Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 100-123, February 2014

ELA

MINIMUM VECTOR RANK AND COMPLEMENT CRITICAL GRAPHS*

XIAOWEI LI†, MICHAEL NATHANSON‡, AND RACHEL PHILLIPS§

Abstract. Given a graph G, a real orthogonal representation of G is a function from its set of vertices to $\mathbb{R}^d$ such that two vertices are mapped to orthogonal unit vectors if and only if they are not neighbors. The minimum vector rank of a graph is the smallest dimension d for which such a representation exists. This quantity is closely related to the minimum semidefinite rank of G, which has been widely studied. Considering the minimum vector rank as an analogue of the chromatic number, this work defines critical graphs as those for which the removal of any vertex decreases the minimum vector rank; and complement critical graphs as those for which the removal of any vertex decreases the minimum vector rank of either the graph or its complement. It establishes necessary and sufficient conditions for certain classes of graphs to be complement critical, in the process calculating their minimum vector rank. In addition, this work demonstrates that complement critical graphs form a sufficient set to prove the Graph Complement Conjecture, which remains open.

Key words. Minimum vector rank, Minimum semidefinite rank, Orthogonal representation, Complement critical graph, Graph complement conjecture.

AMS subject classifications. 05C50, 15A03, 15A18.

1. Introduction.

There is a well-established rich interplay between linear algebra and graph theory. In particular, the association of a graph with a matrix (or a class of matrices) allows properties of the graph to be encoded with algebraic structure. It is also often fruitful to define graph parameters, which are functions from the set of graphs to (for example) the nonnegative integers; perhaps the most famous of these is the *chromatic number* of a graph. The minimum rank of a graph is an example of both of these. It is a graph parameter which is defined in terms of an associated class of matrices (discussed below). This type of inquiry was initiated in [15]; and the specific question of minimizing the rank of a matrix with given zero pattern was the subject of a 2006 American Institute of Mathematics workshop [2] and has generated much interest since then (e.g., [1, 3, 4, 5, 6, 7, 13, 14]). See [10, 11]

*Received by the editors on April 12, 2013. Accepted for publication on January 25, 2014. Handling Editor: Leslie Hogben. Research supported by the Saint Mary's College Summer Science Research Program.

†Department of Mathematics and Computer Science, Saint Mary's College of California, Moraga, CA 94556.

‡Department of Mathematics and Computer Science, Saint Mary's College of California, Moraga, CA 94556 (man6@stmarys-ca.edu).

§Graduate Program in Analytics, University of San Francisco, San Francisco, CA 94117.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 100-123, February 2014

ELA

for thorough surveys of known results and lines of inquiry.

The present work has origins in the work of Haynes, et al. [12], which looked at orthogonal representations of graphs as an analogue of graph coloring, a strategy which we will employ here. In particular, we will look for the smallest dimension in which we can embed an orthogonal representation of a graph. One possible interpretation of such a vector representation is as a set of quantum states, necessarily unit vectors, in which case the associated graph G is the confusability graph of these states [8, 9]. We will not pursue this interpretation here, but it motivates our desire to focus on representations consisting of nonzero vectors. By contrast, much of the literature focuses on the relation between graphs and their associated matrices (e.g., [1, 7, 11]), which do not explicitly give a "coloring" of a graph using vectors. The ideas are closely related, however, and the differences are mostly linguistic.

The main theme of our current work is *complement criticality*. Recall that a critical graph is one such that any proper subgraph has a strictly smaller chromatic number. We wish to look at analogous graphs with respect to the minimal vector rank, defining *vector critical graphs* to be those such that any proper induced subgraph has a strictly smaller minimum vector rank. In addition, it will also be interesting to look at *complement critical graphs*, in which any proper induced subgraph or its complement has a strictly smaller minimum vector rank. That is, if H is a proper induced subgraph of G, then $\mathrm{mvr}(H) + \mathrm{mvr}(\overline{H}) < \mathrm{mvr}(G) + \mathrm{mvr}(\overline{G})$. These terms are more formally defined in Section 3.

On question posed during the 2006 AIM workshop has gained much attention: How large can the sum of the minimum rank of a graph and its complement be? This question (and its conjectured answer) has become known as the Graph Complement Conjecture. We will state this conjecture in Section 2 and show its connections to our current work.

The rest of the paper is organized as follows. Section 2 introduces basic definitions and properties of the minimum vector rank. In Section 3, we give a precise definition of complement critical graphs and explore its immediate consequences. Section 4 establishes formulas for the minimum vector rank of the complement of certain sparse graphs, which allows us in Section 5 to give necessary and sufficient conditions for certain types of graphs to be complement critical. Finally, in Section 6, we briefly restate our results in terms of the minimum semidefinite rank, to illuminate the connection to other work. There is also an Appendix, in which we include some of the more technical proofs from Section 4.

2. The minimum vector rank of a graph. We recall some standard definitions: A *graph* is a pair $G = (V, E)$, where V a finite non-empty set of vertices and $E \subset V \times V$ is the set of edges. We will restrict our focus to *simple* graphs, in which

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 100-123, February 2014

ELA

edges are undirected and no loops or multiple edges are allowed. The *order* of a graph G, denoted $|G|$, is the size of its vertex set. Two vertices u and v in V are *neighbors* if and only if $uv \in E$; the neighborhood of v is denoted $N(v) := \{u \in V : uv \in E\}$. The *degree* of a vertex is its number of neighbors: $\deg(v) = |N(v)|$. A *pendant* vertex has exactly one neighbor, while an *isolated* vertex has no neighbors. Two vertices u and v are *duplicate* vertices if they are neighbors and if they share all other neighbors. That is, $N(u) \cup \{u\} = N(v) \cup \{v\}$.

Building on [2] and works which have followed, we consider the *minimum rank* of a graph G on n vertices (denoted $\mathrm{mr}(G)$) as the minimum rank among $n \times n$ Hermitian matrices M such that for all $i \neq j$, $M_{ij} \neq 0$ if and only if vertices i and j are connected by an edge. (This is equivalent to the fact that the off-diagonal entries of M have the same pattern of zeroes as the adjacency matrix of G.) The *minimum semidefinite rank* $\mathrm{mr}_+(G)$ is similarly defined with the additional requirement that M be a positive semidefinite matrix; this is also sometimes indicated by the alternate notation $\mathrm{msr}(G)$ [14].

In this work, we wish to assign a nonzero vector to every vertex. This is a well-studied idea with, unfortunately, inconsistent notation and terminology. We will use the following definition:

Definition 2.1. Let $G = (V, E)$ be a simple graph. An *orthogonal vector representation* of G is a function

$$\phi : V \to \mathbb{R}^d \text{ such that } \begin{cases} \langle \phi(v_i), \phi(v_i) \rangle & > & 0 & \text{for all } v_i \in V, \\ \langle \phi(v_i), \phi(v_j) \rangle & = & 0 & \text{if } i \neq j \text{ and } v_i v_j \notin E, \\ \langle \phi(v_i), \phi(v_j) \rangle & \neq & 0 & \text{otherwise.} \end{cases}$$

We interpret this as an analogue of vertex coloring in the sense of assigning a vector to each vertex with conditions on adjacent vertices. This is actually a stronger condition than coloring, as it is restricted both by a vertex's neighbors and by its non-neighbors. Note also that our orthogonal representation is defined as mapping to a *real* vector space; and, in fact, [4] has shown a graph for which there exists an orthogonal vector representation in $\mathbb{C}^3$ but not in $\mathbb{R}^3$. However, all of the results in this paper hold whether the underlying field is $\mathbb{R}$ or $\mathbb{C}$.

We note that in other settings, Orthogonal Vector Representations allow some vertices to be assigned the zero vector. We explicitly disallow this here; this means that we can always renormalize any representation to include only unit vectors. These vector representations give us our graph parameter of primary interest for this paper, in which we follow the definition given in [14]:

Definition 2.2. The *minimum vector rank* of a graph G is the smallest integer

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 100-123, February 2014

ELA

d such that there exists an orthogonal vector representation of G in $\mathbb{R}^d$. We denote this as $\mathrm{mvr}(G)$.

The minimum vector rank is closely related to the minimum semidefinite rank described above. To make this connection with other parameters explicit, we offer the following observations:

OBSERVATION 2.3. For a simple graph G, the following hold:

1. $\mathrm{mvr}(G) \geq mr_+(G)$, and $\mathrm{mvr}(G) = \mathrm{mr}_+(G)$ unless G contains isolated vertices.
   In fact, if G contains r isolated vertices, then $r = \mathrm{mvr}(G) - \mathrm{mr}_+(G)$ [14].
2. $\mathrm{mvr}(G) = \mathrm{jmr}_+(\mathrm{G}) := \mathrm{mr}_+(\mathrm{G} \vee \mathrm{K}_1)$, as in [10].
3. $\mathrm{mvr}(G) \geq \chi_v(\overline{G})$, defined in [12].

We can easily adapt known properties of the minimum semidefinite rank (from [10, 12], et al.) to give properties of the minimum vector rank. Recall that if two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ have disjoint sets of vertices, then their *union* $G_1 \cup G_2$ has vertices $V = V_1 \cup V_2$ and edges $E = E_1 \cup E_2$. If G_1 and G_2 are connected graphs, then they are the connected components of $G_1 \cup G_2$. Similarly, $G_1 \vee G_2$ is called the *join* of G_1 and G_2 and contains all the vertices and edges as $G_1 \cup G_2$ but additionally has each vertex in G_1 adjacent to each vertex in G_2. We note that for any nonempty graphs G_1, G_2, $G_1 \vee G_2$ is connected and $\overline{G_1 \vee G_2} = \overline{G_1} \cup \overline{G_2}$.

OBSERVATION 2.4. Let G be a graph on $n \geq 1$ vertices.

1. If H is an induced subgraph of G, then $\mathrm{mvr}(H) \leq \mathrm{mvr}(G)$.
2. If $G = G_1 \cup G_2$, then $\mathrm{mvr}(G) = \mathrm{mvr}(G_1) + \mathrm{mvr}(G_2)$.
3. If $G = G_1 \vee G_2$, then $\mathrm{mvr}(G) = \max(\mathrm{mvr}(G_1), \mathrm{mvr}(G_2))$.
4. In particular, if K_n is the complete graph on n vertices, then $\mathrm{mvr}(K_n) = 1$ and $\mathrm{mvr}(\overline{K_n}) = n$.

Proof. (1) is immediate, since any orthogonal representation of G restricts to an orthogonal representation of H. (2) also follows immediately from the definition. (3) is equivalent to Theorem 3.2 in [11]. (4) follows from the fact that K_n is the join of n copies of K_1 and $\overline{K_n}$ is the union of n copies of K_1. □

3. Complement critical graphs. One can consider the minimum vector rank as a generalization of ordinary graph coloring in which we assign vectors to the vertices instead of colors. This approach was taken in [12], although they defined their quantities differently. The primary study of this paper is a further extension of the analogy with graph coloring. Recall that if $\chi(G)$ denotes the chromatic number of G, a graph G is *critical* if $\chi(H) < \chi(G)$ for all proper subgraphs H of G. We propose

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 100-123, February 2014

ELA

the following as a natural generalization:

DEFINITION 3.1. A graph G is *vector critical* if, for any proper induced subgraph H of G, $\mathrm{mvr}(H) < \mathrm{mvr}(G)$.

We know that $\mathrm{mvr}(H) \leq \mathrm{mvr}(G)$ for any induced subgraph H; G is vector critical if this inequality is strict for all $H \neq G$. We will refer to such graphs simply as *critical* when there is no ambiguity.

The class of critical graphs is fairly restrictive. Since we are looking at induced subgraphs, which are defined in terms of a particular vertex subset, it makes sense to consider a graph and its complement as a pair; the vertices of H determine the structure of $\overline{H}$. This suggests the following definition:

DEFINITION 3.2. A graph G is *complement critical* if, for any proper induced subgraph H of G,

$$\mathrm{mvr}(H) + \mathrm{mvr}(\overline{H}) < \mathrm{mvr}(G) + \mathrm{mvr}(\overline{G}). \tag{3.1}$$

The sum in this inequality is a convenient shorthand for saying that either $\mathrm{mvr}(H) < \mathrm{mvr}(G)$ or $\mathrm{mvr}(\overline{H}) < \mathrm{mvr}(\overline{G})$; removing any vertices from G will decrease at least one of the minimum vector ranks. Complement criticality is clearly a weaker condition than criticality. Every vector critical graph is complement critical, but we will see examples in Section 5 of graphs which are complement critical but not vector critical.

Writing the definition this way also connects it with the larger conversation about the Graph Complement Conjectures (see, e.g., [3, 11]), which were initially proposed at the 2006 AIM workshop. Despite many positive contributions, the conjecture and several variants remain unproven. We state the two primary conjectures and append a third given in the language of minimum vector rank.

CONJECTURE 3.3 (Graph Complement Conjectures). *For any simple graph G on n vertices,*

$$mr(G) + mr(\overline{G}) \leq n + 2, \tag{3.2}$$

$$mr_+(G) + mr_+(\overline{G}) \leq n + 2, \tag{3.3}$$

$$\mathrm{mvr}(G) + \mathrm{mvr}(\overline{G}) \leq n + 2. \tag{3.4}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 100-123, February 2014

ELA

Since $\mathrm{mr}(G) \leq \mathrm{mr}_+(G) \leq \mathrm{mvr}(G)$ for any graph G, it is clear that (3.4) implies (3.3), which in turn implies (3.2). What is less obvious is that if (3.3) holds for all simple graphs G, then so does (3.4). This is good news, since we are considering mvr to be a small tweak of mr_+.

The equivalence follows in a straightforward way: Suppose there exists a graph G for which $\mathrm{mvr}(G) + \mathrm{mvr}(\overline{G}) > n + 2$. Without loss of generality, we assume that G is connected (since for any graph at least one of G and $\overline{G}$ must be connected [16]). If we then look at the graph $G' = G \cup K_1$, then $\mathrm{mr}_+(G') = \mathrm{mr}_+(G) + 1 = \mathrm{mvr}(G) + 1$ since G is connected; and $\mathrm{mr}_+(\overline{G'}) = \mathrm{mr}_+(\overline{G} \vee K_1) = \mathrm{mvr}(\overline{G})$. This implies that

$$\mathrm{mr}_+(G') + \mathrm{mr}_+(\overline{G'}) = \mathrm{mvr}(G) + 1 + \mathrm{mvr}(\overline{G}) > 1 + n + 2 = |G'| + 2. \tag{3.5}$$

So, if there is a counterexample to (3.4), then we can build one for (3.3). Thus, the validity of (3.3) for all graphs is equivalent to the validity of (3.4) for all graphs.

None of these inequalities has been proven to date, but the idea of complement critical graphs gives yet another way to think about this.

PROPOSITION 3.1. *If the inequality (3.4) holds for all complement critical graphs G, then it holds for all simple graphs and Conjecture 3.3 is true.*

This means that in order to prove the validity of (3.4) for all graphs, it is only necessary to verify it for all of the complement critical graphs.

Proof. Suppose G is a graph which is not complement critical. Define the set $\mathcal{S}(G) := \{H : H \text{ is a proper induced subgraph of } G, \mathrm{mvr}(H) + \mathrm{mvr}(\overline{H}) = \mathrm{mvr}(G) + \mathrm{mvr}(\overline{G})\}$. Since G is not complement critical, $\mathcal{S}(G)$ is not empty. Choose a graph $H \in \mathcal{S}(G)$ with the minimum number of vertices. For any proper induced subgraph K of H, $K \notin \mathcal{S}(G)$, which implies that $K \notin \mathcal{S}(H) \subseteq \mathcal{S}(G)$. This means that $\mathcal{S}(H)$ is empty and H is complement critical.

Note that this argument follows from the fact that the minimum vector rank is monotone on induced subgraphs; an analogous property exists for critical graphs based on chromatic number (mentioned in [16]) or on any similarly defined quantity. It is not inherently about the minimum vector rank.

The proposition follows quickly from this simple fact. Suppose our graph G violates the graph complement conjecture: $\mathrm{mvr}(G) + \mathrm{mvr}(\overline{G}) > |G| + 2$. We consider the complement critical induced subgraph H from above and see that

$$\mathrm{mvr}(H) + \mathrm{mvr}(\overline{H}) = \mathrm{mvr}(G) + \mathrm{mvr}(\overline{G}) > |G| + 2 \geq |H| + 2, \tag{3.6}$$

which implies that the graph H also violates the inequality (3.4).

Thus, if (3.4) is violated by a graph which is not complement critical, it must

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 100-123, February 2014

ELA

also be violated for some complement critical graph. Taking the contrapositive gives us Proposition 3.1. □

Proposition 3.1 says that in order to prove the graph complement conjectures, we need only prove (3.4) for complement critical graphs. In the remainder of the paper, we explore classes of complement critical graphs.

4. Minimum vector rank of graph complements. In preparation for our discussion of families of complement critical graphs, it will be useful to establish formulas for the minimum rank of the complement of certain types of graphs. This work starts with the bounds given in [13] on the minimum ranks of several families of graph complements. In particular, if we let C_n be the cycle on n vertices, then $\mathrm{mvr}(\overline{C_n}) \leq 4$; as a result, it is shown in [13] that $\mathrm{mvr}(\overline{U}) \leq 4$ for any unicyclic graph U. We give precise conditions which dictate the minimum vector rank of these graph complements and then extend them to larger classes of graphs.

4.1. Trees and cycles. The simplest sparse graphs are trees. For a tree T, the complement was shown to satisfy $\mathrm{mvr}(\overline{T}) = \mathrm{mr}_+(\overline{T}) \leq 3$ in Theorem 3.16 of [1]. They also show that the minimum rank of $\overline{T}$ depends only on whether T contains an induced copy of P_4; the same logic applies in considering the minimum vector rank. We include the full result for completeness:

PROPOSITION 4.1. *Let T be a tree of order $n \geq 2$, then*

$$\mathrm{mvr}(\overline{T}) = \begin{cases} 2 & \text{if } T = K_{1,n-1}, \\ 3 & \text{otherwise.} \end{cases}$$

Our other starting point is graphs which are cycles. The authors of [1] give a proof that $\mathrm{mr}(\overline{C_n}) = 3$ for $n \geq 5$. Their proof gives an explicit orthogonal representation, which implies that in fact $\mathrm{mr}_+(\overline{C_n}) = \mathrm{mvr}(\overline{C_n}) = 3$ for $n \geq 5$. A different proof of this fact was given in [6]. Direct calculation shows that $\mathrm{mvr}(\overline{C_3}) = \mathrm{mvr}(3K_1) = 3$ and $\mathrm{mvr}(\overline{C_4}) = \mathrm{mvr}(2P_2) = 2$, which gives us a complete characterization of $\mathrm{mvr}(\overline{C_n})$:

PROPOSITION 4.2. *[1, 6] Let C_n be the cycle on $n \geq 3$ vertices. Then* $\mathrm{mvr}(\overline{C_n}) = 3$ *if $n \neq 4$.*

In the case $n = 4$, $\mathrm{mvr}(\overline{C_4}) = 2$.

We note that, in the case $n \neq 4$, the orthogonal representation in [1] consists of pairwise linearly independent vectors, while the minimum dimension for an orthogonal representation for $\overline{C_4}$ is 2 but jumps up to 4 if we insist that all vectors be pairwise linearly independent. Such representations satisfy the hypotheses of Theorem 2.1 in [13], allowing us to extend the methods of that paper relating to the complements of

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 100-123, February 2014

ELA

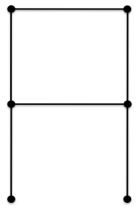

FIG. 4.1. *The unicyclic graph $\overline{L_4}$, whose complement has minimum vector rank 4.*

unicyclic graphs.

PROPOSITION 4.3. *Let U be a unicyclic graph, then*

$$\mathrm{mvr}(\overline{U}) = \begin{cases} 4 & \text{if } \overline{L_4} \text{ is an induced subgraph of } U, \\ 2 & \text{if } U = C_4, \\ 3 & \text{otherwise.} \end{cases}$$

The graph $\overline{L_4}$ is the complement of a 2-tree as described in [13] and is shown in Figure 4.1. Although this proposition can be seen as a special case of Proposition 4.7, the direct proof is an immediate application of the methods in [13], so we include it here:

Proof. Suppose that U is built on a cycle of length $n \neq 4$. It follows from Proposition 4.2 that the complement of this cycle has an orthogonal representation in $\mathbb{R}^3$ in which any two vectors are linearly independent. Using the method of Corollary 3.4 from [13], a unicyclic graph U can be constructed from the cycle C_n by adding one vertex at a time, with the new vertex adjacent to at most one prior vertex in U. The orthogonal representation of U can similarly be built up one vector at a time. Thus, $\mathrm{mvr}(\overline{U}) = \mathrm{mvr}(\overline{C_n}) = 3$ given that $n \neq 4$.

If $n = 4$, then there are two possibilities. If U contains $\overline{L_4}$ as an induced subgraph, then $\mathrm{mvr}(\overline{U}) \geq \mathrm{mvr}(L_4) = 4$. Since $\mathrm{mvr}(\overline{U}) \leq 4$ from [13], we see that $\mathrm{mvr}(\overline{U}) = 4$ if U contains $\overline{L_4}$ as an induced subgraph.

If a unicyclic graph contains a 4-cycle but does not have $\overline{L_4}$ as an induced subgraph, then it contains a pair of diagonally opposite vertices of degree 2, as seen in Figure 4.2. These are duplicate vertices in $\overline{U}$. Removing one of these vertices leaves a tree, which means that $\mathrm{mvr}(\overline{U - v}) = 3$ unless $(U - v)$ is a star, in which case $U = C_4$. We can then reinstate the vertex v and assign it the same vector as its duplicate. Thus, any unicyclic graph that does not have $\overline{L_4}$ as an induced subgraph will have $\mathrm{mvr}(\overline{U}) = 3$ or 2, depending on whether $U = C_4$, which is what we wished to show. □

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 100-123, February 2014

ELA

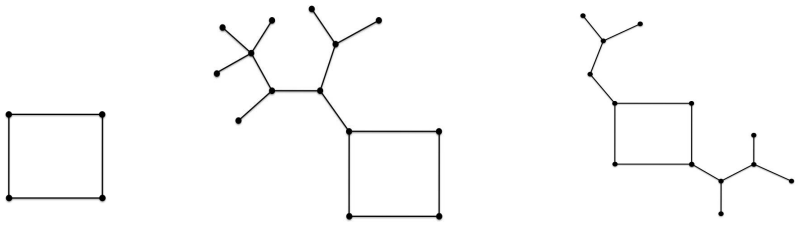

FIG. 4.2. *Examples of unicyclic graphs with 4-cycles but no induced copies of* $\overline{L_4}$.

4.2. Adding long ears to graphs. We would like to extend these ideas to calculate the minimum rank of the complement of graphs with more than one cycle. The following proposition will be a useful tool for this and is of interest in its own right. Recall that an *ear* of a graph is a maximal path subgraph whose internal vertices each have degree 2 in G [16].

PROPOSITION 4.4 (Ear Decomposition). *Let G be a connected graph, and let P be an ear of G containing at least five vertices. Let H be the subgraph of G induced by removing the internal vertices of P.*

If $\overline{H}$ has an orthogonal representation in $\mathbb{R}^3$ consisting of pairwise linearly independent vectors, then so does $\overline{G}$.

Proof. Suppose our ear consists of the path $u, w_1, w_2, \ldots, w_{n-2}, v$, where u and v are vertices in H. We can sequentially build the ear by adding vertices along the path, working from both endpoints, extending our three-dimensional vector representation at every step. Let V' be the set of vertices in H, and $H = G[V']$ be the induced subgraph on those vertices. Using Theorem 2.1 of [13] twice, we find vectors $\mathbf{w}_1$ and $\mathbf{w_{n-2}}$ in $\mathbb{R}^3$ to give a vector representation of $G[V' \cup \{w_1, w_{n-2}\}]$ since w_1 and w_{n-2} are pendant vertices in this graph. We can similarly add vectors for $w_2, w_3, \ldots, w_{n-4}$ in sequence, since these are added as pendant vertices.

When we reach the final step, this method is no longer sufficient, as w_{n-3} will have two neighbors in G and two non-neighbors in $\overline{G}$. To find a vector representation that completes $\overline{G}$, we need Lemma 4.5, which allows for tweaking the vectors assigned to the neighbors of w_{n-3}. This lemma is stated below and proved in Appendix A.1. The requirement that $n \geq 5$ ensures that when the two paths meet in the middle, that vertex is sufficiently isolated from the rest of the graph that we can adjust our representation to include it; and Lemma 4.5 shows us how to do this. This completes the proof that $\text{mvr}(\overline{G}) = \text{mvr}(\overline{G - w_{n-3}}) = 3$. □

LEMMA 4.5. *Let H be a graph with vertices v and w such that*

- $\deg(v) = \deg(w) = 1$;

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 100-123, February 2014

ELA

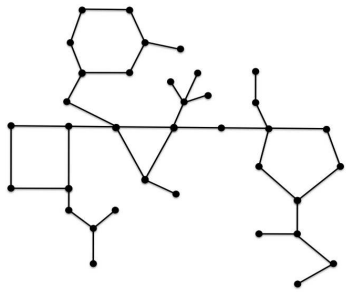

FIG. 4.3. *A necklace graph with four cycles.*

- *v and w have no neighbors in common.*

Let G be the graph formed from H by adding a new vertex u of degree 2 which is adjacent to v and w.

If there is an orthogonal representation of $\overline{H}$ in $\mathbb{R}^3$ which assigns distinct nonzero vectors to each vertex, then there is such an orthogonal representation of $\overline{G}$ in $\mathbb{R}^3$. Hence, $\mathrm{mvr}(\overline{G}) \leq 3$.

We provide another simple result in the style of [13] to cover ears of length 3. This will allow us to attach 3-cycles onto pendant vertices in the next section. The lemma is also proved in the Appendix.

LEMMA 4.6. *Let H be a connected graph with pendant vertex w, and let G be the graph formed by adding two new vertices u and v which are adjacent to w and to each other but have no other neighbors.*

If $\overline{H}$ has an orthogonal representation in $\mathbb{R}^3$ consisting of pairwise linearly independent vectors, then so does $\overline{G}$.

4.3. Applications of Proposition 4.4. Our motivation for Proposition 4.4 was to calculate the minimum vector rank of graphs defined in terms of cycles. The first family of graphs we will study are called necklaces:

DEFINITION 4.1. A *necklace* is a simple connected graph on at least two vertices such that no vertex lies on more than one cycle.

These graphs include trees, cycle graphs, and unicyclic graphs; a generic example is shown in Figure 4.3. Any induced subgraph of a necklace is a union of necklaces and isolated vertices. Necklaces can be built up step by step by either adding a pendant vertex adjacent to an existing one or building a cycle by adding ears between an

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 100-123, February 2014

ELA

existing pair of vertices. Hence, we can use Proposition 4.4 to establish a complement formula.

PROPOSITION 4.7. *Let N be a necklace as defined above. Then*

$$\mathrm{mvr}(\overline{N}) = \begin{cases} 4 & \text{if } \overline{L_4} \text{ is an induced subgraph,} \\ 2 & \text{if } N = C_4 \text{ or } N = K_{1,n} \text{ for } n \geq 1, \\ 3 & \text{otherwise.} \end{cases}$$

Proof. The proposition has already been shown to be true for trees and unicyclic graphs. The proof then proceeds by induction on the number of cycles. Let us assume that the formula works for necklaces with fewer than $c \geq 2$ cycles.

Let N be a necklace with c cycles which has no induced 4-cycles, and let u be a cut-vertex such that $(N - u) = N' \cup F$ is the union of a necklace N' on $c - 1$ cycles and a forest F. (Such a cut-vertex must exist; its cycle is the analogue of a leaf in this "tree of cycles.") By assumption, $\mathrm{mvr}(\overline{N'}) = 3$. If u is on a 3-cycle, we can apply Lemma 4.6 to extend the representation of $\overline{N'}$ to the entire cycle. If u is on a larger cycle and v is one of its neighbors, we can start with the representation of $\overline{N'}$ and use Theorem 2.1 from [13] to extend the representation and show that $\mathrm{mvr}(\overline{N' + u + v}) = 3$. We can then reconstruct the cycle by drawing a long ear from v back to u and use Proposition 4.4 to show that the minimum vector rank is not increased. Finally, we can reinstate the rest of the forest F, one pendant vertex at a time, without increasing the minimum vector rank until we have reconstructed our original graph N. Thus, $\mathrm{mvr}(\overline{N}) = 3$ for all necklace graphs which are 4-cycle-free.

Now, suppose that N is a necklace on $c \geq 2$ cycles which contains 4-cycles but no induced copies of $\overline{L_4}$. Repeating the argument from the proof of Proposition 4.3, every 4-cycle contains a pair of diagonally opposite vertices of degree 2. Removing one vertex from each such pair yields a necklace N' with no 4-cycles and hence $\mathrm{mvr}(\overline{N'}) \leq 3$; and since N has more than one cycle, N' cannot be a star, implying that $\mathrm{mvr}(\overline{N'}) \geq 3$. We then reinstate the duplicate vertices by duplicating its vector assignment and finding that $\mathrm{mvr}(\overline{N}) = 3$.

Finally, we consider the case where N has an induced $\overline{L_4}$ subgraph. This immediately implies that $\mathrm{mvr}(\overline{N}) \geq 4$. We will show that $\mathrm{mvr}(\overline{N}) \leq 4$ by building with a 4-cycle-free necklace. For each 4-cycle in N, split one of its edges into two by introducing a new vertex in the middle: $uv \to \{uw, wv\}$. This turns every four-cycle into a five-cycle and creates a new necklace N' which is 4-cycle-free, implying that $\mathrm{mvr}(\overline{N'}) = 3$. Now, embed this three-dimensional vector representation of $\overline{N}$ in $\mathbb{R}^4$ by setting the fourth coordinate equal to zero. In order to make this a valid vector representation of the original necklace N, for each 4-cycle C_i, we need to adjust the vectors assigned to our vertices u_i and v_i by assigning them respective fourth

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 100-123, February 2014

ELA

coordinates x_i and y_i so that $x_i y_i = -\langle \mathbf{u}_i, \mathbf{v}_i \rangle \neq 0$. Since we have infinitely many independent choices for each pair (x_i, y_i), we can find a labelling such that $\langle \mathbf{u}_i, \mathbf{v}_j \rangle \neq 0$ for $i \neq j$. This guarantees that a representation can be found in $\mathbb{R}^4$, which implies that $\operatorname{mvr}(\overline{N}) = 4$. □

A different generalization of unicyclic graphs are those in which all induced cycles share a single common edge. These are called books in Section 5.5.

DEFINITION 4.2. A *book* is a connected graph on $n \geq 3$ vertices, which has a unique edge e such that the intersection of any two induced cycles of B is exactly the edge e and its vertices. The endpoints of e are called the *binding vertices*.

This definition is discussed more in Section 5.5. Note that the uniqueness of e implies that B contains at least two cycles. The minimum rank of book complements is given in terms of the 4-cycle C_4 and the kite κ, which is shown in Figure 4.4.

PROPOSITION 4.8. *For a book B as defined above,*

$$\operatorname{mvr}(\overline{B}) = \begin{cases} 4 & \text{if either } C_4 \text{ or the kite } \kappa \text{ is an induced subgraph of } B, \\ 3 & \text{otherwise.} \end{cases}$$

Proof. Let B be a book which does not have any induced kite or 4-cycles. Start with a single 3-cycle (if there is one) or else any other induced unicyclic graph. From here, we can build B up one cycle at a time by adding long ears using Proposition 4.4 to show that $\operatorname{mvr}(\overline{B}) = 3$. If B has multiple 3-cycles, then each contains the two binding vertices and a third vertex of degree 2 in B (since B is kite-free). In this case, the set of "third vertices" in $\overline{B}$ consists of all duplicate vertices, and we can assign the same vector to each of them.

The kite κ on five vertices is the only book whose complement is a connected tree. As such, $\operatorname{mvr}(\overline{\kappa}) = |\kappa| - 1 = 4$, implying that $\operatorname{mvr}(\overline{B}) \geq 4$ if it contains an induced kite. Likewise, if B has more than one cycle and contains C_4 as an induced subgraph, it implies that B must have an induced subgraph isomorphic to at least one of the graphs shown in Figure 4.4, each of which has complement minimum vector rank of 4. Since each of these four graphs has an orthogonal representation in $\mathbb{R}^4$ which satisfies the conditions of Theorem 2.2 of [13], we can build up the rest of B one vertex at a time and maintain $\operatorname{mvr}(\overline{B}) \leq 4$. This means that if B contains either a kite or a four-cycle as a subgraph, then $\operatorname{mvr}(\overline{B}) = 4$. □

5. Families of complement critical graphs. In an arbitrary graph G, it is not obvious in general whether it is complement critical. However, if the graph has nice properties, we can sometimes formulate simple criteria. This section focuses on identifying families of vector critical and complement critical graphs. The initial

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 100-123, February 2014

ELA

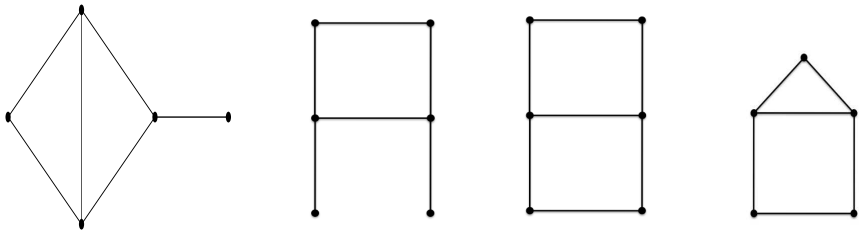

FIG. 4.4. *Books for which* $\mathrm{mvr}(\overline{B}) = 4$*: Kite* κ*,* $\overline{L_4}$*, Domino,* $\overline{P_5}$*.*

section shows that complement critical graphs which are not connected must be built from smaller complement critical graphs. We then derive necessary and sufficient conditions for certain types of graphs to be complement critical based on their induced cycles.

Although the definition of complement critical graphs uses a property of *all* induced subgraphs of G, it is sufficient to consider only those induced by removing a single vertex of G, since G is vector critical if and only if $\mathrm{mvr}(G - v) < \mathrm{mvr}(G)$ for all vertices v. Likewise for complement criticality. We will make use of this fact repeatedly in what follows, writing $(G - v)$ to indicate the induced subgraph of G on all of its vertices except v.

5.1. Disconnected graphs. The first result shows that disconnected complement critical graphs are composed of complement critical components. This will allow us to focus the remainder of our work on connected graphs.

PROPOSITION 5.1. *Let G be a graph which is not connected. We can write $G = G_1 \cup G_2$ with $\mathrm{mvr}(\overline{G_1}) \leq \mathrm{mvr}(\overline{G_2})$. Then G is vector critical if and only if both G_1 and G_2 are vector critical.*

In addition, G is complement critical if and only if the following three conditions are all satisfied:

- $\mathrm{mvr}(\overline{G_1}) < \mathrm{mvr}(\overline{G_2})$*;*
- G_1 *is vector critical;*
- G_2 *is complement critical.*

Note that there is a possible ambiguity in the labelling of G_1 and G_2 if $\mathrm{mvr}(\overline{G_1}) = \mathrm{mvr}(\overline{G_2})$; but in this case, G cannot be complement critical.

Proof. We proceed by removing a single vertex v and looking at the induced subgraph. This vertex may be removed from G_1 or from G_2.

In the case where v is removed from G_1, we note that $\mathrm{mvr}(\overline{G_1 - v}) \leq \mathrm{mvr}(\overline{G_1}) \leq$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 100-123, February 2014

ELA

$\mathrm{mvr}(\overline{G_2})$, which implies that

$$\mathrm{mvr}(\overline{G-v}) = \mathrm{mvr}(\overline{(G_1-v)\vee G_2}) = \mathrm{mvr}(\overline{G_2}) = \mathrm{mvr}(\overline{G}). \tag{5.1}$$

This means that removing a vertex from G_1 will never reduce the minimum vector rank of $\overline{G}$. Likewise, we can see that

$$\mathrm{mvr}(G) - \mathrm{mvr}(G_1) = \mathrm{mvr}(G_2) = \mathrm{mvr}(G-v) - \mathrm{mvr}(G_1-v). \tag{5.2}$$

Therefore, $\mathrm{mvr}(G-v) + \mathrm{mvr}(\overline{G-v}) < \mathrm{mvr}(G) + \mathrm{mvr}(\overline{G})$ if and only if $\mathrm{mvr}(G_1) > \mathrm{mvr}(G_1-v)$. This condition holds for all vertices v of G_1 if and only if G_1 is vector critical.

In the case where v is removed from G_2, we see that

$$\mathrm{mvr}(\overline{G-v}) = \max(\mathrm{mvr}(\overline{G_1}), \mathrm{mvr}(\overline{G_2-v})). \tag{5.3}$$

This means that $\mathrm{mvr}(\overline{G-v}) = \mathrm{mvr}(\overline{G})$ if and only if either $\mathrm{mvr}(\overline{G_1}) = \mathrm{mvr}(\overline{G_2})$ or $\mathrm{mvr}(\overline{G_2}) = \mathrm{mvr}(\overline{G_2-v})$.

So, in order for G to be complement critical, either $\mathrm{mvr}(G_2-v) < \mathrm{mvr}(G_2)$ for all v (i.e., G_2 is vector critical), or else we need G_2 to be complement critical *and* have $\mathrm{mvr}(\overline{G_1}) < \mathrm{mvr}(\overline{G_2})$. □

By repeated application of this result, we can see that every connected component of a complement critical graph must be complement critical; and, in fact, all but one of them must be vector critical as well. This allows us to focus our remaining results on graphs for which both G and $\overline{G}$ are connected.

5.2. Trees. Trees are the simplest connected graphs, and it is straightforward to characterize them in terms of criticality. Recall that a tree is a *star* if it is a complete bipartite graph $K_{1,n-1}$.

PROPOSITION 5.2. *Any tree is complement critical. Any tree which is not a star is vector critical.*

Proof. If T is a tree on n vertices, then $\mathrm{mvr}(T) = n-1$; if we remove a single vertex v, then $\mathrm{mvr}(T-v) \le |T-v| - 1 = n-2$ unless $(T-v)$ contains only isolated vertices. In this case, T must be a star and v must be the central vertex.

Conclusion: If T is not a star, then for any vertex v $\mathrm{mvr}(T-v) \le n-2 < \mathrm{mvr}(T)$, so T is vector critical.

If $T = K_{1,n-1}$ and the vertex being removed is the dominating vertex, then $\mathrm{mvr}(T-v) = \mathrm{mvr}(T)$ but $\mathrm{mvr}(\overline{T-v}) = \mathrm{mvr}(K_{n-1}) = 1$. Since $\mathrm{mvr}(\overline{T}) = 2$ by Proposition 4.1, we see that a star is complement critical. □

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 100-123, February 2014

ELA

5.3. Unicyclic graphs. Following the discussion of trees, which contain no cycles, it is natural to look at graphs which contain only one. For a connected unicyclic graph U, it was shown in [5] that $\mathrm{mvr}(U) = |U|-2$. We use this result and Proposition 4.3 to characterize connected unicyclic graphs which are complement critical.

PROPOSITION 5.3. *A connected unicyclic graph U is complement critical if and only if at least one of the following three conditions is true:*

1. *Each vertex on the cycle of U is adjacent to at least three vertices of degree greater than 1. In this case, U is vector critical.*
2. *$\overline{L_4}$ is an induced subgraph of U.*
3. *$U = S_n^3$, which is the graph formed from the star graph $K_{1,n-1}$ by connecting two of the leaves with an edge, as shown in Figure 5.1.*

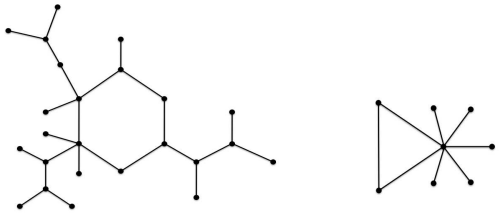

FIG. 5.1. *Generic unicyclic graph (left) and the graph S_8^3 (right).*

Proof. We first observe that if v is not on the cycle of U, then $(U-v)$ is the union of a connected unicyclic graph U' and a (possibly empty) forest F. This implies that

$$\mathrm{mvr}(U-v) = \mathrm{mvr}(U') + \mathrm{mvr}(F) \leq |U'| - 2 + |F| = |U| - 3. \tag{5.4}$$

Thus, removing a vertex not on the cycle will always reduce $\mathrm{mvr}(U)$. This reduces the question of U's complement criticality to what happens when you remove a vertex on the cycle. To do this, we can check directly that each of the three conditions implies that U is complement critical:

Condition (1) implies that for every v on the cycle, $(U-v)$ is a forest containing at least two proper trees. This means that $\mathrm{mvr}(U-v) \leq |U-v| - 2 < \mathrm{mvr}(U)$.

Condition (2) implies that $\mathrm{mvr}(\overline{U}) = 4$. If v is on the cycle, then $(U-v)$ is a forest and $\mathrm{mvr}(\overline{U-v}) \leq 3 < \mathrm{mvr}(\overline{U})$.

Finally, (3) implies that if v is on the cycle, then either $(U-v) = K_{1,n-2}$ or $(U-v) = \overline{K_{n-1}}$. In either case, $\mathrm{mvr}(\overline{U-v}) \leq 2 < \mathrm{mvr}(\overline{U})$.

We conclude that any one of these three conditions implies that U is complement critical. Now, we wish to show that they are in fact necessary. Assume that U is a

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 100-123, February 2014

ELA

connected unicyclic graph which is complement critical for which the conditions (1) and (2) do not hold. We wish to show that $U = S_n^3$ for some n.

Since condition (1) does not hold, there is at least one vertex v on the cycle for which $(U - v)$ is the union of a tree T plus a collection of $r \geq 0$ isolated vertices. This means that $\text{mvr}(U - v) = |T| - 1 + r = |U| - 2 = \text{mvr}(U)$. By assumption, U is complement critical, so $\text{mvr}(\overline{U - v}) < \text{mvr}(\overline{U})$. Condition (2) does not hold, which means that $\text{mvr}(\overline{U}) \leq 3$ by Proposition 4.3. This forces $\text{mvr}(\overline{U - v}) \leq 2$. Proposition 4.1 then tells us that T must be a star.

Let u be the center vertex of T, and let $w \notin \{u, v\}$ be any vertex from the cycle in U. w is necessarily a leaf in T, which means that $\deg(w) \leq 2$ in U. This means that w does not satisfy condition (1), so we can repeat our argument (interchanging the roles of v and w) to show that $(U - w)$ must also include a star with v as a leaf and that $\deg(v) \leq 2$. Since v and w are degree two vertices which are adjacent to each other and to u, we see that U must be built on a 3-cycle and that only one of these three vertices can have degree greater than 2, leading us to conclude that $U = S_n^3$ for some $n \geq 3$.

Conclusion: If U is complement critical and neither condition (1) nor (2) holds, then condition (3) must be true. This gives us the biconditional in the proposition. □

As an immediate consequence, we see that a cycle graph C_n is complement critical if and only if $n = 3$. The three-cycle $C_3 = S_3^3$ is a degenerate form of Figure 5.1; its complement $\overline{C_3} = 3K_1$ is vector critical.

5.4. Necklaces. The next graphs we consider are necklaces, which were defined in Section 4.4. Recall that a necklace is a connected graph in which each vertex belongs to at most one cycle, as shown in Figure 4.3. Graphs in which each *edge* appears in at most one cycle are called cactus graphs and are well-studied [16], but there does not appear to be as much attention to graph in which cycles do not intersect even on the vertices. Necklaces can be thought of as a "tree of cycles," since we can form a tree by collapsing the vertices of each cycle into a single vertex. Such graphs are planar and, since each cycle is in fact an induced cycle, the infinite face of this graph touches every vertex. This implies that necklaces are outerplanar, which allows us to use results from [5].

In order to characterize complement critical necklaces, it will be useful to have formulas for the minimum rank. The minimum rank of the complement was described in Proposition 4.7, while the minimum rank of a necklace itself generalizes the formula for trees and unicyclic graphs:

PROPOSITION 5.4. *If N is a proper necklace on c cycles, then* $\text{mvr}(N) = |N| - c - 1$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 100-123, February 2014

ELA

Proof. Note that the formula specifies to unicyclic graphs and proper trees (with $c = 1$ and $c = 0$, respectively) but does not apply if $N = K_1$. The simplest proof of this formula uses the result from [5], which states that since N is outerplanar and connected, $\mathrm{mvr}(N) = \mathrm{mr}_+(N) = |N| - T(N)$, where $T(N)$ is the minimum number of vertex-disjoint induced trees of N needed to cover all of its vertices. If we choose one vertex from each cycle and delete its two edges on the cycle, what remains is a forest consisting of $c + 1$ induced trees. This is a tree cover, hence $T(N) \leq c + 1$. On the other hand, in any a tree covering of N, each cycle must be covered using at least two induced trees. As a result, the covering must omit at least two edges from each cycle (where the induced trees connect to each other). Since N has $|N| - 1 + c$ edges, the tree covering has at most $|N| - 1 - c$ edges. A union of k trees covering $|N|$ vertices has exactly $|N| - k$ edges, which implies that any tree covering of N uses at least $c + 1$ trees. Thus, $T(N) = 1 + c$ and $\mathrm{mvr}(N) = |N| - c - 1$. □

The following proposition gives the main result of this section.

PROPOSITION 5.5. *A necklace N is vector critical if and only every vertex on a cycle of N is adjacent to at least three vertices of degree greater than 1.*

A necklace N is complement critical if and only if one of the following three conditions is true:

1. *N is vector critical.*
2. *N has $\overline{L_4}$ as an induced subgraph; every induced copy of $\overline{L_4}$ is built on the same 4-cycle; and every vertex which lies on a cycle other than this 4-cycle is adjacent to at least three vertices of degree greater than 1.*
3. *$N = S_n^3$ (see Figure 5.1).*

Proof. This is a generalization of Proposition 5.3, although the conclusions are not as simple. As in the unicyclic case, if v is a vertex which is not on a cycle, then $(N - v) = \bigcup_{i=1}^{k} N_i \cup \overline{K_r}$ is the union of $k \geq 1$ proper necklaces plus $r \geq$ isolated vertices. The total number of cycles is this same as in the original graph. This implies that $\mathrm{mvr}(N-v) = r + \sum_i \mathrm{mvr}(N_i) = r + \sum_i (|N_i| - c_i - 1) = |N| - 1 - c - k < \mathrm{mvr}(N)$ since $k \geq 1$.

Likewise, if v is on a cycle and has at least three neighbors of degree greater than 1, then $(N - v)$ is a union of $k > 1$ proper necklaces plus $r \geq 0$ isolated vertices; and the total number of cycles is now one less. Thus, $\mathrm{mvr}(N - v) = r + \sum_i \mathrm{mvr}(N_i) = |N| - 1 - (c - 1) - k < |N| - c - k < \mathrm{mvr}(N)$.

On the other hand, if v is on a cycle and has only two neighbors of degree greater than 1, then $k = 1$ in the above calculation and we get $\mathrm{mvr}(N - v) = |N| - 1 - (c - 1) - k = \mathrm{mvr}(N)$. This implies that if v is on a cycle, then $\mathrm{mvr}(N - v) < \mathrm{mvr}(N)$ if and only if v has at least three vertices of degree greater than 1. This implies that N

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 100-123, February 2014

ELA

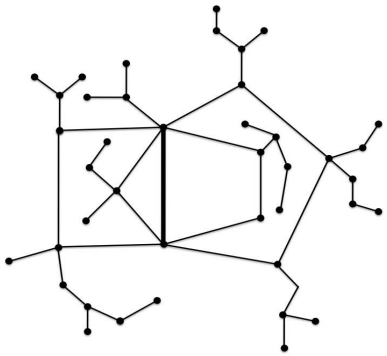

FIG. 5.2. *A book with four induced cycles.*

is vector critical if and only if every vertex that lies on a cycle satisfies the neighbor condition.

When we look at the complement $\overline{N}$, we see that the proof for unicyclic graphs shows both ways in which $\text{mvr}(\overline{N-v}) < \text{mvr}(\overline{N})$:

- $\text{mvr}(\overline{N}) = 4$ and $\text{mvr}(\overline{N-v}) < 4$, which happens when $\overline{L_4}$ is an induced subgraph of N but not $(N-v)$.
- $\text{mvr}(\overline{N}) = 3$ and $\text{mvr}(\overline{N-v}) < 3$, which happens when $(N-v)$ is a star.

This explains why N can only have one copy of $\overline{L_4}$ and limits the possibilities for having $\text{mvr}(\overline{N-v}) < \text{mvr}(\overline{N})$ to those listed in the proposition. □

5.5. Books. The last type of graph we explore is a book, which is defined in Section 4.4 to be a graph with a distinguished edge e such that the intersection of any two induced cycles in B is the edge e and its endpoints. This generalizes the definition in [4], in which it was assumed that all the cycles were the same size and that every edge of B was on a cycle. Figure 5.2 shows a book with a 3-cycle, a 5-cycle, and two 4-cycles; it is apparent that such graphs are always planar. Books are outerplanar only when there are only two induced cycles, but like outerplanar graphs, the chromatic number of the complement is at most 3.

We will assume in what follows that B is connected and that it contains more than one cycle. This uniquely defines the edge e and its vertices, which are called **binding vertices** and denoted as v_b. This is a different direction in which to generalize unicyclic graphs; but we derive results similar to those in previous sections.

First, we observe that the proof of Proposition 3.11 in [4] extends to our graphs, as the binding vertices form a positive zero forcing set for the any book B, which

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 100-123, February 2014

implies that $\mathrm{mvr}(B) \geq |B| - 2$. Since B is not a tree, this is also an upper bound and $\mathrm{mvr}(B) = |B| - 2$. A description of the minimum vector rank of the complement of a book were given in Proposition 4.8. Armed with these results, we can give necessary and sufficient conditions for a book to be complement critical:

PROPOSITION 5.6. *A book B with at least two cycles is complement critical if and only if it satisfies one of the following conditions:*

1. *Each binding vertex has at least one neighbor which does not lie on a cycle and which has degree greater than 1. In this case, B is vector critical.*
2. *Either the four-cycle C_4 or the kite κ is an induced subgraph of B.*
3. *Every cycle of B is a 3-cycle and every vertex of B is adjacent to at least one of the binding vertices.*

Proof. One simplifying observation if that if v is not one of the binding vertices, then $\mathrm{mvr}(B - v) < \mathrm{mvr}(B)$. This is because the induced subgraph $(B - v)$ has a connected component B' which is still a book. We write $(B - v) = B' \cup F$, where F is the (possibly empty) remainder of the graph (which is a forest) and which has the trivial bound $\mathrm{mvr}(F) \leq |F|$. Therefore we have $\mathrm{mvr}(B - v) = \mathrm{mvr}(B') + \mathrm{mvr}(F) \leq |B'| - 2 + |F| = |B| - 3 < \mathrm{mvr}(B)$.

This means that the complement criticality of a book depends solely on its binding vertices. The removal of a binding vertex v_b will give a forest with $t \geq 1$ nontrivial trees and some isolated vertices. This means that $\mathrm{mvr}(B - v_b) = |B| - 1 - t < \mathrm{mvr}(B)$ if and only if $t > 1$. Thus, B will be vector critical if and only if each binding vertex has a neighbor which lies off the cycle and which has degree at least 2.

Otherwise, we need to ask whether $\mathrm{mvr}(\overline{B - v_b}) < \mathrm{mvr}(\overline{B})$. Since $(B - v_b)$ is a forest, $\mathrm{mvr}(\overline{B - v_b}) \leq 3$; thus, if $\mathrm{mvr}(\overline{B}) = 4$, we automatically have complement criticality. Using Proposition 4.8, this gives the second condition.

The only other possibility would be for $\mathrm{mvr}(\overline{B - v_b}) = 2$. This will happen only if each tree in $(B - v_b)$ is a star. In particular, the other binding vertex must be adjacent to every other vertex in its component, which means that each cycle must be a triangle with no extra vertices attached. Also, the remaining binding vertex cannot have any other neighbors of degree greater than 1. This specifically precludes the first condition, which means that if we need $\mathrm{mvr}(\overline{B - v_b}) = 2$ for one of the binding vertices, we need it for the other. This inevitably leads us to the 3-cycle graph B_m^3 ($m \geq 2$), possibly with leaves attached to the binding vertices, as shown in Figure 5.3. □

Because books can be treated as gluing multiple unicyclic graphs together by one common edge, it is not surprising that their properties resemble those of unicyclic

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 100-123, February 2014

ELA

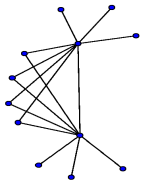

FIG. 5.3. *B_m^3 is a set of m 3-cycles built on a common edge. We can then attach an arbitrary number of pendant vertices to each binding vertex. The graph shown is built from B_4^3.*

graphs. Returning to the examples from [4], we get the following corollary:

COROLLARY 5.7. *The graphs B_m^t described in [4] with $m \geq 2$ are complement critical if and only if $t = 3$ or $t = 4$.*

Proof. These graphs consist of m copies of a t-cycle glued together along a single edge. There are no vertices which do not lie on a cycle. Looking at the Proposition, we see that the graph B_m^3 clearly meets the third condition while B_m^4 meets the second. For any other value of t, B_m^t is complement critical if and only if it meets the first condition, which it cannot since there are no vertices away from the cycles. Thus, no B_m^t is complement critical if $t > 4$. □

6. Complement critical graphs and the minimum semidefinite rank. The notion of a critical graph must always be defined in terms of some graph function; in the previous section, all calculations were done with respect to the minimum vector rank. This brief section is meant to point out that all of our results can be readily adapted using the minimum semidefinite rank (mr_+), which is more widely used in the literature. The only significant difference between the two measures is on a single isolated vertex, where $\mathrm{mvr}(K_1) = 1$ but $\mathrm{mr}_+(K_1) = 0$; and the inequality $\mathrm{mvr}(G) \geq \mathrm{mr}_+(G)$ always holds, with equality guaranteed for connected graphs.

This gives an immediate indication that being complement critical with respect to mvr is a stronger one than being complement critical with respect to mr_+:

PROPOSITION 6.1. *Let G be any connected graph such that $\overline{G}$ is also connected. If G is vector critical with respect to the minimum vector rank, then it is vector critical with respect to the minimum semidefinite rank. If G is complement critical with respect to the minimum vector rank, then it is complement critical with respect to the minimum semidefinite rank.*

The proof is immediate. If G is connected and $\mathrm{mvr}(G - v) < \mathrm{mvr}(G)$, then $\mathrm{mr}_+(G-v) \leq \mathrm{mvr}(G-v) < \mathrm{mvr}(G) = \mathrm{mr}_+(G)$; and the same argument can be made for $\overline{G}$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 100-123, February 2014

ELA

In terms of the results for families of complement critical graphs given in Section 5, we no longer need to worry about the appearance of isolated vertices in induced subgraphs of G. This simplifies some of the conditions:

PROPOSITION 6.2. *Let G be a connected graph which is not complement critical with respect to the minimum vector rank.*

1. *If G is a unicyclic graph or a necklace and each vertex on its cycle has degree at least three, then G is vector critical with respect to the minimum semidefinite rank.*
2. *If G is a necklace graph which has $\overline{L_4}$ as an induced subgraph such that every induced copy of $\overline{L_4}$ is built on the same 4-cycle; and every vertex which lies on a cycle other than this 4-cycle has degree at least 3, then G is complement critical with respect to the minimum semidefinite rank.*
3. *If G is a book in which each binding vertex has at least one neighbor which does not lie on a cycle, then G is critical with respect to the minimum semidefinite rank.*

Examples of graphs which are critical and complement-critical with respect to the minimum semidefinite rank but not the minimum vector rank are shown in Figure 6.1. The simplest of these is the n-sun (discussed in [5, 10]), which is the cycle C_n with a pendant vertex attached to each vertex on the cycle. This graph is critical since for any vertex v on the cycle, $\mathrm{mr}_+(G-v) = (|G|-2)-1 < \mathrm{mr}_+(G)$. However, since the isolated vertex in $(G-v)$ has minimum vector rank equal to 1, $\mathrm{mvr}(G-v) = \mathrm{mvr}(G)$. The figure also shows examples of graphs which meet conditions (2) and (3) from the Proposition.

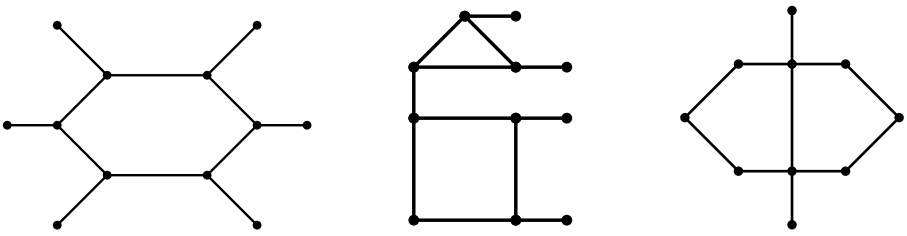

FIG. 6.1. *Examples of graphs which are complement critical with respect to the minimum semidefinite rank but not the minimum vector rank.*

7. Conclusion. In this paper, we introduce the notion of complement critical graphs defined with respect to the minimum vector rank. These objects occupy an important place in the set of graphs when partially ordered by induced subgraphs. We showed a link between complement critical graphs and the Graph Complement Conjectures, showing that these graphs are sufficient to prove or disprove the conjec-

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 100-123, February 2014

ture. We provided a list of general properties of complement graphs and necessary and sufficient conditions for graphs which are built on cycles to be complement critical. In the process, we gave explicit formulas for the complements of certain sparse graphs. We anticipate that this work will be useful in understanding the structure of the set of graphs given by the minimum rank.

Acknowledgment. This work was primarily completed during the Summer Science Research Program at Saint Mary's College of California. The authors gratefully acknowledge the support of this program and its many benefits.

REFERENCES

[1] AIM Minimum Rank–Special Graphs Work Group. Zero forcing sets and the minimum rank of graphs. *Linear Algebra and its Applications*, 428:1628–1648, 2008.
[2] American Institute of Mathematics Workshop. "Spectra of Families of Matrices described by Graphs, Digraphs, and Sign Patterns", Palo Alto, October 23–27, 2006. Material available at `http://aimath.org/WWN/matrixspectrum/matrixspectrum.pdf`.
[3] F. Barioli, et al. On the graph complement conjecture for minimum rank. *Linear Algebra and its Applications*, 436:4373–4391, 2012.
[4] F. Barioli, et al. Zero forcing parameters and minimum rank problems. *Linear Algebra and its Applications*, 433 2: 401–411, 2010.
[5] F. Barioli, S.M. Fallat, L.H. Mitchell, and S.K. Narayan. Minimum Semidefinite Rank of Outerplanar Graphs and the Tree Cover Number. *Electronic Journal of Linear Algebra*, 22:10-21, 2011.
[6] M. Booth, et al. On the minimum semidefinite rank of a simple graph. *Linear and Multilinear Algebra*, 59:483–506, 2011.
[7] M. Booth, et al. On the minimum rank among positive semidefinite matrices with a given graph. *SIAM Journal of Matrix Analysis and Applications*, 30:731–740, 2008.
[8] P.J. Cameron, et al. On the quantum chromatic number of a graph. *Electronic Journal of Combinatorics*, 14, 2007.
[9] R. Duan, S. Severini, and A. Winter. Zero-error communication via quantum channels and a quantum Lovász ϑ-function. In: *2011 IEEE Int. Symposium on Info. Th. Proc.*, 2011.
[10] S. Fallat and L. Hogben. The minimum rank of symmetric matrices described by a graph: A survey. *Linear Algebra and its Applications*, 426:558–582, 2007.
[11] S. Fallat and L. Hogben. Chapter 46: Minimum Rank, Maximum Nullity, and Zero Forcing Numbers of Graphs. In: Leslie Hogben (editor), *Handbook of Linear Algebra*, second edition. CRC Press, Boca Raton, FL 2013. Specific references are made to the version available at `http://arxiv.org/pdf/1102.5142.pdf`, 2010.
[12] G. Haynes, et al. Orthogonal vector coloring. *Electronic Journal of Combinatorics*, 17, 2010.
[13] L. Hogben. Orthogonal representations, minimum rank, and graph complements. *Linear Algebra and its Applications*, 428:2560–2568, 2008.
[14] Y. Jiang, L.H. Mitchell, and S.K. Narayan. Unitary matrix digraphs and minimum semidefinite rank. *Linear Algebra and its Applications*, 428:1685–1695, 2008.
[15] P.M. Nylen. Minimum-Rank Matrices with Prescribed Graph. *Linear Algebra and its Applications*, 248:303–316, 1996.
[16] D.B. West. *Introduction to Graph Theory.* Prentice Hall Inc., Upper Saddle River, NJ, 1996.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 100-123, February 2014

ELA

Appendix A. Appendix on necklace properties.

A.1. Proof of Lemma 4.5. Assume that there exists a vector labelling of the vertices of $\overline{H}$ in $\mathbb{R}^3$ that assigns pairwise linearly independent nonzero vectors to each vertex. Let $\mathbf{v}$ and $\mathbf{w}$ be the vectors corresponding to the vertices v and w; let $\mathbf{y_1}$ and $\mathbf{y_2}$ be the vectors corresponding to the respective neighbors of v and w; and let $\mathbf{y_3}, \ldots, \mathbf{y_k}$ be the vectors corresponding to the remaining vertices of H. This is represented schematically in Figure A.1. Note that the edge y_1y_2 may or may not be present.

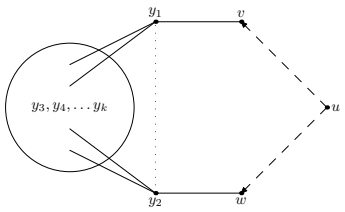

FIG. A.1. *Schematic Representation of Graph in Lemma 4.5.*

It will be convenient to take advantage of being in $\mathbb{R}^3$ and use the cross product. For any real number t, define

$$\mathbf{u}_t := t\mathbf{v} + \mathbf{w}, \qquad \mathbf{v}_t := \mathbf{u}_t \times \mathbf{y_1}, \qquad \mathbf{w}_t := \mathbf{u}_t \times \mathbf{y_2}. \tag{A.1}$$

We wish to show that there exists a real value of t for which $\{\mathbf{u}_t, \mathbf{v}_t, \mathbf{w}_t, \mathbf{y_1}, \ldots, \mathbf{y_k}\}$ is an orthogonal representation of $\overline{G}$.

By construction, for all t,

$$\langle \mathbf{u}_t, \mathbf{v}_t \rangle = \langle \mathbf{u}_t, \mathbf{w}_t \rangle = \langle \mathbf{v}_t, \mathbf{y_1} \rangle = \langle \mathbf{w}_t, \mathbf{y_2} \rangle = 0.$$

Since these are the only non-neighbors of our vertices u, v, w, we require only that no other inner product be zero. Specifically, we need to show that there exists a value of t such that:

(I) $\langle \mathbf{u}_t, \mathbf{y_i} \rangle \neq 0$ for all $i = 1, \ldots, k$;
(II) $\langle \mathbf{v}_t, \mathbf{y_i} \rangle \neq 0$ for $i \neq 1$ and $\langle \mathbf{w}_t, \mathbf{y_i} \rangle \neq 0$ for $i \neq 2$;
(III) $\langle \mathbf{v}_t, \mathbf{w}_t \rangle \neq 0$.

These inner products are linear and quadratic functions of t. The result is proved as long as none of them is identically zero.

Looking at line (I), $\langle \mathbf{u}_t, \mathbf{y_i} \rangle$ is identically zero if and only if $\langle \mathbf{v}, \mathbf{y_i} \rangle = \langle \mathbf{w}, \mathbf{y_i} \rangle = 0$, which would imply that v and w share a common neighbor. In line (II), $\langle \mathbf{v}_t, \mathbf{y_i} \rangle = t \det(\mathbf{v}, \mathbf{y_1}, \mathbf{y_i}) + \det(\mathbf{w}, \mathbf{y_1}, \mathbf{y_i}) = 0$ for all t if and only if $\mathbf{y_i}$ is in both the plane

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 100-123, February 2014

ELA

spanned by $\mathbf{v}$ and $\mathbf{y_1}$ *and* in the plane spanned by $\mathbf{w}$ and $\mathbf{y_1}$. This means that $\mathbf{y_i}$ and $\mathbf{y_1}$ are dependent, which implies that $i = 1$. The second part of (II) is similarly shown.

Finally, in (III), we note that $\langle \mathbf{v}_t, \mathbf{w}_t \rangle = \langle \mathbf{y}_1, \mathbf{y}_2 \rangle \langle \mathbf{u}_t, \mathbf{u}_t \rangle - t \langle \mathbf{y}_1, \mathbf{w} \rangle \langle \mathbf{v}, \mathbf{y}_2 \rangle$. We know that $\langle \mathbf{y}_1, \mathbf{w} \rangle \langle \mathbf{v}, \mathbf{y}_2 \rangle \neq 0$, by assumption, so $\langle \mathbf{v}_t, \mathbf{w}_t \rangle$ is a non-degenerate linear or quadratic function, depending on whether $\langle \mathbf{y}_1, \mathbf{y}_2 \rangle = 0$. Thus, there are at most two values of t for which this is zero.

Since each condition excludes finitely many values of t, there exists a value of t which makes $\{\mathbf{u}_t, \mathbf{v}_t, \mathbf{w}_t, \mathbf{y_1}, \ldots, \mathbf{y_k}\}$ an orthogonal representation of $\overline{G}$ in $\mathbb{R}^3$. □

A.2. Proof of Lemma 4.6. Suppose that we have an orthogonal representation of H in $\mathbb{R}^3$ in which w is represented by the unit vector $\mathbf{w}$. Observation 1.3 in [13] guarantees that there exists a unit vector $\mathbf{u_0}$ in $\mathbb{R}^3$ which is orthogonal to $\mathbf{w}$ but not to any other vector in the representation. Let $\mathbf{v_0} = \mathbf{u_0} \times \mathbf{w}$. As in the previous proof, we define

$$\mathbf{u}_t = \mathbf{u_0} + t\mathbf{v_0}, \qquad \mathbf{v}_t = t\mathbf{u_0} - \mathbf{v_0}.$$

The vectors $\mathbf{u}_t, \mathbf{v}_t$ and $\mathbf{w}$ are mutually orthogonal for all t, and for any other vector $\mathbf{y}$ in the orthogonal representation, $\langle \mathbf{y}, \mathbf{u}_t \rangle$ and $\langle \mathbf{y}, \mathbf{v}_t \rangle$ are not identically zero because $\langle \mathbf{y}, \mathbf{u_0} \rangle \neq 0$. Therefore, we can find a value of t for which $\mathbf{u}_t$ and $\mathbf{v}_t$ extend the orthogonal representation of $\overline{H}$ to $\overline{G}$. □