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INVARIANT NEUTRAL SUBSPACES FOR
HAMILTONIAN MATRICES[*]

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Abstract. Hamiltonian matrices with respect to a nondegenerate skewsymmetric or skewhermitian indefinite inner product in finite dimensional real, complex, or quaternion vector spaces are studied. Subspaces that are simultaneously invariant for the matrices and neutral in the indefinite inner product are of special interest. The dimension of maximal (by inclusion) such subspaces is identified in terms of the canonical forms and sign characteristics. Criteria for uniqueness of maximal invariant neutral subspaces are given. The important special case of invariant Lagrangian subspaces is treated separately. Comparisons are made between real, complex, and quaternion contexts; for example, for complex Hamiltonian matrices with respect to a nondegenerate skewhermitian inner product in a finite dimensional complex vector space, the (complex) dimension of (complex) maximal invariant neutral subspaces is compared to the (quaternion) dimension of (quaternion) maximal invariant neutral subspaces, and necessary and sufficient conditions are given for the two dimensions to coincide (this is not always the case).

Key words. Hamiltonian matrix, Quaternion vector space, Invariant subspace, Skewsymmetric matrix, Skewhermitian matrix, Neutral subspace, Lagrangian subspace.

AMS subject classifications. 15A21, 15A63, 15B57, 15B33.

1. Introduction. Let $\mathbb{F}$ be the real field $\mathbb{R}$, the complex field $\mathbb{C}$, or the skew field of real quaternions $\mathbb{H}$. The set $\mathbb{F}^{n \times 1}$ of n-component column vectors with entries in $\mathbb{F}$ is understood as an $\mathbb{F}$-vector space (right $\mathbb{H}$-vector space in case $\mathbb{F} = \mathbb{H}$), in the standard way. Denote by $\mathbb{F}^{m \times n}$ the set of all $m \times n$ matrices with entries in $\mathbb{F}$, understood as an $\mathbb{F}$-vector space (left $\mathbb{H}$-vector space in case $\mathbb{F} = \mathbb{H}$). Thus, $A \in \mathbb{F}^{m \times n}$ can be interpreted in the standard way as an $\mathbb{F}$-linear transformation on $\mathbb{F}^{n \times 1}$.

Fix an *involution* ϕ of $\mathbb{F}$, in other words, a bijective map $\phi : \mathbb{F} \longrightarrow \mathbb{F}$ having the properties that

$$\phi(xy) = \phi(y)\phi(x), \quad \phi(x + y) = \phi(x) + \phi(y), \quad \text{and} \quad \phi(\phi(x)) = x, \quad \forall \ x, y \in \mathbb{F}.$$

We assume furthermore that ϕ is continuous. (Note that in contrast to the complex case $\mathbb{F} = \mathbb{C}$, every antiautomorphism of $\mathbb{R}$ and of $\mathbb{H}$ is automatically continuous.) In particular, ϕ is the identity map if $\mathbb{F} = \mathbb{R}$, and ϕ is either the identity map or

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the complex conjugation if $\mathbb{F} = \mathbb{C}$. Let $S_\phi \in \mathbb{F}^{n \times m}$ stand for the matrix (or vector if $m = 1$) obtained from $S \in \mathbb{F}^{m \times n}$ by applying entrywise the involution ϕ to the *transposed* matrix $S^T \in \mathbb{F}^{n \times m}$.

It will be convenient to introduce the classification of involutions to be used in the present paper into 5 cases, as follows. Let ϕ be a fixed continuous involution of $\mathbb{F}$. The 5 cases are:

(I) $\mathbb{F} = \mathbb{R}$, $\phi = \mathrm{id}$;   (II) $\mathbb{F} = \mathbb{C}$, $\phi = \mathrm{id}$;   (III) $\mathbb{F} = \mathbb{C}$, $\phi = $ complex conjugation;

(IV) $\mathbb{F} = \mathbb{H}$, $\phi = $ quaternion conjugation;

(V) $\mathbb{F} = \mathbb{H}$, $\phi = $ involution different from quaternion conjugation.

In the sequel, the involutions of $\mathbb{H}$ different from the quaternion conjugation will be termed *nonstandard*. We note that all nonstandard involutions of $\mathbb{H}$ are similar to each other (and are not similar to the quaternion conjugation): If τ_1, τ_2 are two such involutions, then there exists an automorphism σ of $\mathbb{H}$ such that

$$(1.1) \qquad\qquad \tau_1(\alpha) = \sigma^{-1}(\tau_2(\sigma(\alpha))), \quad \forall \ \alpha \in \mathbb{H}.$$

This property, as well as many other properties of involutions to be used later on in the present paper, follows easily from the following description of involutions (see [21] and [22], for example):

PROPOSITION 1.1. *A map* $\phi : \mathbb{H} \longrightarrow \mathbb{H}$ *is an involution if and only if* ϕ *is real linear, and representing* ϕ *as a* 4×4 *real matrix with respect to the basis* $\{1, \mathsf{i}, \mathsf{j}, \mathsf{k}\}$, *where* $\mathsf{i}, \mathsf{j}, \mathsf{k}$ *are the standard imaginary units in* $\mathbb{H}$, *we have:*

$$\phi = \left[\begin{array}{cc} 1 & 0 \\ 0 & T \end{array} \right],$$

where either $T = -I_3$ *(in which case* ϕ *is the quaternion conjugation) or* T *is a* 3×3 *real orthogonal symmetric matrix with eigenvalues* 1, 1, -1.

In view of (1.1), indeed all nonstandard involutions can be treated in one category (V).

We also note that if ϕ is a nonstandard involution of $\mathbb{H}$, then there is a unique (up to multiplication by -1) quaternion $\beta = \beta(\phi)$ such that $\beta^2 = -1$ (this equality holds if and only if β has norm 1 and zero real part) and $\phi(\beta) = -\beta$. Conversely, for every $\beta \in \mathbb{H}$ with $\beta^2 = -1$, there exists a unique nonstandard involution ϕ of $\mathbb{H}$ such that

$$(1.2) \qquad\qquad\qquad \phi(\beta) = -\beta.$$

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Let ϕ be a continuous involution of $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. If $S \in \mathbb{F}^{m \times n}$ we denote by $S_\phi \in \mathbb{F}^{n \times m}$ the matrix obtained by applying ϕ entrywise to the transposed matrix S^T.

1.1. Neutral subspaces. Let there be given an invertible matrix $H \in \mathbb{F}^{n \times n}$ such that $H_\phi = -H$ (thus, n is even if ϕ is of type (I) or (II)).

A subspace $\mathcal{M} \in \mathbb{F}^{n \times 1}$ is said to be (H, ϕ)-*neutral* if $x_\phi H y = 0$ for all $x, y \in \mathcal{M}$. Maximal (by inclusion) (H, ϕ)-neutral subspaces can be identified in terms of their dimensions:

PROPOSITION 1.2. *Let $H = -H_\phi \in \mathbb{F}^{n \times n}$ be invertible. Then an (H, ϕ)-neutral subspace $\mathcal{M} \subseteq \mathbb{F}^{n \times 1}$ is a maximal (H, ϕ)-neutral subspace if and only if:*

(a) *$\dim \mathcal{M} = \lfloor n/2 \rfloor$, where $\lfloor x \rfloor$ denotes the maximal integer not exceeding x, in the cases (I), (II) and (IV);*
(b) *the minimum, call it $\nu(H)$, between the number of positive and the number of negative eigenvalues (counted with multiplicities) of the complex hermitian matrix iH (case (III)), or of the quaternion hermitian matrix $\beta(\phi)H$ (case (V)).*

In particular, there exist (H, ϕ)-neutral subspaces of dimension $\lfloor n/2 \rfloor$ in cases (I), (II), and (IV), and of dimension $\nu(H)$ in cases (III) and (V).

For the case (III) the result of Proposition 1.2 is well known, see [6] and [7], for example; see also [1], where a proof is given in the context of quaternion hermitian matrices. Case (V) is easily reduced to that context using the easily verifiable equality

$$x^*(\beta(\phi)Y)z = \beta(\phi)x_\phi Y z, \qquad \forall \ x, z \in \mathbb{H}^{n \times 1}, \quad \forall \ Y \in \mathbb{H}^{n \times n},$$

where ϕ is a nonstandard involution on $\mathbb{H}$. Finally, in cases (I), (II), and (IV) observe that there cannot be (H, ϕ)-neutral subspaces of dimension larger than $\lfloor n/2 \rfloor$ (this would contradict invertibility of H). We omit the proof that an (H, ϕ)-neutral subspace of dimension smaller than $\lfloor n/2 \rfloor$ is not maximal.

Of particular interest are Lagrangian subspaces. A subspace $\mathcal{M} \subseteq \mathbb{F}^{n \times 1}$ is said to be (H, ϕ)-*Lagrangian* if $\mathcal{M}$ is (H, ϕ)-neutral and $\dim \mathcal{M} = n/2$ (it is assumed here that n is even). Clearly, (H, ϕ)-Lagrangian subspaces (if exist) are maximal (H, ϕ)-neutral. Criteria for existence of (H, ϕ)-Lagrangian subspaces can be easily obtained from Proposition 1.2.

1.2. Hamiltonian matrices. If $H \in \mathbb{F}^{n \times n}$ is an invertible matrix such that $H_\phi = -H$, then a matrix $A \in \mathbb{F}^{n \times n}$ is called (H, ϕ)-*Hamiltonian* if the equality $HA = -A_\phi H$ holds true, in other words if $(HA)_\phi = HA$.

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For an (H,ϕ)-Hamiltonian matrix A, an $\mathbb{F}$-subspace $\mathcal{M} \subseteq \mathbb{F}^{n\times 1}$ is said to be *maximal A-invariant (H,ϕ)-neutral* if it is simultaneously A-invariant and (H,ϕ)-neutral and if no strictly larger subspace is simultaneously A-invariant and (H,ϕ)-neutral. For short, we say that maximal A-invariant (H,ϕ)-neutral subspaces are MIN (A,H,ϕ)-subspaces. Clearly, the set of MIN (A,H,ϕ)-subspaces (for a fixed A, H, ϕ) is closed and hence compact.

Maximal invariant neutral subspaces, and in particular, Lagrangian subspaces that are invariant for (H,ϕ)-Hamiltonian matrices, play an important role in many applications, such as optimal control systems, differential equations, factorizations of matrix valued functions and have been extensively studied (see [4], [12], [16], [17], [20], and [27]).

In this paper, we focus on the MIN (A,H,ϕ)-subspaces. It turns out that (for fixed A, H, and ϕ) all of them have the same dimension. We give formulas for this dimension; in several particular situations, such formulas are known in the literature (see [10], [11], and [12]). Also, parametrization of all MIN (A,H,ϕ)-subspaces is given in terms of certain invariant subspaces of A, under suitable hypotheses. The case of invariant Lagrangian subspaces is of special interest, and necessary and sufficient conditions for existence of those are provided, as well as criteria for uniqueness.

The main tool in our investigation are canonical forms for Hamiltonian matrices. These forms, in various contexts and appearances, are well known for real and complex matrices, and are known (perhaps not well known) for quaternion matrices. Some references for these forms, by no means a complete list, are [2], [8], [9], [14], [15], [22], [23], [24], [28], and [29]. For the reader's convenience, the canonical forms needed are reproduced in the present paper.

Besides the canonical forms, basic results on the quaternion algebra will be used, in particular, the following well-known description of similar quaternions. Quaternions $x,y \in \mathbb{H}$ are said to be *similar* if $x = \alpha^{-1}y\alpha$ for some $\alpha \in \mathbb{H} \setminus \{0\}$. For a quaternion $x = a_0 + a_1\mathsf{i} + a_2\mathsf{j} + a_3\mathsf{k}$, $a_0, a_1, a_2, a_3 \in \mathbb{R}$, we let $\Re(x) := a_0$ and $\mathfrak{V}(x) := a_1\mathsf{i} + a_2\mathsf{j} + a_3\mathsf{k}$ be the *real* and the *vector* parts of x, respectively.

PROPOSITION 1.3. *Two quaternions x and y are similar if and only if $\Re(x) = \Re(y)$ and $|\mathfrak{V}(x)| = |\mathfrak{V}(y)|$.*

We now briefly describe the contents of the paper section by section. Besides the introduction, the paper consists of 10 sections, numbered 2 through 11.

Sections 2–4 are of preparatory character, and the results there apply to all five cases (I)–(V). In Section 2, the root subspaces of square size matrices with entries in $\mathbb{F}$ are introduced and studied. This material is standard for real and complex matrices, but perhaps less so for quaternion matrices. The special case of Hamiltonian matrices

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is given a particular emphasis, and important orthogonality property of root subspaces is proved. In Section 3, the order of neutrality is introduced, based on the result that for a fixed (H, ϕ)-Hamiltonian matrix, the MIN (A, H, ϕ)-subspaces have the same dimension. In Section 4, we collect a few general results on invariant subspaces to be used throughout the paper.

Sections 5–9 are devoted to the detailed treatment of each case (I)–(V) separately. Namely, in Sections 5, 6, 7, 8, and 9, we treat the real case, the complex case with the identity involution, the complex case with the conjugation involution, the quaternion case with the conjugation involution, and the quaternion case with the involution other than conjugation, respectively. In each of these sections, we give a formula for the order of neutrality of Hamiltonian matrices (in terms of the suitable canonical form), parametrization of all MIN (A, H, ϕ)-subspaces (under appropriate conditions), and criteria for uniqueness of such subspaces. In Sections 8 and 9, we also compare with the orders of neutrality of real and complex matrices, in the following sense, as illustrated by example of complex $(H,^{*})$-Hamiltonian matrices A: The matrix A can be also considered as an $(H,^{*})$-Hamiltonian quaternion matrix; how the orders of neutrality in the context of $\mathbb{C}$ and in the context of $\mathbb{H}$ are related? Complete answers to that question and similar ones are given in Sections 8 and 9.

Finally, in the short Section 10, we indicate results on existence and uniqueness of invariant Lagrangian subspaces for Hamiltonian matrices, and in the concluding remarks we show how the results of the paper can be extended to the case of singular H (i.e., degenerate indefinite inner products).

1.3. Notation. We conclude the introduction with notation to be used throughout the paper. The standard imaginary units in $\mathbb{H}$ will be denoted $\mathsf{i}, \mathsf{j}, \mathsf{k}$; thus, $\mathsf{i}^2 = \mathsf{j}^2 = \mathsf{k}^2 = -1$ and $\mathsf{jk} = -\mathsf{kj} = \mathsf{i}$, $\mathsf{ij} = -\mathsf{ji} = \mathsf{k}$, $\mathsf{ki} = -\mathsf{ik} = \mathsf{j}$. We often consider $\mathbb{C}$ as embedded in $\mathbb{H}$, via identifying the complex imaginary unit i with $\mathsf{i} \in \mathbb{H}$. The conjugate quaternion $a_0 - a_1\mathsf{i} - a_2\mathsf{j} - a_3\mathsf{k}$ is denoted by $\overline{x}$ or by x^*, and $|x| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$ stands for the norm of x. We denote by $\mathrm{diag}\,(X_1, X_2, \ldots, X_p)$, or by $X_1 \oplus X_2 \oplus \cdots \oplus X_p$, the block diagonal matrix with diagonal blocks $X_1, \ldots, X_p$ (in that order). The notation A^T, resp., A^*, stands for the transpose, resp., conjugate transpose, of the matrix or vector A. The real subspace of $\mathbb{H}$ spanned by $\alpha, \beta \in \mathbb{H}$ is denoted $\mathrm{Span}_{\mathbb{R}}\,\{\alpha, \beta\}$. More generally, $\mathrm{Span}_{\mathbb{F}}\,\{x_1, \ldots, x_p\}$ stands for the $\mathbb{F}$-vector subspace spanned by the vectors $x_1, \ldots, x_p$.

The *spectrum* $\sigma(A)$ of a matrix $A \in \mathbb{F}^{n \times n}$ is the set of all (right) eigenvalues of A in $\mathbb{C}$ (if $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) or in $\mathbb{H}$ (if $\mathbb{F} = \mathbb{H}$).

$\lfloor x \rfloor$ denotes the maximal integer not exceeding $x \in \mathbb{R}$.

We use $e_{p,q}$ to denote the vector in $\mathbb{F}^{q \times 1}$ with 1 in the pth position and zeros

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elsewhere; for example, $e_{3,5} = [0 \;\; 0 \;\; 1 \;\; 0 \;\; 0]^T$.

The following matrices in standard forms and fixed notation will be used. The subscript in notation for a square size matrix always denotes the size of the matrix.

I and 0 (possibly with subscripts indicating the size) stand for the identity and the zero matrix, respectively.

The Jordan blocks:

$$J_m(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{bmatrix} \in \mathbb{H}^{m \times m}, \quad \lambda \in \mathbb{H}.$$

The real Jordan blocks:

$$J_{2m}(a \pm ib) = \begin{bmatrix} a & b & 1 & 0 & \cdots & 0 & 0 \\ -b & a & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & a & b & \cdots & 0 & 0 \\ 0 & 0 & -b & a & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & 0 & \cdots & -b & a \end{bmatrix} \in \mathbb{R}^{2m \times 2m}, \quad a \in \mathbb{R}, \;\; b \in \mathbb{R} \setminus \{0\}.$$

Real symmetric matrices:

$$(1.3) \qquad F_m = \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & & & 1 & 0 \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ 0 & 1 & & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \quad G_m = \begin{bmatrix} F_{m-1} & 0 \\ 0 & 0_1 \end{bmatrix}.$$

$$(1.4) \qquad \Xi_k = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & \iddots & & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ (-1)^{k-1} & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} = (-1)^{k-1} \Xi_k^T.$$

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Thus, Ξ_k is symmetric if k is odd, and skew-symmetric if k is even. More generally,

$$(1.5) \quad \Xi_m(\alpha) = \begin{bmatrix} 0 & 0 & \cdots & 0 & \alpha \\ 0 & 0 & \cdots & -\alpha & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & (-1)^{m-2}\alpha & \cdots & 0 & 0 \\ (-1)^{m-1}\alpha & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{H}^{m\times m}, \quad \alpha \in \mathbb{H}.$$

2. Root subspaces. Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, and consider a matrix $A \in \mathbb{F}^{n\times n}$. The *minimal polynomial* of A is a polynomial $p_A(t)$ having leading coefficient 1 and minimal degree, with real coefficients (if $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{H}$) or complex coefficients (if $\mathbb{F} = \mathbb{C}$) such that $p_A(A) = 0$. It is easy to see that $p_A(t)$ is uniquely determined by A. Write $p_A(t)$ as product of powers of irreducible factors:

$$(2.1) \qquad\qquad p_A(t) = p_1(t)^{m_1} \cdots p_k(t)^{m_k},$$

where the $p_j(t)$'s are distinct monic irreducible real polynomials (i.e., of the form $t-a$, a real, or of the form $t^2 + pt + q$, where $p, q \in \mathbb{R}$, with no real roots) if $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{H}$, or the $p_j(t)$'s are of the form $t - a$, $a \in \mathbb{C}$, if $\mathbb{F} = \mathbb{C}$, and the m_j's are positive integers. The subspaces

$$\mathcal{M}_j := \{u \,:\, u \in \mathbb{F}^{n\times 1}, \quad p_j(A)^{m_j} u = 0\}, \quad j = 1, 2, \ldots, k,$$

are called the *root subspaces* of A. Since $Ap_j(A)^{m_j} = p_j(A)^{m_j} A$, the root subspaces of A are A-invariant. The root subspace $\mathcal{M}_j$ is said to be associated with eigenvalues a_j of A if $p_j(t) = (t - a_j)^{m_j}$ or eigenvalues $a_j \pm ib_j$, where a_j is real and b_j is positive if $p_j(t) = (t^2 - 2a_j t + a_j^2 + b_j^2)^{m_j}$, and we denote $\mathcal{M}_j = \mathcal{M}_{a_j}(a)$ or $\mathcal{M}_j = \mathcal{M}_{a_j \pm ib_j}(A)$, as appropriate. Note that $a_j \pm ib_j$ are indeed eigenvalues of A in the case $\mathbb{F} = \mathbb{H}$; in general, the eigenvalues of quaternion matrices are closed under quaternion similarity $x \mapsto y^{-1}xy$, $y \in \mathbb{H} \setminus \{0\}$. See, for example, [3] and [30] for more information about eigenvalues, eigenvectors, and Jordan forms of quaternion matrices.

PROPOSITION 2.1. *Let $A \in \mathbb{F}^{n\times n}$, with the minimal polynomial factored as in* (2.1). *Then:*

(a) *The root subspaces decompose $\mathbb{F}^{n\times 1}$ into a direct sum:*

$$(2.2) \qquad\qquad \mathbb{F}^{n\times 1} = \mathcal{M}_1 \dotplus \mathcal{M}_2 \dotplus \cdots \dotplus \mathcal{M}_k.$$

(b) *For every A-invariant subspace $\mathcal{M} \subseteq \mathbb{F}^{n\times 1}$,*

$$(2.3) \qquad\qquad \mathcal{M} = (\mathcal{M} \cap \mathcal{M}_1) \dotplus \cdots \dotplus (\mathcal{M} \cap \mathcal{M}_k).$$

The proposition is well known in the real and complex cases, and can be proved in essentially the same way for the quaternion case.

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2.1. Root subspaces for Hamiltonian matrices and orthogonality. We now specialize the study of root subspaces for Hamiltonian matrices.

Let A be (H, ϕ)-Hamiltonian. Write $p_A(t) = p_o(t) + p_e(t)$, where $p_o(t)$ is the odd part (consisting of terms with odd powers of t) of $p_A(t)$, and $p_e(t)$ is the even part of $p_A(t)$. Since $HA^m = (-1)^m A_\phi^m H$ (m is a nonnegative integer), we have

$$0 = Hp_A(A) = H(p_o(A) + p_e(A)) = (-p_o(A_\phi) + p_e(A_\phi))H.$$

Thus, $-p_o(A_\phi) + p_e(A_\phi) = 0$. If ϕ is one of the types (I), (II), (IV), or (V), then

$$(-p_o(A) + p_e(A))_\phi = -p_o(A_\phi) + p_e(A_\phi) = 0.$$

Hence, $-p_o(A) + p_e(A) = 0$, and by the uniqueness of the minimal polynomial, we conclude that $p_o(t) \equiv 0$ if the degree m of $p_A(t)$ is even and $p_e(t) \equiv 0$ if m is odd. It is easy to see that then we can factorize $p_A(t)$ as follows:

$$
\begin{aligned}
p_A(t) = \pm (q_1(t)q_1(-t))^{m_1} \cdots (q_\ell(t)q_\ell(-t))^{m_\ell} \\
\times t^r(t^2 + \alpha_{\ell+1})^{m_{\ell+1}} \cdots (t^2 + \alpha_{\ell+s})^{m_{\ell+s}}.
\end{aligned}
$$
(2.4)

Here $q_1(t), \ldots, q_\ell(t)$ are distinct irreducible polynomials over $\mathbb{R}$ (if $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{H}$) or over $\mathbb{C}$ (if $\mathbb{F} = \mathbb{C}$) having roots with positive real parts, α_j's are distinct positive numbers, and r is even (odd) if m is even (odd). The sign $\pm$ appears because for $q_j(t) = t - \beta$, $\beta \in \mathbb{R} \setminus \{0\}$, the product $q_j(t)q_j(-t)$ has the form $-t^2 + \beta^2$. Note that the $\ell + s + 1$ polynomials

$$(q_1(t)q_1(-t))^{m_1}, \ldots, (q_\ell(t)q_\ell(-t))^{m_\ell}, t^r, (t^2 + \alpha_{\ell+1})^{m_{\ell+1}}, \ldots, (t^2 + \alpha_{\ell+s})^{m_{\ell+s}}$$

in (2.4) are pairwise coprime. Also, the factorization of the form (2.4) is unique up to permutation of factors.

If ϕ is of type (III) (in particular, $\mathbb{F} = \mathbb{C}$), then, denoting by $p^*(t)$ the polynomial whose coefficients are complex conjugates of the coefficients of the polynomial $p(t)$, we have

$$(-p_o^*(A) + p_e^*(A))_\phi = -p_o(A_\phi) + p_e(A_\phi) = 0,$$

and so $-p_o^*(A) + p_e^*(A) = 0$. Comparing with the minimal polynomial $p_A(t)$ of A, we see that

$$-p_o^*(t) + p_e^*(t) = p_o(t) + p_e(t) \quad \text{if} \quad m \quad \text{is even,}$$

and

$$-p_o^*(t) + p_e^*(t) = -p_o(t) - p_e(t) \quad \text{if} \quad m \quad \text{is odd.}$$

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Assuming first that m is even, we have

$$p_A(-t) = p_e(-t) + p_o(-t) = p_e(t) - p_o(t) = \overline{-p_o^*(\overline{t}) + p_e^*(\overline{t})} = \overline{p_A(\overline{t})},$$

therefore $p_A(t)$ factors as follows:

$$
\begin{aligned}
p_A(t) = \pm\, & ((-t - \alpha_1)(t - \overline{\alpha_1}))^{m_1} \cdots ((-t - \alpha_\ell)(t - \overline{\alpha_\ell}))^{m_\ell} \\
& \times (t - \alpha_{\ell+1})^{m_{\ell+1}} \cdots (t - \alpha_{\ell+s})^{m_{\ell+s}},
\end{aligned}
$$

(2.5)

where $\alpha_1, \ldots, \alpha_\ell$ are distinct complex numbers with positive real parts, $\alpha_{\ell+1}, \ldots, \alpha_{\ell+s}$ are distinct complex numbers with zero real parts (the number zero not excluded), and $m_1, \ldots, m_{\ell+s}$ are positive integers. If m is odd, we analogously obtain

$$p_A(-t) = p_e(-t) + p_o(-t) = p_e(t) - p_o(t) = \overline{-p_o^*(\overline{t}) + p_e^*(\overline{t})} = -\overline{p_A(\overline{t})},$$

and a factorization of type (2.5) follows again.

For an invertible $H \in \mathbb{F}^{n \times n}$ such that $H = -H_\phi$, denote by $[\cdot, \cdot]_{H,\phi}$ the H-inner product with respect to ϕ:

$$[x, y]_{H,\phi} := y_\phi H x = -\phi([y, x]_{H,\phi}), \qquad x, y \in \mathbb{F}^{n \times 1}.$$

If $\mathcal{L}$ is a subset of $\mathbb{F}^{n \times 1}$, we define

$$\mathcal{L}^{[\perp]_{H,\phi}} := \{x \in \mathbb{F}^{n \times 1} : [x, y]_{H,\phi} = 0 \quad \text{for all} \ \ y \in \mathcal{L}\},$$

the *H-orthogonal companion* of $\mathcal{L}$. Clearly, $\mathcal{L}^{[\perp]_{H,\phi}}$ is a subspace of $\mathbb{F}^{n \times 1}$. For an (H, ϕ)-Hamiltonian matrix $A \in \mathbb{F}^{n \times n}$, we have

(2.6)
$$[Ax, y]_{H,\phi} = [x, Ay]_{H,\phi}, \qquad \forall \ \ x, y \in \mathbb{F}^{n \times 1}.$$

Note also that if $\mathcal{L} \subseteq \mathbb{F}^{n \times 1}$ is an A-invariant subspace, then $\mathcal{L}^{[\perp]_{H,\phi}}$ is A-invariant as well. In the sequel, we often abbreviate $[\cdot, \cdot]_H = [\cdot, \cdot]_{H,\phi}$, $\mathcal{L}^{[\perp]_H} = \mathcal{L}^{[\perp]_{H,\phi}}$, with ϕ understood from the context.

THEOREM 2.2. *Assume ϕ is one of the types* (I), (II), (IV), *or* (V), *and let $A \in \mathbb{F}^{n \times 1}$ be (H, ϕ)-Hamiltonian. With factorization* (2.4) *of $p_A(t)$, let $\mathcal{N}_j$ be the sum of root subspaces for A corresponding to the factor $(q_j(t)q_j(-t))^{m_j}$ if $j = 1, 2, \ldots, \ell$, and to the factor $(t^2 + \alpha_j)^{m_j}$ if $j = \ell + 1, \ldots, \ell + s$. Let also $\mathcal{N}_0$ be the root subspace for A corresponding to the factor t^r. For $j = 1, 2, \ldots, \ell$, denote by $\mathcal{M}_j^+$, resp. $\mathcal{M}_j^-$, the root subspace for A corresponding to the factor $(q_j(t))^{m_j}$, resp. $(q_j(-t))^{m_j}$; thus, $\mathcal{N}_j = \mathcal{M}_j^+ \dot{+} \mathcal{M}_j^-$ for $j = 1, 2, \ldots, \ell$.*

Then:

(1) *The subspaces $\mathcal{N}_j$ are mutually (H, ϕ)-orthogonal:*

$$x_\phi H y = 0 \quad \forall \ \ x \in \mathcal{N}_j, \ y \in \mathcal{N}_k, \qquad j, k \in \{0, 1, \ldots, \ell + s\}, \quad j \neq k.$$

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(2) The subspaces $\mathcal{M}_1^+ \dotplus \cdots \dotplus \mathcal{M}_\ell^+$ and $\mathcal{M}_1^- \dotplus \cdots \dotplus \mathcal{M}_\ell^-$ are (H, ϕ)-neutral.
(3) If $\mathcal{M} \subseteq \mathcal{M}_1^\pm \dotplus \cdots \dotplus \mathcal{M}_\ell^\pm$ is an A-invariant subspace, then the subspace

$$(2.7) \qquad \widetilde{\mathcal{M}} := \left(\mathcal{M}^{[\perp]_H} \cap \left(\mathcal{M}_1^\mp \dotplus \cdots \dotplus \mathcal{M}_\ell^\mp \right) \right) \dotplus \mathcal{M}$$

is A-invariant and (H, ϕ)-neutral, and moreover $\widetilde{\mathcal{M}}$ is a $\mathrm{MIN}\,(A', H', \phi)$-subspace, where A' and H' are restrictions of A and H, respectively, to $\mathcal{N}_1 \dotplus \cdots \dotplus \mathcal{N}_\ell$.
Conversely, every $\mathrm{MIN}\,(A', H', \phi)$-subspace $\widetilde{\mathcal{M}}$ has the form (2.7), for some A-invariant subspace

$$\mathcal{M} \subseteq \mathcal{M}_1^\pm \dotplus \cdots \dotplus \mathcal{M}_\ell^\pm,$$

which is uniquely determined by $\widetilde{\mathcal{M}}$.

Proof. Let

$$f_j(t) = \begin{cases} (q_j(t) q_j(-t))^{m_j} & \text{for } j = 1, \dots, \ell; \\ (t^2 + \alpha_j)^{m_j} & \text{for } j = \ell+1, \dots, \ell+s; \\ t^r & \text{for } j = 0, \end{cases}$$

and let $p_j(t) = p_A(t)/f_j(t)$, $j = 0, 1, \dots, \ell+s$. Then for $x \in \mathcal{N}_j$, $y \in \mathcal{N}_k$, where $j, k = 0, 1, \dots, \ell+s$, $j \neq k$, we have $x = p_j(A)x'$, $y = p_k(A)y'$ for some $x, y \in \mathbb{F}^{n \times 1}$, and

$$x_\phi H y = x'_\phi p_j(A_\phi) H p_k(A) y' = \pm x' H p_j(A) p_k(A) y' = 0;$$

the last equality follows because the polynomial $p_j(t) p_k(t)$ is divisible by $p_A(t)$.

For part (2), the proof is analogous: Let $w_j(t) = f_A(t)/(q_j(t)^{m_j})$, $j = 1, 2, \dots, \ell$. Then for $x \in \mathcal{M}_j^+$, $y \in \mathcal{M}_k^+$, where $j, k = 1, \dots, \ell$, we have $x = w_j(A)x'$, $y = w_k(A)y'$ for some $x, y \in \mathbb{F}^{n \times 1}$, and

$$x_\phi H y = x'_\phi w_j(A_\phi) H w_k(A) y' = \pm x' H w_j(-A) w_k(A) y' = 0;$$

the last equality follows again because $w_j(-t) w_k(t)$ is divisible by $p_A(t)$. The proof for $\mathcal{M}_1^- \dotplus \cdots \dotplus \mathcal{M}_\ell^-$ is similar.

Part (3). The subspace (2.7) is clearly A-invariant. It is also (H, ϕ)-neutral in view of parts (1) and (2). To prove the maximality of (2.7), we first notice that

$$\begin{aligned} (2.8) \qquad \mathcal{M}^{[\perp]_H} = {} & \mathcal{N}_0 \dotplus \mathcal{N}_{\ell+1} \dotplus \cdots \dotplus \mathcal{N}_{\ell+s} \\ & + \left((\mathcal{M} \cap \mathcal{N}_1)^{[\perp]_{H_1}} \cap \mathcal{N}_1 \right) \dotplus \cdots \dotplus \left((\mathcal{M} \cap \mathcal{N}_\ell)^{[\perp]_{H_\ell}} \cap \mathcal{N}_\ell \right), \end{aligned}$$

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where H_j is a restriction of H to $\mathcal{N}_j$ (i.e., H_j is a matrix relative to a fixed basis in $\mathcal{N}_j$ such that $[x,y]_{H_j} = [x,y]_H$ for all $x,y \in \mathcal{N}_j$). Observe that each H_j is invertible, in view of invertibility of H and part (1). Equality (2.8) is easily verified using the (H,ϕ)-orthogonality property established in (1). Thus,

$$\mathcal{M}^{[\perp]_H} \cap (\mathcal{M}_1^{\mp} \dotplus \cdots \dotplus \mathcal{M}_\ell^{\mp}) = \dotplus_{j=1}^{\ell} \left((\mathcal{M} \cap \mathcal{N}_j)^{[\perp]_{H_j}} \cap \mathcal{M}_j^{\mp} \right).$$

Therefore, it suffices to prove that for each $j = 1, 2, \ldots, \ell$ and for $\mathcal{M} \subseteq \mathcal{M}_j^{\pm}$, the subspace

$$\left(\mathcal{M}^{[\perp]_{H_j}} \cap \mathcal{M}_j^{\mp} \right) \dotplus \mathcal{M}$$

is maximal H_j-neutral. We may assume that $\ell = 1$ and $\mathcal{N}_1 = \mathbb{F}^{n \times 1}$; thus, $H_1 = H$. Since both $\mathcal{M}_1^+$ and $\mathcal{M}_1^-$ are (H,ϕ)-neutral by (2), we must have $\dim \mathcal{M}_1^+ = \dim \mathcal{M}_1^-$ (otherwise there would be an (H,ϕ)-neutral subspace of dimension larger than $n/2$, a contradiction with the invertibility of H). Therefore n is even and

$$\dim \mathcal{M}_1^+ = \dim \mathcal{M}_1^- = n/2.$$

Next, observe that

$$(2.9) \qquad \mathcal{M}^{[\perp]_H} = \mathcal{M}_1^{\pm} \dotplus (\mathcal{M}^{[\perp]_H} \cap \mathcal{M}^{\mp}).$$

Indeed, the inclusion $\supseteq$ in (2.9) is obvious in view of (H,ϕ)-neutrality of $\mathcal{M}_1^{\pm}$ (recall that we suppose $\mathcal{M} \subseteq \mathcal{M}_1^{\pm}$). For the opposite inclusion, let $x = x_1 + x_2 \in \mathcal{M}^{[\perp]_H}$ with $x_1 \in \mathcal{M}_1^{\pm}$, $x_2 \in \mathcal{M}_1^{\mp}$. Then $x_2 = x - x_1 \in \mathcal{M}^{[\perp]_H}$, and (2.9) follows. Note also that $\dim \mathcal{M}^{[\perp]_H} = n - \dim \mathcal{M}$ (because H is invertible), hence in view of (2.9)

$$\dim \left(\mathcal{M}^{[\perp]_{H_j}} \cap \mathcal{M}_j^{\mp} \right) \dotplus \mathcal{M} = ((n - \dim \mathcal{M}) - n/2) + \dim \mathcal{M} = n/2,$$

and the direct statement of part (3) is proved.

Conversely, if $\widetilde{\mathcal{M}}$ is a MIN (A', H', ϕ)-subspace, then $\widetilde{\mathcal{M}}$ is given by formula (2.7) with

$$(2.10) \qquad \mathcal{M} = \widetilde{\mathcal{M}} \cap \left(\mathcal{M}_1^{\pm} \dotplus \cdots \dotplus \mathcal{M}_\ell^{\pm} \right).$$

Uniqueness of $\mathcal{M}$ is obvious because of formula (2.10). ∎

COROLLARY 2.3. *Assume the hypotheses and notation of Theorem* 2.2. *Then the general form of* MIN (A, H, ϕ)-*subspaces is*

$$(2.11) \qquad \mathcal{M} \dotplus \left(\mathcal{M}^{[\perp]_H} \cap \left(\mathcal{M}_1^{\mp} \dotplus \cdots \dotplus \mathcal{M}_\ell^{\mp} \right) \right) \dotplus \widetilde{\mathcal{M}}_0 \dotplus \widetilde{\mathcal{M}}_{\ell+1} \dotplus \cdots \dotplus \widetilde{\mathcal{M}}_{\ell+s},$$

where $\mathcal{M}$ is an arbitrary A-invariant subspace contained in $\mathcal{M}_1^{\pm} \dotplus \cdots \dotplus \mathcal{M}_\ell^{\pm}$, and $\widetilde{\mathcal{M}}_j$ is an arbitrary MIN (A_j, H_j, ϕ)-*subspace, where A_j and H_j are the restrictions of A and H, respectively, to $\mathcal{N}_j$, for $j = 0, \ell+1, \ldots, \ell+s$.*

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Theorem 2.2 and Corollary 2.3 admit analogues for complex conjugation (i.e., involution of type (III)):

THEOREM 2.4. *Assume ϕ is of the type* (III) *(complex conjugation), and let $A \in \mathbb{C}^{n \times n}$ be $(H,{}^*)$-Hamiltonian. With factorization* (2.5) *of $p_A(t)$, let $\mathcal{N}_j$ be the sum of root subspaces for A corresponding to the factor $((-t-\alpha_j)(t-\overline{\alpha_j}))^{m_j}$ if $j = 1, 2, \ldots, \ell$, and the root subspace corresponding to the factor $(t - \alpha_j)^{m_j}$ if $j = \ell + 1, \ldots, \ell + s$. For $j = 1, 2, \ldots, \ell$, denote by $\mathcal{M}_j^+$, resp. $\mathcal{M}_j^-$, the root subspace for A corresponding to the factor $(t - \overline{\alpha_j})^{m_j}$, resp. $(-t - \alpha_j)^{m_j}$. Then the statements* (1), (2) *and* (3) *of Theorem 2.2 hold true.*

The proof of Theorem 2.4 can be obtained similarly to that of Theorem 2.2. The result of Theorem 2.4 is well known and is found for example in [6] and [7].

COROLLARY 2.5. *Assume the hypotheses and notation of Theorem 2.4. Then the general form of* MIN $(A, H,{}^*)$*-subspaces is given by formula* (2.11), *with $\widetilde{\mathcal{M}}_0$ omitted.*

2.2. Invariant Lagrangian subspaces. Recall that a subspace $\mathcal{M} \subseteq \mathbb{F}^{n \times 1}$ is said to be (H, ϕ)-*Lagrangian* if $\mathcal{M}$ is (H, ϕ)-neutral and $\dim \mathcal{M} = n/2$ (it is assumed here that n is even). Clearly, invariant Lagrangian subspaces are, in particular, maximal invariant neutral subspaces (the maximality of Lagrangian subspaces follows from our standing hypothesis that H is invertible). Corollaries 2.3 and 2.5 when specialized to invariant Lagrangian subspaces, give the following results:

COROLLARY 2.6. *Assume the hypotheses and notation of Theorem 2.2. Then the general form of A-invariant (H, ϕ)-Lagrangian subspaces is*

$$(2.12) \qquad \mathcal{M} \dotplus \left(\mathcal{M}^{[\perp]_H} \cap \left(\mathcal{M}_1^{\mp} \dotplus \cdots \dotplus \mathcal{M}_\ell^{\mp} \right) \right) \dotplus \widetilde{\mathcal{M}}_0 \dotplus \widetilde{\mathcal{M}}_{\ell+1} \dotplus \cdots \dotplus \widetilde{\mathcal{M}}_{\ell+s},$$

where $\mathcal{M}$ is an arbitrary A-invariant subspace contained in $\mathcal{M}_1^{\pm} \dotplus \cdots \dotplus \mathcal{M}_\ell^{\pm}$, and $\widetilde{\mathcal{M}}_j$ is an arbitrary A-invariant (H, ϕ)-neutral subspace in $\mathcal{N}_j$ of dimension $(1/2)\dim \mathcal{N}_j$, for $j = 0, \ell + 1, \ldots, \ell + s$.

In particular, there exist A-invariant (H, ϕ)-Lagrangian subspaces if and only if such subspaces exist for every restriction of A and H to every root subspace for A corresponding to the zero eigenvalue or to a pair of nonzero complex conjugate eigenvalues with zero real parts.

COROLLARY 2.7. *Assume the hypotheses and notation of Theorem 2.4. Then the general form of A-invariant (H, ϕ)-Lagrangian subspaces is given by* (2.12), *with $\widetilde{\mathcal{M}}_0$ omitted. In particular, there exist A-invariant $(H,{}^*)$-Lagrangian subspaces if and only if such subspaces exist for every restriction of A and H to every root subspace for A corresponding to an eigenvalue with zero real part.*

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3. Order of neutrality. Let $\mathbb{F}$ be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and let ϕ be a continuous involution of $\mathbb{F}$.

THEOREM 3.1. *If $H = -H_\phi \in \mathbb{F}^{n \times n}$ is invertible, and A is (H, ϕ)-Hamiltonian, then all $\mathrm{MIN}\,(A, H, \phi)$-subspaces have the same dimension (as $\mathbb{F}$-vector subspaces).*

We omit the proof. For a particular ϕ a proof is given in [11]. In general, the proof follows the same outline as in [11].

The dimension identified in Theorem 3.1 is called the *order of neutrality* of the pair (A, H), and denoted $\gamma_{\mathbb{F}, \phi}(A, H)$, or in short $\gamma(A, H)$ where $\mathbb{F}$ and ϕ are understood from context. (This terminology was introduced in [10] for complex matrices that are selfadjoint with respect to an indefinite inner product.)

4. Preliminary results. The following obvious but important proposition allows us to reduce many proofs to the canonical forms of the pair A, H.

PROPOSITION 4.1. *Let H and A be as in Theorem 3.1, and let $S \in \mathbb{F}^{n \times n}$. Then an A-invariant subspace $\mathcal{M} \in \mathbb{F}^{n \times 1}$ is (H, ϕ)-neutral, resp., $\mathrm{MIN}\,(A, H, \phi)$-subspace or (H, ϕ)-Lagrangian subspace, if and only if the $S^{-1}AS$-invariant subspace $S^{-1}\mathcal{M}$ is $S_\phi H S$-neutral, resp., $\mathrm{MIN}\,(S^{-1}AS, S_\phi H S, \phi)$-subspace or $(S_\phi H S, \phi)$-Lagrangian subspace. In particular,*

$$\gamma(A, H) = \gamma(S^{-1}AS, S_\phi H S).$$

In the study of $\mathrm{MIN}\,(A, H, \phi)$-subspaces, the following result will be useful; in combination with Proposition 4.1 it allows us to reduce sometimes proofs for quaternion cases to complex cases.

THEOREM 4.2. *Let $X \in \mathbb{C}^{n \times n}$, and suppose that X has no real eigenvalues or pairs of complex conjugate nonreal eigenvalues. (The eigenvalues are understood over $\mathbb{C}$, i.e., with complex eigenvectors.) Then every X-invariant subspace $\mathcal{M} \subseteq \mathbb{H}^{n \times 1}$ has a complex basis, i.e., a basis for $\mathcal{M}$ as an $\mathbb{H}$-vector subspace that consists of complex vectors.*

Theorem 4.2 is proved in [26, Theorem 4.5].

Under the hypotheses of Theorem 4.2, we may identify the subspace $\mathcal{M}$ with $\widetilde{\mathcal{M}} \subseteq \mathbb{C}^{n \times 1}$ spanned by a complex basis in $\mathcal{M}$. Note that $\widetilde{\mathcal{M}}$ is independent of the choice of the complex basis. Indeed, let $f_1, \ldots, f_k$ and $g_1, \ldots, g_k$ be two complex bases in $\mathcal{M}$. Then we have

$$[f_1 \quad f_2 \quad \ldots \quad f_k] = [g_1 \quad g_2 \quad \ldots \quad g_k] B$$

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for some invertible $B \in \mathbb{H}^{k \times k}$. Write $B = B_1 + \mathrm{j}B_2$, where $B_1, B_2 \in \mathbb{C}^{k \times k}$. It is easy to see that

$$[f_1 \quad f_2 \quad \ldots \quad f_k] = [g_1 \quad g_2 \quad \ldots \quad g_k]B_1$$

and that B_1 is invertible. Thus, $\mathrm{Span}_{\mathbb{C}}\{f_1, \ldots, f_k\} = \mathrm{Span}_{\mathbb{C}}\{g_1, \ldots, g_k\}$.

We will often implicitly use the easily verifiable fact that a subspace $\mathcal{M}$ of $\mathbb{R}^{n \times 1}$ or of $\mathbb{C}^{n \times 1}$ is (H, ϕ)-neutral if and only if $\mathcal{M}_{\mathbb{H}}$ is (H, ϕ)-neutral as a subspace of $\mathbb{H}$; here $\mathcal{M}_{\mathbb{H}} = \mathrm{Span}_{\mathbb{H}}\{v_1, \ldots, v_p\}$, where $v_1, \ldots, v_p$ is a basis for $\mathcal{M}$.

In the rest of the paper, we consider each case (I)–(V) separately.

5. The real case. In this section, we assume $\mathbb{F} = \mathbb{R}$ and $\phi = \mathrm{id}$, so $A_\phi = A^T$ for any real matrix A. The canonical form of (H, id)-Hamiltonian matrices is given in the next theorem.

THEOREM 5.1. *Let $\mathbb{F} = \mathbb{R}$. Let A be (H, id)-Hamiltonian. Then there is an invertible real matrix S such that $S^{-1}AS$ and $S^T HS$ are block diagonal matrices*

$$(5.1) \qquad S^{-1}AS = A_1 \oplus \cdots \oplus A_s, \quad S^T HS = H_1 \oplus \cdots \oplus H_s,$$

where each pair of diagonal blocks (A_i, H_i) is of one of the following five types:

(i)

$$A_i = J_{2n_1}(0) \oplus J_{2n_2}(0) \oplus \cdots \oplus J_{2n_p}(0), \ \ H_i = \kappa_1 \Xi_{2n_1} \oplus \kappa_2 \Xi_{2n_2} \oplus \cdots \oplus \kappa_p \Xi_{2n_p},$$

where κ_j are signs ± 1;

(ii)

$$A_i = J_{2m_1+1}(0) \oplus -J_{2m_1+1}(0)^T \oplus \cdots \oplus J_{2m_q+1}(0) \oplus -J_{2m_q+1}(0)^T,$$

$$H_i = \begin{bmatrix} 0 & I_{2m_1+1} \\ -I_{2m_1+1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & I_{2m_q+1} \\ -I_{2m_q+1} & 0 \end{bmatrix};$$

(iii)

$$A_i = J_{\ell_1}(a) \oplus -J_{\ell_1}(a)^T \oplus \cdots \oplus J_{\ell_r}(a) \oplus -J_{\ell_r}(a)^T,$$

$$H_i = \begin{bmatrix} 0 & I_{\ell_1} \\ -I_{\ell_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & I_{\ell_r} \\ -I_{\ell_r} & 0 \end{bmatrix},$$

where $a > 0$, and the number a, the total number $2r$ of Jordan blocks, and the sizes $\ell_1, \ldots, \ell_r$ may depend on the particular diagonal block (A_i, H_i);

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(iv)

$$A_i = J_{2k_1}(a \pm ib) \oplus -J_{2k_1}(a \pm ib)^T \oplus \cdots \oplus J_{2k_s}(a \pm ib) \oplus -J_{2k_s}(a \pm ib)^T,$$

$$H_i = \begin{bmatrix} 0 & I_{2k_1} \\ -I_{2k_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & I_{2k_s} \\ -I_{2k_s} & 0 \end{bmatrix},$$

where $a, b > 0$, and again the numbers a and b, the total number $2s$ of Jordan blocks, and the sizes $2k_1, \ldots, 2k_s$ may depend on (A_i, H_i);

(v)

$$A_i = J_{2h_1}(\pm ib) \oplus J_{2h_2}(\pm ib) \oplus \cdots \oplus J_{2h_t}(\pm ib),$$

$$H_i = \eta_1 \begin{bmatrix} 0 & 0 & \cdots & 0 & \Xi_2^{h_1} \\ 0 & 0 & \cdots & -\Xi_2^{h_1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & (-1)^{h_1-2}\Xi_2^{h_1} & \cdots & 0 & 0 \\ (-1)^{h_1-1}\Xi_2^{h_1} & 0 & \cdots & 0 & 0 \end{bmatrix} \oplus \cdots$$

$$\oplus \eta_t \begin{bmatrix} 0 & 0 & \cdots & 0 & \Xi_2^{h_t} \\ 0 & 0 & \cdots & -\Xi_2^{h_t} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & (-1)^{h_t-2}\Xi_2^{h_t} & \cdots & 0 & 0 \\ (-1)^{h_t-1}\Xi_2^{h_t} & 0 & \cdots & 0 & 0 \end{bmatrix},$$

where $b > 0$ and $\eta_1, \ldots, \eta_t$ are signs ± 1. The parameters b, t, $h_1, \ldots, h_t$, and $\eta_1, \ldots, \eta_t$ may depend on the particular diagonal block (A_i, H_i).

The form (5.1) is uniquely determined by the pair (A, H), up to a simultaneous permutation of diagonal blocks in the right hand sides of (5.1).

Theorem 5.1 is found in many sources, see for example [14].

The signs κ_j and η_t constitute the *sign characteristic* of the pair (A, H). Thus, the sign characteristic assigns a sign 1 or -1 to every partial multiplicity of A corresponding to eigenvalues with zero real parts and positive imaginary parts, and to every even partial multiplicity corresponding to the zero eigenvalue (if A is singular). Partial multiplicities of a matrix here and everywhere in the paper are taken with repetitions, i.e., each partial multiplicity is repeated as many times as the number of Jordan blocks in the Jordan form of the matrix with given eigenvalue and given size.

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5.1. Order of neutrality and uniqueness. To identify the order of neutrality, we introduce the following notation: If $A \in \mathbb{R}^{n \times n}$ is (H, ϕ)-Hamiltonian, for a fixed nonzero pure imaginary eigenvalue ib_k with $b_k > 0$ of A, let $\{h_{j,k}\}_{j=1}^{p_k}$ be the partial multiplicities associated with ib_k, each $h_{j,k}$ repeated as many times as the block $J_{2h_{j,k}}(\pm ib_k)$ appears in the real Jordan form of A, and let W_k be the set of all indices j such that $h_{j,k}$ is odd. Then define

$$m(A, H; \pm ib_k) := \frac{1}{2} \dim \mathcal{R}(A; \pm ib_k) - \left| \sum_{j \in W_k} \eta_{j,k} \right|,$$

where $\eta_{j,k}$ is the sign as in type (v) corresponding to the Jordan block $J_{2h_{j,k}}(\pm ib_k)$, and where $\mathcal{R}(A; \pm ib_k) \subseteq \mathbb{R}^{n \times 1}$ is the root subspace of A corresponding to the pair of eigenvalues $\pm ib_k$.

In terms of the canonical form, the order of neutrality is identified as follows:

THEOREM 5.2. *Let $\mathbb{F} = \mathbb{R}$, and let $A \in \mathbb{R}^{n \times n}$ be (H, id)-Hamiltonian. Let S be the sum of root subspaces of A corresponding to all eigenvalues of A except the nonzero pure imaginary eigenvalues. Then*

$$(5.2) \qquad \gamma(A, H) = \sum_{k=1}^{r} m(A, H; \pm ib_k) + \frac{1}{2} \dim S = \frac{1}{2} n - \sum_{k=1}^{r} \left| \sum_{j \in W_k} \eta_{j,k} \right|,$$

where $ib_1, \ldots, ib_r$ are all distinct pure imaginary eigenvalues of A with positive imaginary parts.

Theorem 5.2 was proved in [12].

Next, consider the problem of uniqueness of maximal neutral subspaces.

THEOREM 5.3. *Let $A \in \mathbb{R}^{n \times n}$ be H-Hamiltonian, where $H = -H^T \in \mathbb{R}^{n \times n}$ is invertible. Then:*

(1) *If $\sigma(A) = \{0\}$, then a $\mathrm{MIN}\,(A, H, \mathrm{id})$-subspace is unique if and only if there are no blocks of odd size in the Jordan form of A, and the signs corresponding to the blocks of size divisible by 4 in the sign characteristic of (A, H) are all the same, the signs corresponding to the blocks of even size not divisible by 4 are all the same, and signs corresponding to the blocks of size divisible by 4 are opposite to the signs corresponding to the blocks of even size not divisible by 4.*

(2) *If $\sigma(A) = \{\pm ib\}$, where $b > 0$, then a $\mathrm{MIN}\,(A, H, \mathrm{id})$-subspace is unique if and only if all blocks in the canonical form of (A, H) with even h_i as in Theorem 5.1(v) (i.e., Jordan blocks in the Jordan form of A of size divisible by 4) have the same sign in the sign characteristic of (A, H), and all blocks in the*

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canonical form of (A, H) with odd h_i (i.e., Jordan blocks in the Jordan form of A of (necessarily even) size not divisible by 4) also have the same sign in the sign characteristic of (A, H).

Note that under the conditions in part (2), the blocks with even h_i may have sign different form that of the blocks with odd h_i.

A lemma will be needed for the proof of Theorem 5.3.

LEMMA 5.4. *Let $X \in \mathbb{R}^{p \times p}$, and assume that p is even and $X = [X_{i,j}]_{i,j=1}^{p/2}$ can be partitioned into blocks $X_{i,j} \in \mathbb{R}^{2 \times 2}$, where each $X_{i,j}$ belongs to the subalgebra*

$$\mathbb{C}_0 := \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in \mathbb{R}^{2 \times 2} \, : \, a, b \in \mathbb{R} \right\}.$$

Assume also that X has no real eigenvalues. Then every nonzero X-invariant subspace $\mathcal{L} \subseteq \mathbb{R}^{p \times 1}$ is spanned by the columns of a left invertible matrix Y of the form

$$Y = [Y_{i,j}]_{i,j=1}^{p/2,q/2} \in \mathbb{R}^{p \times q}, \quad \text{where} \quad Y_{i,j} \in \mathbb{C}_0, \quad \forall \; i = 1, 2, \ldots, p; \; j = 1, 2, \ldots, q.$$

Note that $\mathbb{C}_0$ is isomorphic to $\mathbb{C}$.

Proof. The lemma is likely to be known; we provide a proof anyway. To this end, define the map $\chi : \mathbb{C} \to \mathbb{R}^{2 \times 2}$ by

$$\chi(a + \mathrm{i}b) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \quad a, b \in \mathbb{R},$$

and extend it to matrices (also denoted by χ):

$$\chi : \mathbb{C}^{m \times n} \to \mathbb{R}^{2m \times 2n},$$

where

$$\chi \left([a_{j,k} + \mathrm{i}b_{j,k}]_{j,k=1}^{m,n} \right) = [\chi(a_{j,k} + \mathrm{i}b_{j,k})]_{j,k=1}^{m,n}, \qquad a_{j,k}, b_{j,k} \in \mathbb{R}.$$

The map χ is one-to-one, real linear, multiplicative, unital (maps I to I), and $\chi(Z^*) = (\chi(Z))^T$ for every $Z \in \mathbb{C}^{m \times n}$.

Let now X be as in Lemma 5.4, and let K be the (complex) Jordan form of $\chi^{-1}(X)$, so $\chi^{-1}(X)S = SK$ for some invertible $S \in \mathbb{C}^{p/2 \times p/2}$. Applying χ we obtain

(5.3) $$X\chi(S) = \chi(S)\chi(K),$$

and note that $\chi(K)$ is the real Jordan form of X. Now, if $\mathcal{L}$ is X-invariant, then clearly $\mathcal{L}$ is even dimensional, and letting J be the real Jordan form of the restriction

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$X|_{\mathcal{L}}$, we have $XY = YJ$ for some left invertible matrix Y whose columns span $\mathcal{L}$. Combining with (5.3) we see that

$$\chi(K) \cdot \chi(S)^{-1}Y = \chi(S)^{-1}Y \cdot J,$$

and it follows from the proof of [5, Theorem 12.4.2] that $\chi(S)^{-1}Y$ is in the range of χ. Then Y is in the range of χ as well. $\square$

Proof. Theorem 5.3. Without loss of generality we may (and do) assume that A and H are given by the right hand sides of the formulas in (5.1).

Part (1) follows from [19, Theorem 3.2], taking into account that by Theorem 5.2 in this case MIN (A, H, ϕ)-subspaces are Lagrangian.

For the proof of (2) we will use Lemma 5.4. Let A and H be given by the formulas in the right hand side of (v) of Theorem 5.1. We use the map χ defined in the proof of Lemma 5.4, and we take advantage of A and H being in the range of χ, and let

$$\widehat{A} = \chi^{-1}(A) = J_{h_1}(\mathrm{i}b) \oplus \cdots \oplus J_{h_t}(\mathrm{i}b) \in \mathbb{C}^{(n/2)\times(n/2)},$$

$$\widehat{H} = \chi^{-1}(H) = \eta_1\Xi_{h_1}(\mathrm{i}^{h_1}) \oplus \cdots \oplus \eta_t\Xi_{h_t}(\mathrm{i}^{h_t}) \in \mathbb{C}^{(n/2)\times(n/2)}.$$

Then $\widehat{H}$ is skewhermitian invertible and $\widehat{H}\widehat{A}$ is hermitian. Note that the pair $(\widehat{A}, \widehat{H})$ is already in the canonical form of Theorem 6.1. Letting $\widehat{A} = \mathrm{i}\widetilde{A}$, $\widehat{H} = \mathrm{i}\widetilde{H}$, where $\widetilde{A}$, $\widetilde{H}\widetilde{A}$ are hermitian, we see from the proof of Theorem 6.1 that the sign characteristic of $(\widetilde{A}, \widetilde{H})$ coincides with that of the pair $(\widehat{A}, \widehat{H})$. Thus, the conditions on the sizes of Jordan blocks and the sign characteristic in (2) can be recast as follows: The signs in the sign characteristic of $(\widetilde{A}, \widetilde{H})$ corresponding to blocks of even size h_i are all the same, and the signs in the sign characteristic of $(\widetilde{A}, \widetilde{H})$ corresponding to blocks of odd size h_i are all the same. By Theorems 2.2 and 2.4 in [18] this is the exact criterion for uniqueness of a MIN $(\widetilde{A}, \widetilde{H}, \phi)$-subspace, or what is the same, a MIN $(\widehat{A}, \widehat{H}, \phi)$-subspace. (The context in [18] is complex selfadjoint matrices in indefinite inner products and maximal invariant positive (or negative) semidefinite subspaces, but the proofs are essentially the same for MIN $(\widetilde{A}, \widetilde{H},^*)$-subspaces.) It remains to observe that by Lemma 5.4 there is a one-to-one correspondence between $\widehat{A}$-invariant $\widehat{H}$-neutral subspaces $\widehat{\mathcal{M}}$ and A-invariant H-neutral subspaces $\mathcal{M}$ given by the formula

$$\widehat{\mathcal{M}} = \mathrm{Span}_{\mathbb{C}}\{v_1, \ldots, v_p\} \quad \Longleftrightarrow \quad \mathcal{M} = \mathrm{Span}_{\mathbb{R}}\{\text{columns of } [\chi(v_1)\ \chi(v_2)\ \ldots\ \chi(v_p)]\},$$

where $v_1, \ldots, v_p \in \mathbb{C}^{(n/2)\times 1}$ are linearly independent (over $\mathbb{C}$). $\square$

Combining Corollary 2.3 with Theorem 5.3, we obtain:

THEOREM 5.5. *Assume that $H = -H^T \in \mathbb{R}^{n\times n}$ is invertible, $A \in \mathbb{R}^{n\times n}$ is H-Hamiltonian, and the conditions in Theorem 5.3(1) are satisfied (if A is singular), as*

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well as the conditions in Theorem 5.3(2) for every pure imaginary nonzero eigenvalue of A with positive imaginary part. Let $\lambda_0 = 0$ and $\lambda_{\ell+1}, \ldots, \lambda_{\ell+s}$ be all distinct pure imaginary eigenvalues of A with positive imaginary part, and for $j = 0, \ell+1, \ldots, \ell+s$, let $\widetilde{\mathcal{M}}_j$ be the unique (in view of Theorem 5.3) MIN (A_j, H_j, id)-subspace, where A_j and H_j are the restrictions of A and H, respectively, to the root subspace for A corresponding to λ_j.

Then all MIN (A, H, id)-subspaces $\widetilde{\mathcal{M}}$ are parameterized by the A-invariant subspaces $\mathcal{M}$ such that $\mathcal{M}$ is contained in the sum of root subspaces for A corresponding to the eigenvalues with positive real parts. The parametrization is given by the formula

$$\mathcal{M} \dotplus \left(\mathcal{M}^{[\perp]_H} \cap \mathcal{R}_- \right) \dotplus \widetilde{\mathcal{M}}_0 \dotplus \widetilde{\mathcal{M}}_{\ell+1} \dotplus \cdots \dotplus \widetilde{\mathcal{M}}_{\ell+s},$$

where $\mathcal{R}_-$ is the sum of the root subspaces for A corresponding to the eigenvalues with negative real parts.

In a similar fashion, under the hypotheses and notation of Theorem 5.5, a description of all MIN (A, H, id)-subspaces can be given, parameterized by the A-invariant subspaces $\mathcal{M}$ contained in the sum of root subspaces for A corresponding to the eigenvalues with negative real parts.

6. The case of ϕ of type (III). We assume here $\mathbb{F} = \mathbb{C}$ and ϕ the complex conjugation. Most of this case is essentially known in the literature. We present the results in the form suitable for the present paper, with a view of using these results later on.

We start with a canonical form.

THEOREM 6.1. Let $\widehat{H} \in \mathbb{C}^{n \times n}$ be an invertible skewhermitian matrix, and let $\widehat{X} \in \mathbb{C}^{n \times n}$ be $(\widehat{H}, {}^*)$-Hamiltonian. Then for some invertible complex matrix S, the matrices $S^* \widehat{H} S$ and $S^{-1} \widehat{X} S$ have simultaneously the following form:

$$(6.1) \qquad S^* \widehat{H} S = \bigoplus_{j=1}^{r} \eta_j \Xi_{\ell_j}(\mathrm{i}^{\ell_j}) \oplus \bigoplus_{v=1}^{s} \begin{bmatrix} 0 & F_{p_v} \\ -F_{p_v} & 0 \end{bmatrix} \oplus \bigoplus_{u=1}^{q} \zeta_u \Xi_{m_u}(\mathrm{i}^{m_u}),$$

$$(6.2) \qquad S^{-1} \widehat{X} S = \bigoplus_{j=1}^{r} J_{\ell_j}(0) \oplus \bigoplus_{v=1}^{s} \begin{bmatrix} -J_{p_v}(\overline{\alpha_v}) & 0 \\ 0 & J_{p_v}(\alpha_v) \end{bmatrix} \oplus \bigoplus_{u=1}^{q} J_{m_u}(\gamma_u),$$

where η_j, ζ_u are signs ± 1, the complex numbers $\alpha_1, \ldots, \alpha_s$ have positive real parts, and the complex numbers $\gamma_1, \ldots, \gamma_q$ are nonzero with zero real parts.

The form (6.1), (6.2) is uniquely determined by the pair $(\widehat{X}, \widehat{H})$, up to a permutation of associated pairs of primitive blocks.

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Conversely, if $\widehat{H}$, $\widehat{X}$ have the forms (6.1), (6.2), then $\widehat{H}$ is invertible skewhermitian and the equality $\widehat{H}\widehat{X} = (\widehat{H}\widehat{X})^*$ holds.

The signs η_j, ζ_u form the *sign characteristic* of the pair $(\widehat{X}, \widehat{H})$. Thus, the sign characteristic attaches a sign ± 1 to every Jordan block in the Jordan form of $\widehat{X}$ corresponding to a pure imaginary or zero eigenvalue of $\widehat{X}$.

Proof. Let $\widehat{X} = \mathrm{i}\widetilde{X}$, $\widehat{H} = \mathrm{i}\widetilde{H}$, then $\widetilde{H}$ is complex hermitian and $\widetilde{H}\widetilde{X}$ is hermitian. The well-known canonical form (given in many sources, see for example [6] and [7]) for the pair $(\widetilde{X}, \widetilde{H})$ under the transformations

$$\widetilde{X} \mapsto S^{-1}\widetilde{X}S, \qquad \widetilde{H} \mapsto S^*\widetilde{H}S, \qquad S \in \mathbb{C}^{n \times n} \text{ invertible},$$

has primitive pairs of blocks of the following two types:

(1) $\widetilde{X}_1 = J_p(\lambda)$, $\qquad \widetilde{H}_1 = \pm F_p$, where λ is real and p positive integer;
(2) $\widetilde{X}_2 = J_p(\overline{\alpha}) \oplus J_p(\alpha)$, $\qquad \widetilde{H}_2 = F_{2p}$, where α is nonreal with negative imaginary part, and p positive integer.

For $(\widetilde{X}_1, \widetilde{H}_1)$, we have

$$(6.3) \qquad S^{-1}(\mathrm{i}\widetilde{X}_1)S = J_p(\mathrm{i}\lambda), \quad S^*(\mathrm{i}\widetilde{H}_1)S = \begin{cases} \pm \Xi_p(\mathrm{i}^p) & \text{if } p \text{ is odd,} \\ \mp \Xi_p(\mathrm{i}^p) & \text{if } p \text{ is even,} \end{cases}$$

where $S = \mathrm{diag}\,(1, -\mathrm{i}, (-\mathrm{i})^2, \ldots, (-\mathrm{i})^{p-1})$. For $(\widetilde{X}_2, \widetilde{H}_2)$, we have

$$T^{-1}\begin{bmatrix} \mathrm{i}J_p(\overline{\alpha}) & 0 \\ 0 & \mathrm{i}J_p(\alpha) \end{bmatrix}T = \begin{bmatrix} -J_p(\overline{\gamma}) & 0 \\ 0 & J_p(\gamma) \end{bmatrix},$$

$$(6.4) \qquad T^*\begin{bmatrix} 0 & \mathrm{i}F_p \\ \mathrm{i}F_p & 0 \end{bmatrix}T = \begin{bmatrix} 0 & F_p \\ -F_p & 0 \end{bmatrix},$$

where $\gamma = \mathrm{i}\alpha$ and

$$T = \begin{bmatrix} \mathrm{diag}\,(1, \mathrm{i}, \ldots, \mathrm{i}^{p-1}) & 0 \\ 0 & \mathrm{diag}\,((-\mathrm{i})^s, (-\mathrm{i})^{s+1}, \ldots, (-\mathrm{i})^{s+p-1}) \end{bmatrix},$$

and where the integer s is adjusted so that $\mathrm{i}(-\mathrm{i})^{s+2} = 1$. Formulas (6.3), (6.4) transform the canonical form for pairs $(\widetilde{X}, \widetilde{H})$ such that the matrices $\widetilde{H}, \widetilde{H}\widetilde{X}$ are hermitian with invertible $\widetilde{H}$ to the form (6.1), (6.2).

The uniqueness statement follows from that of the above mentioned canonical form for the pair $(\widetilde{X}, \widetilde{H})$. $\quad \square$

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6.1. Order of neutrality and uniqueness. A formula for the order of neutrality is given in the next theorem.

THEOREM 6.2. *For invertible skewhermitian matrix $\widehat{H} \in \mathbb{C}^{n\times n}$ and $(\widehat{H},{}^{*})$-Hamiltonian matrix $\widehat{X} \in \mathbb{C}^{n\times n}$, the order of neutrality of the pair $(\widehat{X},\widehat{H})$ is equal to*

$$(6.5) \qquad \gamma(\widehat{X},\widehat{H}) := p + \sum_{k=1}^{a}\left(\left(\sum_{j=1}^{\infty}\lfloor\frac{j}{2}\rfloor p_{k,j}\right) + \min\{\sum_{j \text{ odd}} p_{k,j}^{+}, \sum_{j \text{ odd}} p_{k,j}^{-}\}\right).$$

Here, in reference to the canonical form (6.1), (6.2) of $(\widehat{X},\widehat{H})$, $\gamma_1,\ldots,\gamma_a$ are all the distinct γ_u's, together with zero (if blocks $\eta_j \Xi_{\ell_j}(\mathrm{i}^{\ell_j})$, $J_{\ell_j}(0)$ are present in (6.1), (6.2)), for each γ_k, where $k = 1,2,\ldots,a$, we let $p_{k,j}$ be the number of Jordan blocks in (6.2) of size $j \times j$ and eigenvalue γ_k, where among those blocks $p_{k,j}^{+}$ and $p_{k,j}^{-}$ stands for the number of blocks that have sign $+1$ and -1, respectively, in the corresponding block of (6.1); finally, $p = p_1 + \cdots + p_s$ is half of the sum of the sizes of Jordan blocks with eigenvalues with nonzero real parts in (6.2).

See [25] for a particular case of formula (6.5), given in the context of matrices $-\mathrm{i}\widehat{H}$ (hermitian) and $-\mathrm{i}\widehat{X}$.

Proof. It will be convenient to work instead with the matrices $\widetilde{H} = -\mathrm{i}\widehat{H}$ and $\widetilde{X} = -\mathrm{i}\widehat{X}$. In view of Corollary 2.5, we need to consider only two cases: (1) $\sigma(\widetilde{X})$ is nonreal; (2) $\sigma(\widetilde{X}) = \{\lambda\}$ is real singleton. The first case is handled by the same Corollary 2.5 (the order of neutrality in this case is equal to $n/2$). Consider the case (2), and using the canonical form of the pair $(\widetilde{X},\widetilde{H})$ we can assume without loss of generality (Proposition 4.1) that $\lambda = 0$, and

$$\widetilde{X} = J_{\ell_1}(0) \oplus \cdots \oplus J_{\ell_r}(0), \quad \widetilde{H} = \eta_1 F_{\ell_1} \oplus \cdots \oplus \eta_r F_{\ell_r},$$

where we assume without loss of generality, using simultaneous permutation of constituent blocks in $\widetilde{X}$ and $\widetilde{H}$ if necessary, that $\ell_1,\ldots,\ell_s$ are even, $\ell_{s+1},\ldots,\ell_r$ are odd, $\ell_{s+2v-1} \geq \ell_{s+2v}$ for $v = 1,2,\ldots,u$, and the signs are as follows:

$$\eta_{s+1} \cdot \eta_{s+2} = -1, \ldots, \eta_{s+2u-1} \cdot \eta_{s+2u} = -1,$$

and all η_j's with $j = s + 2u + 1,\ldots,r$ are the same; here u is the minimum between the number of 1's among the η_j's for those indices j for which ℓ_j is odd, and the number of -1's among such η_j's. Formula (6.5) gives

$$(6.6) \qquad\qquad \gamma(\widetilde{X},\widetilde{H}) = \left(\sum_{j=1}^{r}\lfloor\frac{\ell_j}{2}\rfloor\right) + u.$$

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Clearly, there cannot be $\widetilde{X}$-invariant $\widetilde{H}$-neutral subspace of dimension larger than $\gamma(\widetilde{X}, \widetilde{H})$, because it is not difficult to see that the right hand side of (6.6) is equal to the maximal possible dimension of $\widetilde{H}$-neutral subspaces (Proposition 1.2). To construct a $\widetilde{X}$-invariant $\widetilde{H}$-neutral subspace $\mathcal{M}$ of dimension equal to $\gamma(\widetilde{X}, \widetilde{H})$, we define subspaces in $\mathbb{C}^{n \times 1}$ as follows:

$$
\mathcal{M}_j = \begin{cases}
\mathrm{Span}_{\mathbb{C}}\{e_{1,\ell_j}, \ldots, e_{\ell_j/2,\ell_j}\} & \text{for } j = 1, \ldots, s; \\[1mm]
\mathrm{Span}_{\mathbb{C}}\{e_{1,\ell_{s+2v-1}+\ell_{s+2v}}, \ldots, e_{(1/2)(\ell_{s+2v-1}-1),\ell_{s+2v-1}+\ell_{s+2v}}, \\
\quad e_{\ell_{s+2v-1}+1,\ell_{s+2v-1}+\ell_{s+2v}}, \ldots, e_{\ell_{s+2v-1}+(1/2)(\ell_{s+2v}-1),\ell_{s+2v-1}+\ell_{s+2v}}, \\
\quad e_{(1/2)(\ell_{s+2v-1}+1),\ell_{s+2v-1}+\ell_{s+2v}} + e_{\ell_{s+2v-1}+(1/2)(\ell_{s+2v}+1),\ell_{s+2v-1}+\ell_{s+2v}}\} \\
\quad \text{for } v = 1, 2, \ldots, u, \text{ and we set } j = s + 2v - 1; \\[1mm]
\mathrm{Span}_{\mathbb{C}}\{e_{1,\ell_j}, \ldots, e_{(1/2)(\ell_j-1),\ell_j}\} & \text{for } j = s+2u+1, \ldots, r.
\end{cases}
$$

Let

$$\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_s \oplus \mathcal{M}_{s+1} \oplus \mathcal{M}_{s+3} \oplus \cdots \oplus \mathcal{M}_{s+2u-1} \oplus \mathcal{M}_{s+2u+1} \oplus \mathcal{M}_{s+2u+2} \oplus \cdots \oplus \mathcal{M}_r.$$

A straightforward inspection shows that $\mathcal{M}$ has all the required properties. $\square$

The cases of uniqueness of a MIN $(\widehat{X}, \widehat{H}, ^*)$-subspace are described as follows:

THEOREM 6.3. *Let $\widehat{X}$ and $\widehat{H}$ be as in Theorem 6.1. Assume that $\sigma(\widehat{X}) = \{ib\}$, where $b \in \mathbb{R}$, a singleton. Then a MIN $(\widehat{X}, \widehat{H}, ^*)$-subspace is unique if and only if the signs in the sign characteristic of $(\widehat{X}, \widehat{H})$ corresponding to the Jordan blocks of even size in the Jordan form of $\widehat{X}$ are all the same, and the signs corresponding to the Jordan blocks of odd size in the Jordan form of $\widehat{X}$ are also all the same.*

For the proof of Theorem 6.3 see the proofs of Theorems 2.2 and 2.4 in [18]. (The context in [18] is complex selfadjoint matrices in indefinite inner products and maximal invariant positive (or negative) semidefinite subspaces, but the proofs are essentially the same for MIN $(\widehat{X}, \widehat{H}, ^*)$-subspaces; at some point in the proof formula (6.5) will be used.)

Analogously to Theorem 5.5, a parametrization can be given of all MIN $(A, H, ^*)$-subspaces in terms of A-invariant subspace contained in the sum of all root subspaces corresponding to eigenvalues with positive real parts, alternatively with negative real parts, under the conditions described in Theorem 6.3. We leave the statement of this parametrization to the interested readers.

7. The case of ϕ of type (II). In this section, we assume $\mathbb{F} = \mathbb{C}$ and $\phi = \mathrm{id}$; thus, $A_\phi = A^T$.

We start with the canonical form for complex (H, id)-Hamiltonian matrices which

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is available in many sources, e.g. [15], and in terms of pairs of complex symmetric/skewsymmetric matrices it is given in [29], for example. We present the canonical form as given in [16].

THEOREM 7.1. *Let* $H \in \mathbb{C}^{n \times n}$ *be skew-symmetric and invertible (in particular,* n *is even), and let* $A \in \mathbb{C}^{n \times n}$ *be* (H, id)*-Hamiltonian. Then there exists an invertible matrix* $P \in \mathbb{C}^{n \times n}$ *such that* $P^{-1}AP$ *and* $P^T HP$ *are block diagonal matrices*

$$(7.1) \qquad P^{-1}AP = A_1 \oplus A_2 \oplus A_3, \quad P^T HP = H_1 \oplus H_2 \oplus H_3,$$

where the blocks have the following forms:

(i) $\qquad A_1 = J_{2n_1}(0) \oplus \cdots \oplus J_{2n_p}(0), \quad H_1 = \Xi_{2n_1} \oplus \cdots \oplus \Xi_{2n_p},$

with $n_1, \ldots, n_p$ *are positive integers;*

(ii) $\quad A_2 = \begin{bmatrix} J_{2m_1+1}(0) & 0 \\ 0 & J_{2m_1+1}(0) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} J_{2m_q+1}(0) & 0 \\ 0 & J_{2m_q+1}(0) \end{bmatrix},$

$$H_2 = \begin{bmatrix} 0 & \Xi_{2m_1+1} \\ -\Xi_{2m_1+1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & \Xi_{2m_q+1} \\ -\Xi_{2m_q+1} & 0 \end{bmatrix},$$

with $m_1, \ldots, m_q$ *nonnegative integers;*

(iii) $\qquad A_3 = A_{3,1} \oplus \cdots \oplus A_{3,k}, \quad H_3 = H_{3,1} \oplus \cdots \oplus H_{3,k} ,$

where

$$A_{3,j} = \begin{bmatrix} J_{\ell_{j,1}}(\lambda_j) & 0 \\ 0 & -J_{\ell_{j,1}}(\lambda_j)^T \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} J_{\ell_{j,q_j}}(\lambda_j) & 0 \\ 0 & -J_{\ell_{j,q_j}}(\lambda_j)^T \end{bmatrix},$$

$$H_{3,j} = \begin{bmatrix} 0 & I_{\ell_{j,1}} \\ -I_{\ell_{j,1}} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & I_{\ell_{j,q_j}} \\ -I_{\ell_{j,q_j}} & 0 \end{bmatrix},$$

with positive integers $\ell_{j,1}, \ldots, \ell_{j,q_j}$, *and* $\lambda_j \in \mathbb{C}$ *with* $\Re(\lambda_j) > 0$ *or* $\Re(\lambda_j) = 0$ *and* $\Im(\lambda_j) > 0$ *for* $j = 1, \ldots, k$. *Moreover,* $\lambda_1, \ldots, \lambda_k$ *are pairwise distinct.*

The form (7.1) *is uniquely determined by the pair* (A, H), *up to a simultaneous permutation of primitive diagonal blocks in the right hand sides of* (7.1).

By inspection of the canonical form (7.1), and using Proposition 4.1, we obtain the following formula for the order of neutrality:

THEOREM 7.2. *Let* (A, H) *be as Theorem* 7.1. *Then the order of neutrality of* (A, H) *is equal to* $n/2$.

Thus, in this case, the MIN (A, H, id)-subspaces are J-Lagrangian.

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A description of all A-invariant (H, id)-Lagrangian subspaces was obtained in [4], using the Schur canonical form (different from the canonical form above). In particular, we have the following criterion for uniqueness of MIN (A, H, id)-subspaces.

THEOREM 7.3. *Let (A, H) be as in Theorem 7.1. If $\sigma(A) = \{\lambda, -\lambda\}$, where λ is nonzero with zero real part, then a MIN (A, H, id)-subspace is non-unique. If $\sigma(A) = \{0\}$, then a MIN (A, H, id)-subspace is unique if and only if A is unicellular, i.e., its Jordan form consists of only one Jordan block, necessarily of even size.*

Proof. We verify only the part of Theorem 7.3 under the hypothesis that $\sigma(A) = \{0\}$. The "if" part is obvious: If A is unicellular, then there exists a unique A-invariant subspace of dimension $n/2$. We prove the "only if" part. Without loss of generality we may assume that A and H are given by the right hand sides of formulas (ii) in Theorem 7.1. If there are blocks of odd size in the Jordan form of A, then an inspection of the formulas (ii) reveals the non-uniqueness of a MIN (A, H, id)-subspace. Assume now that A has more than Jordan block of even size. For notational convenience suppose

$$A = J_{2n_1}(0) \oplus J_{2n_2}(0), \qquad H = \Xi_{2n_1} \oplus \Xi_{2n_2},$$

and assume without loss of generality that $n_1 \geq n_2$. Then

$$\mathcal{M}_1 := \mathrm{Span}_{\mathbb{C}} \{e_{1,2n_1+2n_2}, \ldots, e_{n_1,2n_1+2n_2}, e_{2n_1+1,2n_1+2n_2}, \ldots, e_{2n_1+n_2,2n_1+2n_2}\}$$

and

$$\mathcal{M}_2 := \mathrm{Span}_{\mathbb{C}} \{e_{n_1+1,2n_1+2n_2} + \mathrm{i} e_{2n_1+n_2+1,2n_1+2n_2}, e_{n_1,2n_1+2n_2} + \mathrm{i} e_{2n_1+n_2,2n_1+2n_2},$$
$$e_{n_1-1,2n_1+2n-2}, \ldots, e_{1,2n_1+2n_2}, e_{2n_1+n_2-1,2n_1+2n_2}, \ldots, e_{2n_1+1,2n_1+2n_2}\}$$

are two different $(n_1 + n_2)$-dimensional $J_{2n_1}(0) \oplus J_{2n_2}(0)$-invariant $(\Xi_{2n_1} \oplus \Xi_{2n_2}, \mathrm{id})$-neutral subspaces. ∎

THEOREM 7.4. *Let A and H be as in Theorem 7.1. Assume that A has no nonzero pure imaginary eigenvalues, and $\dim(\mathrm{Ker}\, A) \leq 1$. Then all MIN (A, H, id)-subspaces, equivalently all A-invariant H-Lagrangian subspaces, are parameterized by the A-invariant subspaces $\mathcal{M}$ contained in the sum of root subspaces for A corresponding to the eigenvalues with positive real parts. The parametrization is given by the formula*

$$\mathcal{M} \dotplus \left(\mathcal{M}^{[\perp]_H} \cap \mathcal{R}_-\right) \dotplus \widetilde{\mathcal{M}}_0,$$

where $\mathcal{R}_-$ is the sum of the root subspaces for A corresponding to the eigenvalues with negative real parts, and (if A is singular) $\widetilde{\mathcal{M}}_0$ is the unique A-invariant subspace of dimension $(1/2)\dim(\mathrm{Ker}(A^n))$.

Theorem 7.4 is an immediate corollary of Theorem 7.3 and Corollary 2.3.

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8. The case of ϕ of type (IV). In this section, we assume that $\mathbb{F} = \mathbb{H}$ and ϕ is the conjugation. Fix an invertible $H = -H^* \in \mathbb{H}^{n \times n}$, and let $A \in \mathbb{H}^{n \times n}$ be H-Hamiltonian. We study here subspaces $\mathcal{M} \subseteq \mathbb{H}^{n \times 1}$ that are MIN $(A, H,^*)$.

A canonical form comes first.

THEOREM 8.1. *Let $H \in \mathbb{H}^{n \times n}$ be an invertible skewhermitian matrix, and let $X \in \mathbb{H}^{n \times n}$ be $(H,^*)$-Hamiltonian. Then for some invertible quaternion matrix S, the matrices $S^* H S$ and $S^{-1} X S$ have simultaneously the following form:*

$$(8.1) \qquad S^* H S = \bigoplus_{j=1}^{r} \eta_j \Xi_{\ell_j}(\mathrm{i}^{\ell_j}) \oplus \bigoplus_{v=1}^{s} \begin{bmatrix} 0 & F_{p_v} \\ -F_{p_v} & 0 \end{bmatrix} \oplus \bigoplus_{u=1}^{q} \zeta_u \Xi_{m_u}(\mathrm{i}^{m_u}),$$

$$(8.2) \qquad S^{-1} X S = \bigoplus_{j=1}^{r} J_{\ell_j}(0) \oplus \bigoplus_{v=1}^{s} \begin{bmatrix} -J_{p_v}(\overline{\alpha_v}) & 0 \\ 0 & J_{p_v}(\alpha_v) \end{bmatrix} \oplus \bigoplus_{u=1}^{q} J_{m_u}(\gamma_u),$$

where η_j, ζ_u are signs ± 1 with the additional condition that $\eta_j = 1$ if ℓ_j is odd, the complex numbers $\alpha_1, \dots, \alpha_s$ have positive real parts, and the complex numbers $\gamma_1, \dots, \gamma_q$ are with zero real parts and positive imaginary parts.

The form (8.1), (8.2) is uniquely determined by the pair (X, H), up to an arbitrary simultaneous permutation of primitive blocks in each of the three parts:

$$\left(\bigoplus_{j=1}^{r} \eta_j \Xi_{\ell_j}(\mathrm{i}^{\ell_j}), \ \bigoplus_{j=1}^{r} J_{\ell_j}(0) \right), \quad \left(\bigoplus_{u=1}^{q} \zeta_u \Xi_{m_u}(\mathrm{i}^{m_u}), \ \bigoplus_{u=1}^{q} J_{m_u}(\gamma_u) \right),$$

and

$$\left(\bigoplus_{v=1}^{s} \begin{bmatrix} 0 & F_{p_v} \\ -F_{p_v} & 0 \end{bmatrix}, \ \bigoplus_{v=1}^{s} \begin{bmatrix} -J_{p_v}(\overline{\alpha_v}) & 0 \\ 0 & J_{p_v}(\alpha_v) \end{bmatrix} \right),$$

and up to replacements of some α_k's with their complex conjugates.

Conversely, if H, X have the forms (8.1), (8.2), then H is invertible skewhermitian, and X is $(H,^*)$-Hamiltonian.

Theorem 8.1 is given in many sources, see, for example, [2], [24], and [28]. The formulation as in Theorem 8.1 is taken from [24].

REMARK 8.2. As seen from the canonical form (8.1), (8.2), the *sign characteristic* of an H-Hamiltonian matrix A assigns a sign ± 1 to every even partial multiplicity of A corresponding to the zero eigenvalue, and to all partial multiplicities of every nonzero eigenvalue with zero real part (recall that under quaternions, if $\lambda_0 \in \mathbb{H}$ is an eigenvalue of A, then so is every $\mu_0 \in \mathbb{H}$ with $\Re(\mu_0) = \Re(\lambda_0)$ and $|\mathfrak{V}(\mu_0)| = |\mathfrak{V}(\lambda_0)|$).

REMARK 8.3. One can take all γ_u in Theorem 8.1 to have negative imaginary parts. Indeed, the following formulas make explicit the replacement of γ_u by its

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complex conjugate:

$$S^{-1} J_{m_u}(-\gamma_u) S = J_{m_u}(\gamma_u),$$

$$S^*(\Xi_{m_u}(\mathrm{i}^{m_u})) S = \begin{cases} -\Xi_{m_u}(\mathrm{i}^{m_u}) & \text{if } m_u \text{ is odd,} \\ \Xi_{m_u}(\mathrm{i}^{m_u}) & \text{if } m_u \text{ is even,} \end{cases}$$

where $S = \mathrm{j}I$. Note that under this replacement ζ_u reverses to its negative if m_u is odd, and remains invariant if m_u is even.

8.1. Formula for the order of neutrality. Given an $(H,^*)$-Hamiltonian $X \in \mathbb{H}^{n \times n}$, in the next theorem we identify the order of neutrality $\gamma(X,H)$ in terms of the canonical form. Let (8.1), (8.2) be the canonical form of the pair (X,H). Let $\gamma_1, \ldots, \gamma_a$ be all the distinct γ_u's. For each γ_k, $k = 1, 2, \ldots, a$, let $p_{k,j}$ be the number of blocks $J_{m_u}(\gamma_u)$ in (8.2) of size $j \times j$ and eigenvalue γ_k. Among those blocks let $p_{k,j}^+$ and $p_{k,j}^-$ be the number of blocks that have sign $+1$ and -1, respectively, in the corresponding block $\zeta_u \Xi_{m_u}(\mathrm{i}^{m_u})$ of (8.1); thus, $p_{k,j} = p_{k,j}^+ + p_{k,j}^-$, for $j = 1, 2, \ldots$, and $k = 1, 2, \ldots, a$. For the eigenvalue zero, let r be the number of nilpotent Jordan blocks in (8.2), let q_m be the number of blocks $J_{\ell_j}(0)$ in (8.2) of size $m \times m$. Finally, let $p = p_1 + \cdots + p_s$, half of the sum of the sizes of blocks with eigenvalues with nonzero real parts in (8.2).

THEOREM 8.4. *The order of neutrality of (X,H) is given by*

$$\gamma(X,H) = p + \sum_{k=1}^{a} \left(\left(\sum_{j=1}^{\infty} \lfloor \frac{j}{2} \rfloor p_{k,j} \right) + \min\{ \sum_{j \text{ odd}} p_{k,j}^+, \sum_{j \text{ odd}} p_{k,j}^- \} \right)$$

$$(8.3) \qquad + \left(\sum_{m \text{ even}} (m/2) q_m \right) + \left(\sum_{m \text{ odd}} ((m-1)/2) q_m \right) + \lfloor (\sum_{m \text{ odd}} q_m)/2 \rfloor.$$

Proof. We assume that H and X are given by the right hand sides of (8.1) and (8.2), respectively (Proposition 4.1). For m odd, we replace $\lfloor (\sum_{m \text{ odd}} q_m)/2 \rfloor$ pairs of blocks $\Xi_m(\mathrm{i}^m)$, $J_m(0)$ in (8.1), (8.2) by $-\Xi_m(\mathrm{i}^m)$, $J_m(0)$, using the formulas

$$(-\mathrm{j})I_m \cdot J_m(0) \cdot \mathrm{j}I_m = J_m(0), \qquad (-\mathrm{j})I_m \cdot \Xi_m(\mathrm{i}^m) \cdot \mathrm{j}I_m = -\Xi_m(\mathrm{i}^m), \quad m \text{ odd.}$$

Formula (6.5) now yields an X-invariant H-neutral complex subspace $\mathcal{M}$ of dimension equal to the right hand side of (8.3). It is easy to see that complex vectors are linearly independent over $\mathbb{C}$ if and only if they are linearly independent over $\mathbb{H}$. Thus, $\mathcal{M}$ has the same dimension as a subspace of $\mathbb{H}^{n \times 1}$.

It remains to show that there do not exist quaternion X-invariant H-neutral subspaces of larger dimension. In view of Corollary 2.3 we need only to consider two

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cases:

$$(1) \quad X = \bigoplus_{u=1}^{q} J_{m_u}(\gamma), \quad H = \bigoplus_{u=1}^{q} \zeta_u \Xi_{m_u}(\mathrm{i}^{m_u}),$$

where γ is a complex number with zero real part and positive imaginary part;

$$(2) \quad X = \bigoplus_{j=1}^{r} J_{\ell_j}(0), \quad H = \bigoplus_{j=1}^{r} \eta_j \Xi_{\ell_j}(\mathrm{i}^{\ell_j}),$$

where the signs η_j are taken to be 1 for odd ℓ_j.

If (1) holds true, then by Theorem 4.2 we are done in view of the formula (6.5). It remains to consider case (2). In this case, observe that the right hand side of (8.3) is equal to $n/2$ if n is even, and to $(n-1)/2$ if n is odd. This is exactly the maximal dimension of an H-neutral subspace (Proposition 1.2), so obviously no X-invariant H-neutral subspace can have a larger dimension. ☐

8.2. Uniqueness of maximal invariant neutral subspaces. Theorem 4.2, combined with Theorem 6.3, and taking advantage of the canonical form of (X, H) given in Theorem 8.1, yields the first part of the following uniqueness result:

THEOREM 8.5. *Let H, X be as in Theorem 8.1.*

(a) *Assume $\sigma(X) \cap \mathbb{C} = \{\pm \mathrm{i}b\}$, $b > 0$. Then a $\mathrm{MIN}\,(X, H,^*\,)$-subspace is unique if and only if the signs in the sign characteristic of (X, H) corresponding to the Jordan blocks of even size in the Jordan form of X are all the same, and the signs corresponding to the Jordan blocks of odd size in the Jordan form of X are also all the same.*

(b) *Assume X is nilpotent. Then a $\mathrm{MIN}\,(X, H,^*\,)$-subspace is unique if and only if the signs in the sign characteristic of (X, H) corresponding to the Jordan blocks of even size in the Jordan form of X are all the same, and there is at most one Jordan block of odd size.*

Proof. It remains to prove (b). The "only if" part. If the condition on the sign characteristic is not satisfied then by Theorem 6.3 a $\mathrm{MIN}\,(A, H,^*\,)$-subspace is not unique already among complex subspaces. If there are at least two Jordan blocks of odd size, then by using the formulas

$$(-\mathrm{j}I) J_\ell(0)(\mathrm{j}I) = J_\ell(0), \quad (\mathrm{j}I) \Xi_\ell(\mathrm{i}^\ell)(\mathrm{j}I), \qquad \ell \text{ odd},$$

we replace one pair of corresponding blocks $\Xi_\ell(\mathrm{i}^\ell)$, $J_\ell(0)$ with $-\Xi_\ell(\mathrm{i}^\ell)$, $J_\ell(0)$, thereby creating two opposite signs in the sign characteristic corresponding to nilpotent Jordan

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blocks of odd sizes. Now clearly a MIN $(X, H,^*)$-subspace is not unique as it is not unique already among complex subspaces, by Theorem 6.3.

The "if" part. We argue analogously to the proof of Theorem 1 in [25]. Replacing if necessary H with $-H$, we may assume that

$$X = J_{\ell_1} \oplus \cdots \oplus J_{\ell_r}, \qquad H = \Xi_{\ell_1}(\mathrm{i}^{\ell_1}) \oplus \cdots \oplus \Xi_{\ell_{r-1}}(\mathrm{i}^{\ell_{r-1}}) \oplus \eta \Xi_{\ell_r}(\mathrm{i}^{\ell_r}),$$

where $\ell_1, \ldots, \ell_{r-1}$ are even and the sign η is equal to 1 if ℓ_r is even as well. (If ℓ_r is odd, then $\eta = \pm 1$.) For $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, \ell_i$, denote by $f_{i,j}$ the $(\ell_1 + \cdots + \ell_{i-1} + j)$th unit coordinate vector ($\ell_0 = 0$ by convention). Thus, $f_{i,1}, \ldots, f_{i,\ell_i}$ is a Jordan chain of X.

Now let $\mathcal{N} \subseteq \mathbb{H}^{n \times 1}$ be an X-invariant $(H,^*)$-neutral subspace, and let

$$x = \sum_{i=1}^{r} \sum_{j=1}^{\ell_i} f_{i,j} x_{i,j} \in \mathcal{N}, \qquad x_{i,j} \in \mathbb{H}.$$

We claim that if $j > \lfloor \ell_i/2 \rfloor$, then $x_{i,j} = 0$. Suppose not. Let K be the set of all indices i, $1 \le i \le r$, for which the set

$$\{j \,:\, \lfloor \ell_i/2 \rfloor < j \le \ell_i, \quad x_{i,j} \ne 0\}$$

is non-void, and the difference

$$(\max \{j \,:\, \lfloor \ell_i/2 \rfloor < j \le \ell_i, \quad x_{i,j} \ne 0\}) - \lfloor \ell_i/2 \rfloor$$

is maximal. Denote this maximal difference by ν. Since $\mathcal{N}$ is X-invariant, we have

$$y := \sum_{i=1}^{r} \sum_{j=\nu}^{\ell_i} f_{i,j-\nu+1} x_{i,j} \in \mathcal{N}, \quad z := \sum_{i=1}^{r} \sum_{j=\nu+1}^{\ell_i} f_{i,j-(\nu+1)+1} x_{i,j} \in \mathcal{N}.$$

A computation shows that

$$y^* H z = - \sum_{i \in K, \ i \le r-1} |x_{i,\nu+\ell_i/2}|^2.$$

Since $\mathcal{N}$ is H-neutral, it follows that $x_{i,\nu+\ell_i/2} = 0$ for all $i \in K$, $i < r$. Thus, we must have $K = \{r\}$. However, then

$$y^* H y = \pm x^*_{r,\nu+(\ell_r-1)/2} \mathrm{i} x_{r,\nu+(\ell_r-1)/2},$$

which must be zero, and we obtain a contradiction with the definition of K. Thus, $x_{i,j} = 0$ if $j > \lfloor \ell_i/2 \rfloor$, for $i = 1, 2, \ldots, r$.

On the other hand, the $\mathbb{H}$-vector subspace $\mathcal{N}_0$ spanned by $f_{i,j}$ with $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, \lfloor \ell_i/2 \rfloor$, is clearly X-invariant H-neutral. In view of the claim proved in the preceding paragraph, $\mathcal{N}_0$ must be a unique MIN $(X, H,^*)$-subspace. $\square$

A result analogous to Theorem 5.5 (a parametrization of all $\mathrm{MIN}\,(A, H,{}^*\,)$-subspaces for quaternion matrices in cases of uniqueness when restricted to a root subspace corresponding to eigenvalue zero or to similar nonzero eigenvalues with zero real parts) is valid. We omit the statement of this result.

8.3. Comparison with the complex case (III). As a corollary of Theorem 8.4 we can characterize the difference (if any) between the order of neutrality for *complex* and for *quaternion* X-invariant H-neutral subspaces. If $\widehat{H} \in \mathbb{C}^{n \times n}$ is invertible skewhermitian and $\widehat{X} \in \mathbb{C}^{n \times n}$ is $(\widehat{H},{}^*\,)$-Hamiltonian, we denote temporarily by $\gamma_{\mathbb{H}}(\widehat{X}, \widehat{H})$ and $\gamma_{\mathbb{C}}(\widehat{X}, \widehat{H})$ the order of neutrality of $(\widehat{X}, \widehat{H})$ when the quaternion subspaces and when only the subspaces in $\mathbb{C}^{n \times 1}$ are considered, respectively. Clearly, $\gamma_{\mathbb{H}}(\widehat{X}, \widehat{H}) \geq \gamma_{\mathbb{C}}(\widehat{X}, \widehat{H})$.

We need some preparation to state the result. Let $\widehat{H} \in \mathbb{C}^{n \times n}$ be invertible skewhermitian, and let $\widehat{X} \in \mathbb{C}^{n \times n}$ be $(\widehat{H},{}^*\,)$-Hamiltonian, having the canonical form as in Theorem 6.1. Let

$$\gamma_1, -\gamma_1, \gamma_2, -\gamma_2, \ldots, \gamma_b, -\gamma_b$$

be all the distinct pairs of nonzero complex conjugate eigenvalues of $\widehat{X}$ with zero real parts (if any), where we take $\gamma_1, \ldots, \gamma_b$ to have positive imaginary parts. For each pair $(\gamma_k, -\gamma_k)$, and for every positive odd integer j, let $p_{k,j}$ be the number of blocks of size j with eigenvalue γ_k in (6.2), and let $p_{k,j}^+$ and $p_{k,j}^-$ be the number of Jordan blocks of size j with eigenvalue γ_k that have sign ζ_u equal to $+1$ and to -1, respectively, in the corresponding block of (6.1); thus, $p_{k,j} = p_{k,j}^+ + p_{k,j}^-$. We let $p_{k,j}^{\pm}$ be zero if there are no Jordan blocks of size j with eigenvalue γ_k having sign ± 1. Analogously, for every positive odd integer j, let $q_{k,j}$ be the number of Jordan blocks of size j with eigenvalue $-\gamma_k$ in (6.2), and let $q_{k,j}^+$ and $q_{k,j}^-$ be the number of such blocks that have sign ζ_u equal to $+1$ and to -1, respectively, in the corresponding block of (6.1). Define the number

$$
\Gamma_k := -\min\left\{ \sum_{j\ \text{odd}} p_{k,j}^+, \ \sum_{j\ \text{odd}} p_{k,j}^- \right\} - \min\left\{ \sum_{j\ \text{odd}} q_{k,j}^+, \ \sum_{j\ \text{odd}} q_{k,j}^- \right\}
$$
$$
+ \min\left\{ \sum_{j\ \text{odd}} (p_{k,j}^+ + q_{k,j}^-), \ \sum_{j\ \text{odd}} (p_{k,j}^- + q_{k,j}^+) \right\},
$$

and $\Gamma := \Gamma_1 + \cdots + \Gamma_b$.

COROLLARY 8.6. *Let $p^{\pm}$ be the number of Jordan blocks $J_{\ell_j}(0)$ of odd size with the sign $\eta_j = \pm 1$ in the canonical form (6.1), (6.2) of $(\widehat{X}, \widehat{H})$. Then*

$$(8.4) \qquad \gamma_{\mathbb{H}}(\widehat{X}, \widehat{H}) - \gamma_{\mathbb{C}}(\widehat{X}, \widehat{H}) = \Gamma + \lfloor (p^+ + p^-)/2 \rfloor - \min\{p^+, p_-\}.$$

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Proof. It follows from Remark 8.3 that the canonical form of the pair $\widehat{X}$, $\widehat{H}$ under simultaneous congruence with quaternion congruence matrices (Theorem 8.1) is obtained from (6.1), (6.2) upon replacing $-\gamma_k$ with γ_k ($k = 1, 2, \ldots, b$) and changing the signs for odd size blocks, as well as changing the signs -1 (if any) for nilpotent blocks of odd sizes into $+1$. Comparing the formulas for the order of neutrality of $(\widehat{X}, \widehat{H})$ under simultaneous congruence with quaternion congruence matrices (8.3) and that with complex congruence matrices (6.5), the result follows. $\square$

COROLLARY 8.7. *In the notation defined before and in Corollary 8.6, the equality*

$$(8.5) \qquad\qquad \gamma_{\mathbb{H}}(\widehat{X}, \widehat{H}) = \gamma_{\mathbb{C}}(\widehat{X}, \widehat{H})$$

holds true if and only if

$$(8.6) \qquad \left(\sum_{j \text{ odd}} \left(p_{k,j}^+ - p_{k,j}^- \right) \right) \cdot \left(\sum_{j \text{ odd}} \left(q_{k,j}^+ - q_{k,j}^- \right) \right) \le 0$$

for $k = 1, \ldots, b$, and

$$(8.7) \qquad\qquad \min\{p^+, p_-\} = \lfloor (p^+ + p^-)/2 \rfloor.$$

Proof. By Corollary 8.6, the equality (8.5) holds true if and only if (8.7) and

$$\min \left\{ \sum_{j \text{ odd}} p_{k,j}^+, \sum_{j \text{ odd}} p_{k,j}^- \right\} + \min \left\{ \sum_{j \text{ odd}} q_{k,j}^+, \sum_{j \text{ odd}} q_{k,j}^- \right\}$$

$$(8.8) \qquad = \min \left\{ \sum_{j \text{ odd}} (p_{k,j}^+ + q_{k,j}^-), \sum_{j \text{ odd}} (p_{k,j}^- + q_{k,j}^+) \right\}$$

hold true, for $k = 1, \ldots, b$. Letting

$$p_k := \sum_{j \text{ odd}} (p_{k,j}^+ - p_{k,j}^-), \quad q_k := \sum_{j \text{ odd}} (q_{k,j}^+ - q_{k,j}^-),$$

the equality (8.8) takes the following form (after subtracting

$$\left(\sum_{j \text{ odd}} p_{k,j}^- \right) + \left(\sum_{j \text{ odd}} q_{k,j}^- \right)$$

from both sides):

$$(8.9) \qquad \min\{p_k, 0\} + \min\{q_k, 0\} = \min\{p_k, q_k\}.$$

A straightforward analysis shows that (8.9) is equivalent to $p_k q_k \leq 0$. $\blacksquare$

EXAMPLE 8.8. Let

$$\widehat{H} = \begin{bmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{bmatrix}, \qquad \widehat{X}_1 = 0_2, \qquad \widehat{X}_2 = \mathrm{i} \begin{bmatrix} a & b + \mathrm{i}c \\ b - \mathrm{i}c & d \end{bmatrix} \neq 0,$$

where $a, b, c, d \in \mathbb{R}$. Then $\widehat{H}$ has no nonzero neutral complex subspaces, but $\mathrm{Span}_{\mathbb{H}} \begin{bmatrix} 1 \\ \mathsf{j} \end{bmatrix}$ is a one-dimensional quaternion $\widehat{H}$-neutral subspace, in accordance with formula (8.4), which for this example gives $\gamma_{\mathbb{H}}(\widehat{X}_1, \widehat{H}) - \gamma_{\mathbb{C}}(\widehat{X}_1, \widehat{H}) = 1$.

On the other hand, all nonzero $\widehat{H}$-neutral subspaces are of the form

$$(8.10) \qquad \mathrm{Span}_{\mathbb{H}} \begin{bmatrix} 1 \\ x\mathsf{j} + y\mathsf{k} \end{bmatrix}, \quad \text{where} \ \ x, y \in \mathbb{R} \ \ \text{and} \ \ x^2 + y^2 = 1.$$

A calculation shows that all the subspaces (8.10) are $\widehat{X}_2$-invariant if and only if one of them is A-invariant which happens if and only if $a + d = 0$. Indeed, the equality

$$(8.11) \qquad \mathrm{i} \begin{bmatrix} a & b + \mathrm{i}c \\ b - \mathrm{i}c & d \end{bmatrix} \begin{bmatrix} 1 \\ x\mathsf{j} + y\mathsf{k} \end{bmatrix} = \begin{bmatrix} 1 \\ x\mathsf{j} + y\mathsf{k} \end{bmatrix} z, \quad z \in \mathbb{H},$$

gives $z = \mathrm{i}a + \mathrm{i}(b + \mathrm{i}c)(x\mathsf{j} + y\mathsf{k})$, which upon substituting in the bottom equation in (8.11) yields $a + d = 0$. Observe that $a + d = 0$ happens if and only if either $\widehat{X}_2 = 0$ or $\widehat{X}_2$ has two nonzero complex conjugate eigenvalues. This confirms the result of Corollary 8.6.

8.4. Comparison with the real case. We conclude this section with the comparison with the real case. We start with a lemma.

LEMMA 8.9. (1) *Let*

$$A = J_{2h}(\pm \mathrm{i}b), \quad H = \eta \begin{bmatrix} 0 & 0 & \cdots & 0 & \Xi_2^h \\ 0 & 0 & \cdots & -\Xi_2^h & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & (-1)^{h-2}\Xi_2^h & \cdots & 0 & 0 \\ (-1)^{h-1}\Xi_2^h & 0 & \cdots & 0 & 0 \end{bmatrix}$$

be real matrices, where $b \neq 0$ *and* $\eta = \pm 1$ *(thus, A is (H, id)-Hamiltonian). Then there exists an invertible matrix* $S \in \mathbb{H}^{2h \times 2h}$ *such that*

$$(8.12) \qquad S^{-1}AS = J_h(\mathrm{i}b) \oplus J_h(\mathrm{i}b), \qquad S^*HS = \zeta_1 \Xi_h(\mathrm{i}^h) \oplus \zeta_2 \Xi_h(\mathrm{i}^h),$$

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where the signs ζ_1 and ζ_2 are opposites if h is odd, and the signs ζ_1 and ζ_2 are equal to η if h is even.

(2) *Let*

$$A = J_{2m+1}(0) \oplus -J_{2m+1}(0)^T, \qquad H = \begin{bmatrix} 0 & I_{2m+1} \\ -I_{2m+1} & 0 \end{bmatrix};$$

thus, A is (H, id)-Hamiltonian. Then there exists an invertible matrix $S \in \mathbb{H}^{(4m+2)\times(4m+2)}$ such that

$$(8.13) \quad S^{-1}AS = J_{2m+1}(0) \oplus J_{2m+1}(0), \quad S^*HS = \Xi_{2m+1}(\mathrm{i}^{2m+1}) \oplus \Xi_{2m+1}(\mathrm{i}^{2m+1}).$$

Proof. Part (2) is evident, because the right hand sides of (8.13) must form the canonical matrices of the pair (A, H) under simultaneous congruence over the quaternions (Theorem 8.1); note that $J_{2m+1}(0) \oplus J_{2m+1}(0)$ is the Jordan form of $J_{2m+1}(0) \oplus -J_{2m+1}(0)^T$.

Part (1). The existence of such S, for some signs ζ_1 and ζ_2 follows from the canonical form (8.1), (8.2). To determine the signs, consider the real symmetric matrix HA. The signature (i.e., the difference between the number of positive eigenvalues and the number of negative eigenvalues, counted with multiplicities) of HA is zero; indeed, the central 2×2 block in HA is $\begin{bmatrix} 0 & b \\ b & 1 \end{bmatrix}$ if h is even, and $\begin{bmatrix} 0 & -b \\ -b & 0 \end{bmatrix}$ if h is odd, the signature of both 2×2 matrices being zero. Because of the inertia theorem (which is valid for quaternion hermitian matrices), we should have also the signature of the matrix Y equal to zero, where $Y = S^*HS \cdot S^{-1}AS$ is the product of matrices in the right hand sides of (8.12). A calculation shows that for odd h, the signature of $\Xi_h(\mathrm{i}^h)J_h(\mathrm{i}b)$ is 1 if $b < 0$ and -1 if $b > 0$. Thus, we must have ζ_1 and ζ_2 opposites to ensure that the signature of Y is zero.

Now assume h is even. Then the signature of $\Xi_h(\mathrm{i}^h)J_h(\mathrm{i}b)$ is zero, so the signature of Y is zero no matter the signs ζ_j. So we have to argue differently. We will construct explicitly the matrix S in two steps. The first step is adapted from [13]. Define an invertible $2h \times 2h$ matrix

$$C_{2h} = \frac{1}{\sqrt{2}} \cdot \left(\begin{bmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{bmatrix} \right),$$

whose inverse is easily computed:

$$C_{2h}^{-1} = \frac{\sqrt{2}}{2i} \left(\begin{bmatrix} \mathrm{i} & -1 \\ \mathrm{i} & 1 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathrm{i} & -1 \\ \mathrm{i} & 1 \end{bmatrix} \right).$$

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Use the unit coordinate vectors $e_{j,2h}$, $j = 1, 2, \ldots, 2h$, to define the permutation matrix

$$D_{2h} = [e_{1,2h} \ e_{3,2h} \ e_{5,2h} \ldots e_{2h-1,2h} \ e_{2,2h} \ e_{4,2h} \ldots e_{2h,2h}] \in \mathbb{R}^{2h \times 2h},$$

and observe that $D_{2h}^{-1} = D_{2h}^T$. A computation serves to establish the equalities

$$(C_{2h}D_{2h})^{-1}J_{2h}(\pm ib)(C_{2h}D_{2h}) = J_h(-ib) \oplus J_h(ib)$$

and

$$(C_{2h}D_{2h})^* \begin{bmatrix} 0 & 0 & \cdots & 0 & \Xi_2^h \\ 0 & 0 & \cdots & -\Xi_2^h & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & (-1)^{h-2}\Xi_2^h & \cdots & 0 & 0 \\ (-1)^{h-1}\Xi_2^h & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$\times \ (C_{2h}D_{2h}) = (-1)^{h/2} \begin{bmatrix} \Xi_h & 0 \\ 0 & \Xi_h \end{bmatrix}.$$

In the second step, use the equalities

$$(8.14) \quad (-jI)J_h(-ib)(jI) = J_h(ib), \quad (-jI) \cdot \Xi_h \cdot (jI) = \Xi_h = (-1)^{h/2}\Xi_h(i^h).$$

Combining the two steps, we obtain formula (8.12) for h even. $\square$

Theorems 5.2 and 8.4 yield, taking into account Lemma 8.9:

COROLLARY 8.10. *Let $A \in \mathbb{R}^{n \times n}$ be (H, id)-Hamiltonian, where $H = -H^T \in \mathbb{R}^{n \times n}$ is invertible. Then the maximal dimension $\gamma_{\mathbb{R}}(A, H)$ of a real A-invariant (H, id)-neutral subspace, and the maximal (quaternion) dimension $\gamma_{\mathbb{H}}(A, H)$ of a quaternion A-invariant $(H,{}^*)$-neutral subspace, are related as follows:*

$$\gamma_{\mathbb{H}}(A, H) - \gamma_{\mathbb{R}}(A, H) = \sum_{k=1}^{r} \big| \text{the sum of signs corresponding}$$

(8.15)
$$\text{to blocks } J_{2h_{j,k}}(\pm ib_k) \text{ with odd } h_{j,k} \big|,$$

where $ib_1, \ldots, ib_r$ are all distinct eigenvalues of A with positive $b_1, \ldots, b_r$. Here the blocks $J_{2h_{j,k}}(\pm ib_k)$ refer to the canonical form of (A, H) given in Theorem 5.1.

Proof. By Corollary 2.3 we may assume that either $\sigma(A) = \{\pm ib\}$, $b \in \mathbb{R} \setminus \{0\}$, or $\sigma(A) = \{0\}$. Suppose the latter case holds. Formula (5.2) gives $\gamma_{\mathbb{R}}(A, H) = n/2$. On the other hand, Lemma 8.9(2) together with formula (8.3) gives also $\gamma_{\mathbb{H}}(A, H) = n/2$. Suppose $\sigma(A) = \{\pm ib\}$. Lemma 8.9(1) and formula (8.3) yield $\gamma_{\mathbb{H}}, (A, H) = n/2$, and applying (5.2) the result follows. $\square$

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9. The case of ϕ having form (V). In this section, we assume $\mathbb{F} = \mathbb{H}$ and ϕ is a fixed involution of $\mathbb{H}$ different from the quaternion conjugation. Then there is $\beta = \beta(\phi) \in \mathbb{H}$ such that $\beta^2 = -1$ and $\phi(\beta) = -\beta$; moreover such β is uniquely determined by ϕ up to negation (i.e., replacement of β with $-\beta$). We denote by $\mathrm{Inv}\,(\phi)$ the set of quaternions α which are fixed by ϕ: $\phi(\alpha) = \alpha$. The set $\mathrm{Inv}\,(\phi)$ is a three dimensional real vector space of $\mathbb{H}$ that contains the reals (see Proposition 1.1).

Again, we start with the canonical form. We describe first the primitive pairs of blocks.

(A) $L = \kappa\beta(\phi)F_k$, $A = \beta(\phi)J_k(0)$, where $\kappa = 1$ if k is even, and $\kappa = \pm 1$ if k is odd;

(B)
$$L = \begin{bmatrix} 0 & F_\ell \\ -F_\ell & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -J_\ell(\alpha) & 0 \\ 0 & J_\ell(\alpha) \end{bmatrix},$$

where $\alpha \in \mathrm{Inv}\,(\phi)$, $\mathfrak{R}(\alpha) > 0$.

(C) $L = \delta\beta(\phi)F_s$, $A = \beta(\phi)J_s(\tau)$, where $\delta = \pm 1$ and τ is a negative real number.

REMARK 9.1. One can replace "negative" by "positive" in (C). Indeed, if $\gamma \in \mathbb{H}$ is such that $\gamma^{-1}\beta(\phi)\gamma = -\beta(\phi)$ and $|\gamma| = 1$, then

$$\left(\mathrm{diag}\,(\gamma^{-1}, -\gamma^{-1}, \ldots, (-1)^{s-1}\gamma^{-1})\right)\,\beta(\phi)J_s(\tau)\left(\mathrm{diag}\,(\gamma, -\gamma, \ldots, (-1)^{s-1}\gamma)\right)$$

$$(9.1) \hspace{3cm} = \beta(\phi)J_s(-\tau),$$

and

$$\left(\mathrm{diag}\,(\phi(\gamma), -\phi(\gamma), \ldots, (-1)^{s-1}\phi(\gamma))\right)\,F_s\,(\mathrm{diag}\,(\gamma, -\gamma, \ldots, (-1)^s\gamma)) = (-1)^{s-1}F_s,$$

where $\tau > 0$. Note that $\gamma^2 = -1$ and $\phi(\gamma) = \gamma$. Also,

$$\left(\mathrm{diag}\,(\phi(\gamma), -\phi(\gamma), \ldots, (-1)^{s-1}\phi(\gamma))\right)\,(\beta(\phi)F_s)\,\left(\mathrm{diag}\,(\gamma, -\gamma, \ldots, (-1)^{s-1}\gamma)\right)$$

$$= (-1)^{s-1}(\beta(\phi)F_s).$$

Note that the replacement of τ by its negative in (C) will reverse the sign δ if s is even, and will leave the sign invariant if s is odd.

THEOREM 9.2. Let $A \in \mathbb{H}^{n\times n}$ be (H, ϕ)-Hamiltonian, where $H = -H_\phi \in \mathbb{H}^{n\times n}$ is invertible. Then there exists an invertible quaternion matrix S such that $S_\phi HS$ and $S^{-1}AS$ have the following block diagonal form:

$$(9.2) \hspace{1cm} S_\phi HS = L_1 \oplus L_2 \oplus \cdots \oplus L_m, \quad S^{-1}AS = A_1 \oplus A_2 \oplus \cdots \oplus A_m,$$

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where each pair of matrices (L_i, A_i) has one of the forms (A), (B), (C). *Moreover, the form* (9.2) *is uniquely determined by the pair* (A, H), *up to an arbitrary simultaneous permutation of blocks and up to a replacement of α in each block of the form* (B) *with a similar quaternion α' such that $\phi(\alpha') = \alpha'$.*

Conversely, if H and A are given as in formula (9.2), *then $H = -H_\phi$ is invertible and A is (H, ϕ)-Hamiltonian.*

The result of Theorem 9.2 is found in [2], [23], and [28], among many sources. For a detailed proof of the theorem, see for example [23].

The *sign characteristic* of an (H, ϕ)-Hamiltonian matrix A assigns a sign ± 1 to every partial multiplicity corresponding to a nonzero eigenvalue of A with zero real part (if any), and to every odd partial multiplicity corresponding to the eigenvalue zero of A (if A is not invertible). Note that the set of (right) eigenvalues of a quaternion matrix is invariant under similarity: If α is an eigenvalue, then so is every quaternion of the form $\gamma^{-1}\alpha\gamma$, where $\gamma \in \mathbb{H} \setminus \{0\}$. If A is (H, ϕ)-Hamiltonian, then every nonzero eigenvalue λ of A with zero real part is similar to $\beta(\phi)\tau$ for some (unique) negative τ, and the signs in the sign characteristic of (A, H) which are attributed to the partial multiplicities corresponding to λ are, by definition, those that appear in the canonical form (9.2) for the eigenvalue $\beta(\phi)\tau$.

9.1. Order of neutrality and uniqueness of maximal invariant neutral subspaces. The next theorem gives a formula for the order of neutrality in terms of the canonical form of Theorem 9.2.

THEOREM 9.3. *Let $A \in \mathbb{H}^{n \times n}$ be (H, ϕ)-Hamiltonian. Then the order of neutrality $\gamma(A, H)$ is given by formula* (6.5), *where in reference to the canonical form* (9.2) *of (A, H), $\gamma_1, \ldots, \gamma_a$ are all the distinct eigenvalues of A of the form $\beta(\phi)\tau$, where τ is a nonpositive real number, and $p_{k,j}$, $p_{k,j}^+$, $p_{k,j}^-$ have the same meaning as in* (6.5).

For the proof of Theorem 9.3, as well as for later proofs, we need comparison between the primitive blocks of the canonical form of Theorem 9.2 and those of Theorem 6.1. This comparison is accomplished in the following lemma.

LEMMA 9.4. *For each of the following quadruples of matrices (A_i, H_i, A_i', H_i'), $i = 1, 2, 3, 4$, there exists an invertible quaternion matrix S_i such that*

$$S_i^{-1} A_i S_i = A_i', \qquad (S_i)_\phi H_i S_i = H_i',$$

where in parts (2), (3), *and* (4) *it is assumed that ϕ is a nonstandard involution such that $\phi(\mathrm{i}) = -\mathrm{i}$ and $\beta(\phi) = \mathrm{i}$:*

(1)

$$A_1 = J_\ell(0), \quad H_1 = \eta \Xi_\ell(\mathrm{i}^\ell), \quad A_1' = \beta(\phi) J_\ell(0), \quad H_1' = \beta(\phi) F_\ell,$$

where ℓ is even and η is sign ± 1;

(2)

$$A_2 = J_{\ell_j}(0), \quad H_2 = \eta\Xi_\ell(\mathrm{i}^\ell), \quad A_2' = \beta(\phi)J_\ell(0), \quad H_2' = \eta\beta(\phi)F_\ell,$$

where ℓ is odd and η is sign ± 1;

(3) $A_3 = \begin{bmatrix} -J_p(\overline{\lambda}) & 0 \\ 0 & J_p(\lambda) \end{bmatrix}, \quad H_3 = \begin{bmatrix} 0 & F_p \\ -F_p & 0 \end{bmatrix},$

$A_3' = \begin{bmatrix} -J_p(\alpha) & 0 \\ 0 & J_p(\alpha) \end{bmatrix}, \quad H_3' = \begin{bmatrix} 0 & F_p \\ -F_p & 0 \end{bmatrix},$

where λ is a complex number with positive real part, and $\alpha \in \mathrm{Inv}(\phi)$ is such that $\Re(\alpha) = \Re(\lambda)$ and $|\mathfrak{V}(\alpha)| = |\mathfrak{V}(\lambda)|$.

(4)

$$A_4 = J_m(\gamma), \quad H_4 = \zeta\Xi_m(\mathrm{i}^m), \quad A_4' = \beta(\phi)J_m(\tau), \quad H_4' = \zeta'\beta(\phi)F_m,$$

where the complex number γ is nonzero with zero real part, $\tau = -|\gamma| < 0$, and ζ, ζ' are signs ± 1 related as follows: If m is odd or if m is even and the imaginary part of γ is positive, then $\zeta' = \zeta$. If m is even and the imaginary part of γ is negative, then $\zeta' = -\zeta$.

Note that for a given nonstandard involution ϕ, a normalized quaternion q such that $\phi(q) = -q$ is unique up to negation; thus, the condition $\phi(\mathrm{i}) = -\mathrm{i}$ forces $\beta(\phi) = \mathrm{i}$ or $\beta(\phi) = -\mathrm{i}$. We make the choice $\beta(\phi) = \mathrm{i}$.

Proof. Part (1) follows from the following equalities (where ℓ is even):

$$\mathrm{diag}\,(1, -\beta(\phi), \cdots, (-\beta(\phi))^{\ell-1})J_\ell(0)\mathrm{diag}\,(1, \beta(\phi), \cdots, \beta(\phi)^{\ell-1}) = \beta(\phi)J_\ell(0),$$

$$\mathrm{diag}\,(1, -\beta(\phi), \cdots, (-\beta(\phi))^{\ell-1})\Xi_\ell\mathrm{diag}\,(1, \beta(\phi), \cdots, \beta(\phi)^{\ell-1}) = \pm\beta(\phi)F_\ell,$$

with sign $+1$ if $\ell/2$ is odd and -1 if $\ell/2$ is even; and

$$\mathrm{diag}\,(q, -q, q, \ldots, -q)\beta(\phi)J_\ell(0)\mathrm{diag}\,(-q, q, -q, \ldots, q) = \beta(\phi)J_\ell(0),$$

$$\mathrm{diag}\,(-q, q, -q, \ldots, q)\beta(\phi)F_\ell\mathrm{diag}\,(-q, q, -q, \ldots, q) = -\beta(\phi)F_\ell,$$

where $q \in \mathbb{H}$ is such that $q^2 = -1$, $\phi(q) = q$ and $q\beta(\phi)q = \beta(\phi)$. (Such q exists: In terms of Proposition 1.1, q is any normalized eigenvector of T corresponding to the eigenvalue 1.)

For part (2) take $S_2 = \mathrm{diag}\,(1, \beta(\phi), \ldots, \beta(\phi)^{\ell-1})$.

Part (3). Observe that α is similar to λ, therefore the Jordan form of A_3 and of A_3' over the quaternions is the same. Thus, the canonical form of the pair (A_3, H_3)

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under the transformations $A_3 \mapsto S^{-1}A_3 S$, $H_3 \mapsto S_\phi H_3 S$, where $S \in \mathbb{H}^{2p \times 2p}$ is invertible, must be (A_3', H_3'), by Theorem 9.2.

Part (4). Let $\gamma = \mathrm{i}\tau'$, where $\tau' \in \mathbb{R} \setminus \{0\}$. Then, with

$$S_4 = \mathrm{diag}\,(1, \beta(\phi), \ldots, \beta(\phi)^{m-1}),$$

we have

$$(S_4)_\phi \Xi(\mathrm{i}^\ell) S_4 = \begin{cases} \mathrm{i}F_\ell & \text{if } \ell \text{ is odd;} \\ -\mathrm{i}F_\ell & \text{if } \ell \text{ is even;} \end{cases}$$

and

$$S_4^{-1} J_m(\gamma) S_4 = \beta(\phi) J_m(\tau').$$

If $\tau' < 0$ we are done. Otherwise, use Remark 9.1 to replace τ' with $-\tau'$. $\square$

Proof. Theorem 9.3. By Corollary 2.3 we need to consider only two cases: (1) A is nilpotent (only one eigenvalue, and it is equal to zero); (2) A has only one eigenvalue (up to similarity), and this eigenvalue is nonreal. In the second case, we identify the real linear span of 1 and $\beta(\phi)$ with $\mathbb{C}$ (note that ϕ acts as complex conjugation in $\mathrm{Span}_\mathbb{R}\{1, \beta(\phi)\}$), and assume by Proposition 4.1 that A and H are given by their canonical form (9.2). It follows that there exists an A-invariant (H, ϕ)-neutral subspace of the dimension required by formula (6.5), because by Theorem 6.2 such subspace exists already among subspaces of $\mathbb{C}^{n \times n}$. Note that by Lemma 9.4 the number produced by (6.5) is the same whether we use the canonical form (9.2) or the canonical form (6.1) for (A, H). Conversely, there cannot be an A-invariant (H, ϕ)-neutral subspace of larger dimension, in view of the same Theorem 6.2 and Theorem 4.2.

Suppose now A is nilpotent. As in case (2) we see that there exists an A-invariant (H, ϕ)-neutral subspace of dimension

$$(9.3) \qquad \left(\sum_{j=1}^\infty \lfloor \frac{j}{2} \rfloor p_j\right) + \min\left\{\sum_{j \text{ odd}} p_j^+, \sum_{j \text{ odd}} p_j^-\right\},$$

where p_j is the number of Jordan blocks in the Jordan form of A of size $j \times j$, and, for j odd, among those blocks p_j^+ and p_j^- are the numbers of blocks that have sign $+1$ and -1, respectively, in the sign characteristic of (A, H). It remains to show that there cannot be an A-invariant (H, ϕ)-neutral subspace of larger dimension. To this end observe that by Proposition 1.2, the maximal dimension of an (H, ϕ)-neutral subspace is equal to (9.3), for

$$H = \oplus_{j=1}^\infty \left(\delta_{j,1}\beta(\phi)F_j \oplus \cdots \oplus \delta_{j,p_j}\beta(\phi)F_j\right),$$

where $\delta_{j,q}$ are signs ± 1, and for each j, exactly $p_j^{\pm}$ of them are equal to ± 1's. $\square$

Uniqueness of MIN (A, H, ϕ)-subspaces is characterized similarly to Theorem 6.3:

THEOREM 9.5. *Let A and H be as in Theorem 9.2.*

(a) *Assume that A has only one eigenvalue in the closed upper complex half-plane and that this eigenvalue has zero real part and positive imaginary part. Then a MIN (A, H, ϕ)-subspace is unique if and only if the signs in the sign characteristic of (A, H) corresponding to the Jordan blocks of even size in the Jordan form of A are all the same, and the signs corresponding to the Jordan blocks of odd size in the Jordan form of A are also all the same.*

(b) *Assume that A is nilpotent. Then a MIN (A, H, ϕ)-subspace is unique if and only if the signs in the sign characteristic of (A, H) corresponding to the Jordan blocks of odd size in the Jordan form of A are all the same.*

Proof. We may assume (Proposition 4.1) that A and H are given in the canonical form of Theorem 9.2. If (A_i', H_i') is a primitive pair of blocks in (A, H), we associate with it a pair of blocks (A_i, H_i) as in Lemma 9.4, choosing all signs η in part (1) to be equal, and choosing all γ's in part (4) to have imaginary parts of the same sign, either all positive or all negative. Collecting the blocks (A_i, H_i) in a direct sum, we obtain a canonical form given by the right hand sides of (6.1) and (6.2) as in Theorem 6.1. Denote this canonical form by $\widehat{H}$ and $\widehat{A}$, respectively. In view of Lemma 9.4 and Theorems 6.2 and 9.3, the dimensions of MIN (A, H, ϕ)-subspaces and of MIN $(\widehat{A}, \widehat{H},^*)$-subspaces is the same. Also, by Proposition 4.1, a MIN (A, H, ϕ)-subspace is unique if and only if a MIN $(\widehat{A}, \widehat{H},^*)$-subspace is unique among quaternion subspaces.

We now consider the two parts (a) and (b) separately. Suppose the hypotheses of (a) hold true. If the condition on sign characteristic of (A, H) in part (a) is satisfied, then by Lemma 9.4 the condition on the sign characteristic of $(\widehat{A}, \widehat{H})$ given in Theorem 6.3 is satisfied (here it is essential that the γ's have imaginary parts of the same sign). Thus, by Theorem 6.3 a MIN $(\widehat{A}, \widehat{H},^*)$-subspace is unique among complex subspaces. But by Theorem 4.2 the MIN $(\widehat{A}, \widehat{H},^*)$-subspace is unique also among quaternion subspaces, and the uniqueness in Theorem 9.5(a) follows in view of the observations made in the preceding paragraph. If a MIN (A, H, ϕ)-subspace is unique, then a MIN $(\widehat{A}, \widehat{H},^*)$-subspace is unique among quaternion subspaces, hence a fortiori it is unique among complex subspaces (cf. the observations made in the preceding paragraph), and so the conditions on the sign characteristic of Theorem 6.3 are satisfied for $(\widehat{A}, \widehat{H})$. By Lemma 9.4, these conditions translate to the conditions on the sign characteristic of (A, H) given in part (a).

Suppose now the hypotheses of (b) hold true. If a MIN (A, H, ϕ)-subspace is

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unique, then arguing as in the case (a), we obtain that the conditions on the sign characteristic given in (b) are satisfied. Conversely, if the conditions given in (b) are satisfied, then an argument similar to that employed in the proof of Theorem 8.5 (the "if" part) shows that a MIN (A, H, ϕ)-subspace is unique. $\square$

Combining Corollary 2.3 with Theorem 9.5, we obtain:

THEOREM 9.6. *Let A and H be as in Theorem* 9.2. *Assume that the conditions in Theorem* 9.5(2) *are satisfied (if A is singular), as well as the conditions in Theorem* 9.5(1) *for every complex pure imaginary nonzero eigenvalue of A with positive imaginary part. Let $\lambda_0 = 0$ and $\lambda_{\ell+1}, \ldots, \lambda_{\ell+s}$ be all distinct pure imaginary eigenvalues of A with positive imaginary part, and for $j = 0, \ell+1, \ldots, \ell+s$, let $\widetilde{\mathcal{M}}_j$ be the unique (in view of Theorem* 9.5) *MIN (A_j, H_j, ϕ)-subspace, where A_j and H_j are the restrictions of A and H, respectively, to the root subspace for A corresponding to λ_j.*

Then all MIN (A, H, ϕ)-subspaces $\widetilde{\mathcal{M}}$ are parameterized by the A-invariant subspaces $\mathcal{M}$ such that $\mathcal{M}$ is contained in the sum of root subspaces for A corresponding to the eigenvalues with positive real parts. The parametrization is given by the formula

$$\mathcal{M} \dotplus \left(\mathcal{M}^{[\perp]_H} \cap \mathcal{R}_- \right) \dotplus \widetilde{\mathcal{M}}_0 \dotplus \widetilde{\mathcal{M}}_{\ell+1} \dotplus \cdots \dotplus \widetilde{\mathcal{M}}_{\ell+s},$$

where $\mathcal{R}_-$ is the sum of the root subspaces for A corresponding to the eigenvalues with negative real parts.

An analogous result holds if "positive real" and "negative real" are interchanged in the statement of Theorem 9.6.

9.2. Comparison with the complex case (III). Let $\widehat{H} \in \mathbb{C}^{n \times n}$ be an invertible skewhermitian matrix, and let $\widehat{A}$ be $(\widehat{H}, {}^*)$-Hamiltonian. We may also consider $\widehat{H}$ as an $n \times n$ quaternion ϕ-skewhermitian matrix, and $\widehat{A}$ as a $(\widehat{H}, \phi)$-Hamiltonian matrix, where ϕ is a nonstandard involution such that $\phi(\mathsf{i}) = -\mathsf{i}$. Denoting by $\gamma_\mathbb{C}(\widehat{A}, \widehat{H})$ the order of neutrality of $(\widehat{H}, \widehat{A})$ when the matrices are considered over $\mathbb{C}$, and by $\gamma_\mathbb{H}(\widehat{A}, \widehat{H})$ the order of neutrality of the same pair of matrices considered over $\mathbb{H}$ as above, we have actually the equality

$$(9.4) \qquad\qquad \gamma_\mathbb{C}(\widehat{A}, \widehat{H}) = \gamma_\mathbb{H}(\widehat{A}, \widehat{H}).$$

Indeed, Lemma 9.4 shows that the canonical forms of (A, H) given by Theorems 6.1 and 9.2 can possibly differ only in the signs corresponding to Jordan blocks of even sizes associated with eigenvalues with zero real parts. But these signs do not play a role in the formulas for the order of neutrality ((6.5), Theorem 9.3), so equality (9.4) follows.

9.3. Comparison with the real case. In this subsection, we compare with real Hamiltonian matrices. Thus, let $H = -H^T \in \mathbb{R}^{n \times n}$ be invertible, and let $A \in \mathbb{R}^{n \times n}$

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be (H, id)-Hamiltonian. Then we have $\gamma_{\mathbb{R}}(A, H) := \gamma(A, H)$ given by Theorem 5.2. On the other hand, considering A as a (H, ϕ)-Hamiltonian quaternion matrix, where ϕ is a nonstandard involution, we have the order of neutrality $\gamma_{\mathbb{H}}(A, H)$ given by Theorem 9.3. It turns out that

$$(9.5) \qquad \gamma_{\mathbb{R}}(A, H) = \gamma_{\mathbb{H}}(A, H) - \sum_{k=1}^{r} \left| \sum_{j \in W_k} \eta_{jk} \right|,$$

where r, W_k, $\eta_{j,k}$ are defined as in (5.2) (thus, only nonzero pure imaginary eigenvalues of A may contribute to the sum in (9.5)).

As in the preceding subsection, to verify formula (9.5), we need to compare the canonical forms of the pair (A, H) over the reals and over the quaternions. This is done in the next lemma.

LEMMA 9.7. *For each of the following quadruples of matrices (A_i, H_i, A'_i, H'_i),* $i = 1, 2, 3, 4, 5$, *there exists an invertible quaternion matrix S_i such that*

$$S_i^{-1} A_i S_i = A'_i, \qquad (S_i)_\phi H_i S_i = H'_i :$$

(1) $A_1 = J_{2m+1}(0) \oplus -J_{2m+1}(0)^T, \qquad H_1 = \begin{bmatrix} 0 & I_{2m+1} \\ -I_{2m+1} & 0 \end{bmatrix},$
$\quad A'_1 = \beta(\phi)(J_{2m+1}(0) \oplus J_{2m+1}(0)), \quad H'_1 = (\beta(\phi)F_{2m+1}) \oplus (-\beta(\phi)F_{2m+1});$

(2)
$$\qquad A_2 = J_{2m}(0), \quad H_1 = \kappa \Xi_{2m}, \quad A'_2 = \beta(\phi)J_{2m}(0), \quad H'_2 = \beta(\phi)F_{2m},$$

$\quad$*where $\kappa = \pm 1$;*

(3) $A_3 = J_\ell(a) \oplus -J_\ell(a)^T, \qquad H_3 = \begin{bmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{bmatrix},$

$\quad A'_3 = (-J_\ell(a)) \oplus J_\ell(a), \qquad H'_3 = \begin{bmatrix} 0 & F_\ell \\ -F_\ell & 0 \end{bmatrix},$ *where $a > 0$;*

(4) $A_4 = J_{2k}(a \pm \mathrm{i}b) \oplus (-J_{2k}(a \pm \mathrm{i}b)^T), \qquad H_4 = \begin{bmatrix} 0 & I_{2k} \\ -I_{2k} & 0 \end{bmatrix},$

$\quad A'_4 = (-J_k(\overline{\alpha})) \oplus J_k(\alpha) \oplus (-J_k(\overline{\alpha})) \oplus J_k(\alpha),$
$\quad H'_4 = \begin{bmatrix} 0 & F_k \\ -F_k & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & F_k \\ -F_k & 0 \end{bmatrix},$
$\quad$*where $a > 0$, $b > 0$, and $\alpha = a + \mathrm{i}b$;*

(5) $A_5 = J_{2h}(\pm \mathrm{i}b), \qquad H_5 = \eta \begin{bmatrix} 0 & 0 & \cdots & 0 & \Xi_2^h \\ 0 & 0 & \cdots & -\Xi_2^h & 0 \\ \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots & \vdots \\ 0 & (-1)^{h-2}\Xi_2^h & \cdots & 0 & 0 \\ (-1)^{h-1}\Xi_2^h & 0 & \cdots & 0 & 0 \end{bmatrix},$

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$$A'_5 = \beta(\phi)J_h(\tau) \oplus \beta(\phi)J_h(\tau), \qquad H'_5 = \beta(\phi)F_h \oplus (-\beta(\phi)F_h),$$

where $b > 0$ and $\tau = -b$.

Note that (A_i, H_i), resp. (A'_i, H'_i), represent the primitive pairs of blocks of the canonical form of (A, H) under simultaneous real, resp. quaternion ϕ-, congruence.

Proof. Part (2) follows from the proof of Part (1) in Lemma 9.4.

For part (3), take

$$S_3 = \begin{bmatrix} 0 & I_\ell \\ -F_\ell & 0 \end{bmatrix}.$$

Part (1). We identify $\mathrm{Span}_{\mathbb{R}}\{1, \beta(\phi)\}$ with $\mathbb{C}$ via the identification of $\beta(\phi)$ with i. Then ϕ acts as complex conjugation on $\mathrm{Span}_{\mathbb{R}}\{1, \beta(\phi)\}$. The canonical form represented by the right hand sides of (6.1) and (6.2) of both pairs (A_1, H_1) and (A'_1, H'_1) is easily seen to be the same, namely,

$$J_{2m+1}(0) \oplus J_{2m+1}(0), \quad (\eta_1 \Xi_{2m+1}(\mathrm{i}^{2m+1})) \oplus \eta_2 \Xi_{2m+1}(\mathrm{i}^{2m+1}),$$

where the signs η_1 and η_2 must be opposite (otherwise, the complex hermitian matrices

$$\mathrm{i}H_1, \quad \mathrm{i}H_2, \quad \text{and} \quad \mathrm{i}((\eta_1 \Xi_{2m+1}(\mathrm{i}^{2m+1})) \oplus \eta_2 \Xi_{2m+1}(\mathrm{i}^{2m+1}))$$

would not have the same inertia, which is precluded by the inertia theorem). In view of Theorem 6.1 existence of S_1 with the required properties is guaranteed.

Part (4). The complex Jordan form of both A_4 and A'_4 is $J_k(\overline{\alpha}) \oplus J_k(-\overline{\alpha}) \oplus J_k(\alpha) \oplus J_k(-\alpha)$. By Theorem 6.1 the H_4-Hamiltonian matrix A_4 and the H'_4-Hamiltonian matrix A'_4 have the same canonical form over the complex numbers, and existence of S_4 follows.

Part (5). Note that A_5 and A'_5 are similar over the quaternions, therefore by Theorem 9.2 there exists invertible $S \in \mathbb{H}^{2h \times 2h}$ such that

(9.6) $$S^{-1}A_5 S = A'_5, \quad S_\phi H_5 S = \delta_1 \beta(\phi)F_h \oplus \delta_2 \beta(\phi)F_h,$$

where δ_1, δ_2 are signs ± 1. It remains to prove that $\delta_1 \delta_2 = -1$.

Suppose first h is odd. It is easy to see by inspection that there is an h-dimensional (H_5, ϕ)-neutral subspace. By (9.6), the same should hold for $\delta_1 \beta(\phi)F_h \oplus \delta_2 \beta(\phi)F_h$, which in view of Proposition 1.2 is possible only if $\delta_1 \neq \delta_2$.

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Second, suppose h is even. We follow the argument in the proof of Lemma 8.9 part (1), the case of even h, where we substitute $\beta(\phi)$ for i, and instead of (8.14) use the equalities

$$(-q)J_h(\beta(\phi)b)(qI) = J_h(-\beta(\phi)b),$$

(9.7) $$\qquad (qI) \cdot \Xi_h \cdot (qI) = -\Xi_h = (-1)^{h/2-1}\Xi_h(\beta(\phi)^h),$$

where $q \in \mathbb{H}$ is such that $q^2 = -1$ and $q\beta(\phi) = -\beta(\phi)q$. As a result, we can replace A_5 and H_5 with

$$A_5'' := J_h(-\beta(\phi)b) \oplus J_h(-\beta(\phi)b) \quad \text{and} \quad H_5'' := \Xi_h(\beta(\phi)^h) \oplus (-\Xi_h(\beta(\phi)^h)),$$

respectively. Finally, to transform the pair of matrices (A_5'', H_5'') to $(A_5, \pm H_5)$, use the matrix

$$S := \operatorname{diag}(1, \beta(\phi), \ldots, \beta(\phi)^{h-1}) \oplus \operatorname{diag}(1, \beta(\phi), \ldots, \beta(\phi)^{h-1}).$$

This completes the proof of Lemma 9.7. $\square$

Proof. (9.5). Lemma 9.7 and Theorem 9.3 show that $\gamma_{\mathbb{H}}(A, H) = n/2$. Now (9.5) follows from Theorem 5.2. $\square$

9.4. Comparison with the complex case (II). If $A \in \mathbb{C}^{n \times n}$ is (H, id)-Hamiltonian, where $H = -H^T \in \mathbb{C}^{n \times n}$ is invertible, then by Theorem 7.2 the index of neutrality of (A, H) is $n/2$. Let ϕ be a nonstandard involution on $\mathbb{H}$ such that $\phi(\mathrm{i}) = \mathrm{i}$. Considering A as an (H, ϕ)-Hamiltonian quaternion matrix, we obtain that its index of neutrality is at least $n/2$. But it cannot exceed $n/2$, because $n/2$ is the maximal dimension of an (H, ϕ)-neutral subspace, so it must be equal to $n/2$.

10. Invariant Lagrangian subspaces. Using Corollaries 2.6 and 2.7, combined with the results of Sections 5 through 9, we obtain information regarding invariant Lagrangian subspaces. For example:

THEOREM 10.1. *Let $\mathbb{F} = \mathbb{H}$ and let ϕ be the quaternion conjugation. For invertible $H = -H^* \in \mathbb{H}^{n \times n}$ and for an $(H, ^*)$-Hamiltonian matrix X, we have:*

(a) *There exists an X-invariant $(H, ^*)$-Lagrangian subspace if and only if, in the notation of Theorem 8.4,*

$$\sum_{j \text{ odd}} p_{k,j}^+ = \sum_{j \text{ odd}} p_{k,j}^-, \quad k = 1, 2, \ldots, a,$$

and the number of nilpotent Jordan blocks of odd sizes in the Jordan form of X is even.

(b) *Let $\mathcal{M}_+$ be the sum of root subspaces of X corresponding to the eigenvalues with positive real parts. Then for every A-invariant subspace $\mathcal{M} \subseteq \mathcal{M}_+$ there exists a unique A-invariant $(H, {}^*)$-Lagrangian subspace $\mathcal{M}'$ containing $\mathcal{M}$ if and only if X has no Jordan blocks of odd size associated with eigenvalues having zero real part (including the zero eigenvalue), and for every such eigenvalue λ_0 the signs in the sign characteristic associated with λ_0 are all the same.*

Note that the necessary and sufficient condition in part (b) does not exclude the possibility that different complex eigenvalues λ with different nonnegative imaginary parts will have different signs in their sign characteristics. Analogous result holds true if "positive" is replaced by "negative" in Theorem 10.1(b).

The proof is immediate from Theorems 8.4 and 8.5.

In a similar fashion one obtains results concerning invariant Lagrangian subspaces in settings (I), (II), (III), and (V). We omit the statements of such results.

11. Concluding remarks. The results of this paper can be easily extended to the situations where the matrix H is singular. If $H \in \mathbb{F}^{n \times n}$ such that $H_\phi = -H$, where ϕ is a continuous involution of $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, then a matrix $A \in \mathbb{F}^{n \times n}$ will be called (H, ϕ)-*Hamiltonian* if the equality $HA = -A_\phi H$ holds. Using the well known canonical forms of H under the transformation $H \mapsto S_\phi H S$, where S is invertible, we may assume without loss of generality that H has the form

$$(11.1) \qquad H = \begin{bmatrix} H_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad H_1 \in \mathbb{F}^{m \times m} \text{ invertible.}$$

Decomposing an (H, ϕ)-Hamiltonian matrix $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ conformably with (11.1) we find that $A_{21} = 0$ and A_{22} is (H_1, ϕ)-Hamiltonian. In particular, $\operatorname{Ker} H$ is A-invariant. Clearly, an A_{22}-invariant subspace $\mathcal{M} \subseteq \mathbb{F}^{m \times 1}$ is (H_1, ϕ)-neutral if and only if the A-invariant subspace $\mathbb{F}^{(n-m) \times 1} \oplus \mathcal{M}$ is (H, ϕ)-neutral. It follows that the subspace $\mathcal{M}$ is MIN (A_{22}, H_1, ϕ) if and only if the subspace $\mathbb{F}^{(n-m) \times 1} \oplus \mathcal{M}$ is MIN (A, H, ϕ). In particular, the order of neutrality of the pair (A, H) is equal to the order of neutrality of (A_{22}, H_1) plus the dimension of the kernel of H.

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