authors thank the anonymous referee for valuable comments, which led to an improvement of the presentation of the paper.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 124-145, February 2014

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