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HURWITZ-RADON'S SYMPLECTIC ANALOGY AND HUA'S CYCLIC RECURRENCE RELATION*

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Abstract. A symplectic version of the classical Hurwitz-Radon theorem is presented in this paper. This problem was first approached but without full success by L.K. Hua in a paper on geometry of matrices, in 1947. The present paper solves Hua's problem in a complete and elementary way. As a consequence, a direct matrix proof of a related result of D.B. Shapiro, which is of independent interest, is given. It turns out that the sympletic version is closely related to Hurwitz and Radon's original orthogonal version via a remarkable observation of Hua. Hua's cyclic recurrence relation and its unitary version are also presented.

Key words. Hurwitz-Radon equations, Hurwitz-Radon theorem, Symplectic Hurwitz-Radon theorem, Hua's cyclic recurrence relation.

AMS subject classifications. 15A04, 51N30.

1. Introduction. Let $G = GL(n, F), O(n, F)$ or $Sp(n, F)$. The following two equations for elements $A_1, \dots, A_r$ (or $B_1, \dots, B_r$, respectively) of G have been studied extensively:

$$A_iA_j + A_jA_i = -2\delta_{ij}, \tag{1.1}$$

and

$$B_iB_j + B_jB_i = 2\delta_{ij}, \tag{1.2}$$

where δ_{ij} is the Kronecker symbol and stands for the corresponding scalar matrix. (Here we always let F be a field of characteristic other than 2, and hence, the above two equations are different.)

For the case of $G = GL(n, F)$ and $G = O(n, F)$, the following problem has been studied by many authors since Hurwitz's pioneering work in 1898: For any fixed field F and positive integer n, determine the largest possible value of r for which there exist r matrices satisfying (1.1) (or (1.2), respectively). Some related results including the famous Hurwitz-Radon theorem (for the case of orthogonal group) are reviewed in the following section, as they are needed in later sections.

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For the case of $G = Sp(n, F)$, the corresponding problem for equation (1.1) was first studied by L.K. Hua [6] with background on geometry of matrices in 1947. By his surpassing matrix skills, Hua successfully found a remarkable connection between this problem for $G = Sp(n, \mathbb{C})$ and the Hurwitz's original problem for $G = O(n, \mathbb{C})$, which reduced the former problem to the latter one. However, it seems that Hua was not aware of the classical Hurwitz-Radon theorem and tried to resolve this problem on his own. Since this work was buried in a geometrical discussion, Hua's contribution on the Hurwitz-Radon theorem was rarely known. But in the long run, his idea and approach turned out to be really useful and powerful.

This paper is to remedy Hua's original incomplete treatment to obtain a symplectic version of the Hurwitz-Radon theorem and the corresponding result for equation (1.2). These results concerning symplectic groups are new. They can be used to prove a related theorem of D.B. Shapiro, which was also viewed as a symplectic version of the Hurwitz-Radon theorem. However, note that the present version is more precise and powerful than the Shapiro's. The key point in our proof of these new results is an elementary observation of Hua, which is completed here as a theorem called Hua's cyclic recurrence relations. These remarkable relations and their unitary version also lead to a simple proof of the classical Hurwitz-Radon theorem.

2. Review of some related work.

2.1. The case of the orthogonal group. For the case of $G = O(n, F)$, equations (1.1) are known as the Hurwitz-Radon equations, which are connected with the problems on compositions of quadratic forms. The fundamental result can be stated in the following compact form due to Radon [11].

THEOREM 2.1 (Hurwitz-Radon theorem). *Let $F = \mathbb{C}$ or $\mathbb{R}$. There exist r matrices $A_1, \ldots, A_r \in O(n, F)$ satisfying the Hurwitz-Radon equations* (1.1) *if and only if $r \leq R(n)$, where $R(n) = 8^a + 2b - 1$ for $n = 2^{4a+b}c$, $b = 0, 1, 2, 3$ and c is odd.*

Radon [11] proved this theorem for $F = \mathbb{R}$, while the corresponding result for $F = \mathbb{C}$ was obtained by Hurwitz [7]. Since Hurwitz's and Radon's independent but closely related pioneering works, these equations appeared naturally in various settings, such as projective geometry, composition algebras, vector product in Euclidean spaces, etc. In particular, Eckmann [3] found an interesting interpretation of the Hurwitz-Radon number $R(n)$ in terms of the largest number of orthogonal linear vector fields on the sphere in $\mathbb{R}^n$. Besides so many interesting proofs of Hurwitz-Radon's theorem (cf. Lam [8] and Lin [9]), Shapiro [12] also extended this result to an *arbitrary* field of characteristic different from 2 in 1977. Moreover, since Shapiro considered (1.1) and (1.2) simultaneously, a similar result for equation (1.2) had been also obtained. We will see that Shapiro's remarkable generalization can be viewed as a variation of an

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old result due to Newman and Williamson independently. From this point of view, the present author refined and simplified Shapiro's original abstract treat by supplying a direct proof of the following concrete matrix result (which can be viewed as a "mixed Hurwitz-Radon theorem") in [9]:

THEOREM 2.2 (Mixed Hurwitz-Radon theorem). *Let F be a field of characteristic different from 2. Suppose there are $r+s$ matrices $A_1,\ldots,A_r,B_1,\ldots,B_s \in O(n,F)$ satisfying*

$$\begin{cases} A_iA_j + A_jA_i = -2\delta_{ij} & (i,j=1,\ldots,r) \\ B_kB_l + B_lB_k = 2\delta_{kl} & (k,l=1,\ldots,s) \\ A_iB_k = -B_kA_i & (i=1,\ldots,r,\ k=1,\ldots,s), \end{cases} \tag{2.1}$$

then $r+s \le 2q+1$, here q is defined by the condition $n = 2^q n_0$, where n_0 is odd. Moreover, there exist $r+s = 2q+1$ matrices $A_1,\ldots,A_r,B_1,\ldots,B_s \in O(n,F)$ satisfying the above equations if and only if $s \in [0,2q+1]$ satisfies $s \equiv q+1 \pmod 4$.

It is easy to deduce the general Hurwitz-Radon theorem from the above result. The proof (cf. Lin [9]) suggests that Theorem 2.1 can be restated in the following equivalent form:

THEOREM 2.3. *Suppose F is a field of characteristic different from 2. Then there exist r matrices $A_1,\ldots,A_r \in O(n,F)$ satisfying the Hurwitz-Radon equations (1.1) if and only if $r \le R(n)$, where*

$$R(n) = \begin{cases} 2q & \textit{if} \quad q \equiv 0 \pmod 4 \\ 2q-1 & \textit{if} \quad q \equiv 1 \pmod 4 \\ 2q-1 & \textit{if} \quad q \equiv 2 \pmod 4 \\ 2q+1 & \textit{if} \quad q \equiv 3 \pmod 4 \end{cases}$$

for $n = 2^q n_0$, n_0 is odd.

Similarly, one can deduce the following result form Theorem 2.2, which can be viewed as a dual version of Theorem 2.3:

THEOREM 2.4. *Suppose F is a field of characteristic different from 2. Then there exist s matrices $B_1,\ldots,B_s \in O(n,F)$ satisfying the dual Hurwitz-Radon equations (1.2) if and only if $s \le S(n)$, where*

$$S(n) = \begin{cases} 2q+1 & \textit{if} \quad q \equiv 0 \pmod 4 \\ 2q & \textit{if} \quad q \equiv 1 \pmod 4 \\ 2q-1 & \textit{if} \quad q \equiv 2 \pmod 4 \\ 2q-1 & \textit{if} \quad q \equiv 3 \pmod 4 \end{cases}$$

for $n = 2^q n_0$, n_0 is odd.

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2.2. The case of general linear group. For the case of $G = GL(n, F)$, these equations appeared first in the work of some famous physicists such as Dirac, Eddington and Pauli. They studied the equations (together with the corresponding differential versions) for $GL(2, \mathbb{C})$ and $GL(4, \mathbb{C})$ respectively. In particular, Eddington studied a mixed equation combining (1.1) and (1.2) in $GL(n, \mathbb{R})$ and obtained an interesting result. Eddington's result was generalized to $GL(n, \mathbb{C})$ and $GL(n, \mathbb{R})$ by Newman [10] and Williamson [15] independently. In 1953, Dieudonné [2] obtained a similar version of the above Hurwtiz-Radon theorem for $GL(n, K)$, where K is a general division ring. Dieudonné noted the similarity between Newman's result and Hurwitz's classical result (it seems that he had overlooked Radon's and Williamson's contributions). However, Dieudonné neither realized that the classical Hurwitz-Radon theorem could be proved easily by transferring of Newman-Williamson theorem to $O(n, F)$ as shown in Lin [9], nor noticed that original Newman-Williamson theorem for $GL(n, \mathbb{R})$ could be generalized faithfully to $GL(n, K)$, where K is a division ring such that $-1 = x^2 + y^2, xy = yx$ is unsolvable in K. Here we only state the result for the commutative case.

Theorem 2.5. *Let F be a field of characteristic different from 2. Suppose $x^2+y^2 = -1$ has no solution in F. If there are $r+s$ matrices $A_1, \ldots, A_r, B_1, \ldots, B_s \in GL(n, F)$ satisfying* (2.1) *then $r + s \leq 2q + 1$, here q is defined by the condition $n = 2^q n_0$, where n_0 is odd. Moreover, there exist $r+s = 2q+1$ matrices $A_1, \ldots, A_r, B_1, \ldots,$ $B_s \in GL(n, F)$ satisfying the above equations if and only if the integer $s \in [0, 2q + 1]$ satisfies $s \equiv q + 1 \pmod 4$.*

The proof is omitted, since it is similar to the proof of Theorem 2.2 given in Lin [9]. One can derive the following two corollaries due to Dieudonné, which is also presented in Lam's famous book on quadratic forms cf. [8, pp. 126–127].

Theorem 2.6. *Let F be a field of characteristic different from 2, suppose $x^2 + y^2 = -1$ has no solution in F. Then there exist r matrices $A_1, \ldots, A_r \in GL(n, F)$ satisfying* (1.1) *if and only if $r \leq R(n)$, where $R(n)$ is defined in Theorem 2.3.*

Theorem 2.7. *Let F be a field of characteristic different from 2, suppose $x^2 + y^2 = -1$ has no solution in F. Then there exist s matrices $B_1, \ldots, B_s \in GL(n, F)$ satisfying* (1.2) *if and only if $s \leq S(n)$, where $S(n)$ is defined in Theorem 2.4.*

3. The case of the symplectic group.

3.1. Hua's reduction lemma. Hua found a useful result which reduces the problem for the symplectic group to the problem for orthogonal group (at least for an algebraically closed field). To state this result, we introduce a terminology: A set of $r + s$ matrices $A_1, \ldots, A_r, B_1, \ldots, B_s$ satisfying (2.1) is called *an (r, s)-family.* Then

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we have the following lemma which is essentially due to Hua [6].

LEMMA 3.1. *Suppose F is an algebraically closed field and $r \geq 2$. Then there exists an (r,s)-family in $Sp(n,F)$ if and only if there exists an $(s,r-2)$-family in $O(n/2,F)$.*

The proof was based on Hua's following observation on the canonic form of a $(2,0)$-family in $Sp(n,F)$ (cf. Hua [6, Theorem 35]):

LEMMA 3.2. *Let F be an algebraically closed field. Suppose that $A_1, A_2 \in Sp(n,F)$ satisfy*

$$A_1^2 = A_2^2 = -I \qquad \text{and} \qquad A_1A_2 = -A_2A_1.$$

Then there exists $P \in Sp(n,F)$ such that

$$P^{-1}A_1P = \begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix} \qquad \text{and} \qquad P^{-1}A_2P = \begin{bmatrix} 0 & iI \\ iI & 0 \end{bmatrix}.$$

Now we can prove Lemma 3.1 as follows:

Proof. By Lemma 3.2, without loss of generality, we assume that

$$A_1 = \begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix} \qquad \text{and} \qquad A_2 = \begin{bmatrix} 0 & iI \\ iI & 0 \end{bmatrix}.$$

It is not hard to show that any matrix X anti-commuting with A_1 and A_2 simultaneously is of the form

$$X = \begin{bmatrix} 0 & Y \\ -Y & 0 \end{bmatrix}, \qquad \text{where} \qquad Y \in M(n/2,F).$$

Moreover, $X \in Sp(n,F)$ if and only if $Y \in O(n/2,F)$, $X^2 = \pm I$ if and only if the corresponding Y satisfies $Y^2 = \mp I$, and X_1, X_2 anti-commute if and only if the corresponding Y_1, Y_2 anti-commute. □

For later application we also need the following result on the canonic form of a $(0,2)$-family in $Sp(n,F)$, which was missed by Hua.

LEMMA 3.3. *Let F be an algebraically closed field. Suppose that $B_1, B_2 \in Sp(n,F)$ satisfy*

$$B_1^2 = B_2^2 = I \qquad \text{and} \qquad B_1B_2 = -B_2B_1.$$

Then $n = 4m$ and there exists $P \in Sp(n,F)$ such that

$$P^{-1}B_1P = \begin{bmatrix} 0 & -J \\ J & 0 \end{bmatrix} \qquad \text{and} \qquad P^{-1}B_2P = \begin{bmatrix} 0 & iJ \\ iJ & 0 \end{bmatrix}.$$

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Proof. Let $A = B_1B_2$ and $B = B_1$, and $V_\pm = \{\xi \in F^{2n}, A\xi = \pm i\xi\}$. For any pair of $\xi \in V_+, \eta \in V_-$ such that $[\xi, \eta] = 1$, the subspace W spanned by $\xi, B\eta, \eta, -B\xi$ is invariant under A and B. In fact, we have

$$A\xi = i\xi,\ A(B\eta) = -B(A\eta) = -B(-i\eta) = B\eta,\ A\eta = -i\eta,\ A(-B\xi) = BA\xi = i(B\xi),$$

and

$$B\xi = -(-B\xi), \quad B(B\eta) = \eta, \quad B(\eta) = B\eta, \quad B(-B\xi) = -\xi.$$

Moreover, from these relations it can be shown that $\xi, B\eta, \eta, -B\xi$ is a symplectic basis for W. Hence, by induction, we obtain a symplectic basis of the form $\xi_1, B\eta_1, \ldots, \xi_m, B\eta_m, \eta_1, -B\xi_1, \ldots, \eta_m, -B\xi_m$. If we take

$$P = [\xi_1, B\eta_1, \ldots, \xi_m, B\eta_m, \eta_1, -B\xi_1, \ldots, \eta_m, -B\xi_m],$$

then it is easy to verify that $P \in Sp(n, F)$ has the desired property. □

This lemma has the following consequence.

Lemma 3.4. *Suppose F is an algebraically closed field and $s \geq 2$. Then there exists an (r, s)-family in $Sp(n, F)$ if and only if there exists an $(s - 2, r)$-family in $Sp(n/2, F)$.*

Proof. Without loss of generality, by Lemma 3.4, we assume that

$$B_1 = \begin{bmatrix} 0 & -J \\ J & 0 \end{bmatrix} = \widetilde{B_1} \qquad \text{and} \qquad B_2 = \begin{bmatrix} 0 & iJ \\ iJ & 0 \end{bmatrix} = \widetilde{B_2}.$$

Then any matrix X anti-commuting with $\widetilde{B_1}$ and $\widetilde{B_2}$ simultaneously is of the form

$$X = \begin{bmatrix} iY & 0 \\ 0 & iJYJ \end{bmatrix},$$

where $Y \in M(m, F), (n = 2m = 4l)$. Moreover, $X \in Sp(n, F) \iff Y \in Sp(m, F)$. $X^2 = \pm I \iff Y^2 = \mp I$. $X_1, X_2 \in Sp(n, F)$ anti-commute if and only if the corresponding $Y_1, Y_2 \in Sp(m, F)$ anti-commute. □

3.2. The symplectic Hurwitz-Radon theorem and its duality: Part I. In this subsection, we present a symplectic version of the mixed Hurwitz-Radon theorem (Theorem 3.5), and use it to prove a symplectic analogy of the Hurwitz-Radon theorem (Theorem 3.7).

Theorem 3.5. *Let F be a field of characteristic different from 2. Suppose $x^2 + y^2 = -1$ has a solution in F. If there are $r + s$ matrices $A_1, \ldots, A_r, B_1, \ldots, B_s \in$*

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$Sp(n, F)$ satisfying (2.1), then $r + s \le 2q + 1$, here q is defined by the condition $n = 2^q n_0$, where n_0 is odd. Moreover, there exist $r+s = 2q+1$ matrices $A_1, \dots, A_r, B_1, \dots, B_s \in Sp(n, F)$ satisfying (2.1) if and only if the integer $s \in [0, 2q+1]$ satisfies $s \equiv q-1 \pmod 4$.

Note that there is a difference between Theorem 2.2 and Theorem 3.5: Theorem 2.2 is true for any field of characteristic different from 2, while Theorem 3.5 is true only for those fields F such that $x^2 + y^2 = -1$ is solvable in F (for example, any algebraic field or any Galois filed).

In the following, we reduce the proof of Theorem 3.5 to Theorem 2.2.

Proof. We divide the proof into two steps.

Step 1. We first prove the theorem under the assumption that F is algebraically closed.

The case for $n = 2n_0$. We have to show that if there exists an (r, s)-family in $Sp(2n_0, F)$, then $r + s \le 3$ and the equality holds if and only if $(r, s) = (3, 0)$.

Suppose there is a given (r, s)-family in $Sp(2n_0, F)$, then by Lemma 3.3, we must have $s \le 1$.

If $s = 1$, then we must have $r = 0$, since a $(1, 1)$-family A, B can be converted to a $(0, 2)$ family by replacing A with AB. In this case $r + s = 0 + 1 = 1 < 3$.

If $s = 0$, then we will show that the maximum value of r is 3. Suppose $r \ge 2$. Then by Lemma 3.1, there exists a $(0, r-2)$ family in $O(n_0, F)$. By Theorem 2.4 or a direct argument, $r - 2 \le S(n_0) = 1$, i.e., $r \le 3$. Finally, we can construct a $(3, 0)$-family in $Sp(2n_0, F)$ via Lemma 3.1 (see the proof of Lemma 3.2) as follows:

$$A_1 = \begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & iI \\ iI & 0 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Now we consider the case of $n = 2^q n_0$ where $q \ge 2$.

By Lemma 3.1, we can construct a $(q+2, q-1)$-family in $Sp(n, F)$ via a $(q-1, q)$-family in $O(n/2, F)$ as follows:

$$\begin{aligned} \mathcal{A}_1 &= \begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix}, \qquad \mathcal{A}_2 = \begin{bmatrix} 0 & iI \\ iI & 0 \end{bmatrix}, \\ \mathcal{A}_3 &= \begin{bmatrix} 0 & B_1 \\ -B_1 & 0 \end{bmatrix}, \quad \dots, \quad \mathcal{A}_{q+2} = \begin{bmatrix} 0 & B_q \\ -B_q & 0 \end{bmatrix}, \\ \mathcal{B}_1 &= \begin{bmatrix} 0 & A_1 \\ -A_1 & 0 \end{bmatrix}, \quad \dots, \quad \mathcal{B}_{q-1} = \begin{bmatrix} 0 & A_{q-1} \\ -A_{q-1} & 0 \end{bmatrix}, \end{aligned}$$

where $A_1, \dots, A_{q-1}, B_1, \dots, B_q$ is a $(q-1, q)$-family in $O(n/2, F)$. To obtain another

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$(2q+1-s, s)$-family with $s \equiv q-1 \pmod 4$, we only need to use the following Newman-Williamson trick described in Lemma 3.6 repeatedly.

Now we show that any (r, s)-family in $Sp(n, F)$ should satisfy $r+s \le 2q+1$. If $r \ge 2$, then by Lemma 3.1, there exists an $(s, r-2)$-family in $O(n/2, F)$, hence by Theorem 2.2, we have $s+(r-2) \le 2(q-1)+1$, i.e., $r+s \le 2q+1$. Now suppose for the given maximal (r, s)-family in $Sp(2^q n_0, F)$ we have $r \le 1$, then $s \ge (2q+1)-r \ge 5-1=4$, hence by Lemma 3.6, we can transform it to an $(r+4, s-4)$-family in $Sp(2^q n_0, F)$. And from the previous discussion it follows that $(r+4)+(s-4) \le 2q+1$, i.e., $r+s \le 2q+1$.

Step 2. If $x^2=-1$ has a solution in F, then the above $(q+2, q-1)$-family is already contained in $Sp(n, F)$. For the general case, suppose $\lambda^2+\mu^2=-1$ for $\lambda, \mu \in F$. If we replace the two pure imaginary matrices

$$\mathcal{A}_1 = \begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix} \quad \text{and} \quad \mathcal{A}_2 = \begin{bmatrix} 0 & iI \\ iI & 0 \end{bmatrix}$$

in the above construction by

$$\widetilde{\mathcal{A}_1} = \frac{1}{i}(\lambda \mathcal{A}_1 + \mu \mathcal{A}_2) = \begin{bmatrix} \lambda I & \mu I \\ \mu I & -\lambda I \end{bmatrix} \text{ and } \widetilde{\mathcal{A}_2} = \frac{1}{i}(-\mu \mathcal{A}_1 + \lambda \mathcal{A}_2) = \begin{bmatrix} -\mu I & \lambda I \\ \lambda I & \mu I \end{bmatrix}$$

and keep the other matrices unchanged, then we get a $(q+2, q-1)$-family in $Sp(n, F)$. ☐

LEMMA 3.6. *Let G be a matrix group containing $-I$. Suppose there exists an (r, s)-family in G. If $r \ge 4$, then there exists an $(r-4, s+4)$-family in G. If $s \ge 4$, then there exists an $(r+4, s-4)$-family in G.*

The following proof is essentially due to M.H.A. Newman [10], who had acknowledged John Williamson in the *Correction* part of [10].

Proof. Given any $(4, 0)$-family A_1, A_2, A_3, A_4, if we construct a new set of four matrices B_1, B_2, B_3, B_4 as follows:

$$B_1 = A_2A_3A_4, \quad B_2 = A_1A_3A_4, \quad B_3 = A_1A_2A_4, \quad B_4 = A_1A_2A_3,$$

then these four matrices B_1, B_2, B_3, B_4 will form a $(0, 4)$-family. Moreover, it is easy to see that a matrix X anti-commutes with A_1, A_2, A_3, A_4 simultaneously if and only if X anti-commutes with B_1, B_2, B_3, B_4 simultaneously. Thus, an $(r+4, s)$-family can be converted to an $(r, s+4)$-family by the above transformation. By symmetry, a $(r, s+4)$-family can be converted to an $(r+4, s)$-family by a similar transformation. ☐

As in the orthogonal case, Theorem 3.5 also has two important corollaries as follows:

THEOREM 3.7 (Symplectic Hurwitz-Radon theorem (I)). *Let F be a field of characteristic different from 2. Suppose $x^2+y^2=-1$ has a solution in F. Then there*

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are r matrices $A_1,\dots,A_r \in Sp(n,F)$ satisfying (1.1) *if and only if $r \le P(n)$, where*

$$P(n) = \begin{cases} 2q-1 & \textit{if} \quad q \equiv 0 \pmod 4 \\ 2q+1 & \textit{if} \quad q \equiv 1 \pmod 4 \\ 2q & \textit{if} \quad q \equiv 2 \pmod 4 \\ 2q-1 & \textit{if} \quad q \equiv 3 \pmod 4 \end{cases}$$

for $n = 2^q n_0$, n_0 is odd.

Proof. Set $n = 2^q n_0$, where n_0 is odd. There are four cases according to the possible values of q module 4.

If $q \equiv 0 \pmod 4$, then by Theorem 3.5, there exists a $(2q-2,3)$-family in $Sp(n,F)$, as a corollary, $P(n) \ge 2q-2$. Note that if there are $2q-2$ matrices $A_1,\dots,A_{2q-2}$ satisfying (1.1), then we can add one more matrix $A_{2q-1} = A_1 \cdots A_{2q-2}$ to this set to obtain a set of $2q-1$ matrices satisfying (1.1). Hence, $P(n) \ge 2q-1$. We claim that we have $P(n) = 2q-1$ in this case. If it is not the case, then we have $P(n) \ge 2q$. Let $A_1,\dots,A_{2q}$ be a $(2q,0)$-family, set $B_1 = A_1 \cdots A_{2q}$, then it is easy to show that $A_1,\dots,A_{2q},B_1$ is a $(2q,1)$-family. Thus, by Theorem 3.5, q must satisfy $q \equiv 3 \pmod 4$, which is contradictory to the hypothesis $q \equiv 0 \pmod 4$. Hence, $P(n) = 2q-1$.

If $q \equiv 1 \pmod 4$, then by Theorem 3.5, there exists a $(2q+1,0)$-family in $Sp(n,F)$, and $r = 2q+1$ attains the maximum value of r, hence $P(n) = 2q+1$ in this case.

If $q \equiv 2 \pmod 4$, then by Theorem 3.5, there exists a $(2q,1)$-family in $Sp(n,F)$. As a corollary, there exists a $(2q,0)$-family in $Sp(n,F)$, and therefore, $P(n) \ge 2q$. We only need to show that $P(n) \le 2q$. If it is not the case, then $P(n) = 2q+1$, hence there would be a $(2q+1,0)$-family in $Sp(n,F)$. Thus, by Theorem 3.5 we have $q \equiv 1 \pmod 4$, which is contradictory to the hypothesis $q \equiv 2 \pmod 4$. Hence, $P(n) = 2q$.

If $q \equiv 3 \pmod 4$, then by Theorem 3.5, there exists a $(2q-1,2)$-family in $Sp(n,F)$, as a corollary, $P(n) \ge 2q-1$. We claim that we have $P(n) = 2q-1$ in this case. If it is not the case, then $P(n) \ge 2q$. Let $A_1,\dots,A_{2q}$ be a $(2q,0)$-family, set $A_{2q+1} = A_1 \cdots A_{2q}$, then it is easy to see that $A_{2q+1} \in Sp(n,F)$ and $A_{2q+1}^2 = -I$, moreover, A_{2q+1} anti-commuts with each $A_i (i = 1,\dots,2q)$. In other words, $A_1,\dots,A_{2q+1}$ form a $(2q+1,0)$-family. Thus, by Theorem 3.5, q must satisfy $q \equiv 1 \pmod 4$, which is contradictory to the hypothesis $q \equiv 3 \pmod 4$. Hence, $P(n) = 2q-1$. ∎

Theorem 3.8. *Let F be a field of characteristic different from 2. Suppose $x^2 + y^2 = -1$ has a solution in F. Then there are s matrices $B_1,\dots,B_s \in Sp(n,F)$*

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satisfying (1.2) *if and only if* $s \leq Q(n)$, *where*

$$Q(n) = \begin{cases} 2q-1 & \textit{if } q \equiv 0 \pmod 4 \\ 2q-1 & \textit{if } q \equiv 1 \pmod 4 \\ 2q+1 & \textit{if } q \equiv 2 \pmod 4 \\ 2q & \textit{if } q \equiv 3 \pmod 4 \end{cases}$$

for $n = 2^q n_0$, n_0 *is odd.*

The proof is similar to the above proof, and hence, it is omitted.

3.3. The symplectic Hurwitz-Radon theorem and its duality (II). For a field F such that $x^2 + y^2 = -1$ has no solution in F, the corresponding results for Theorem 3.7 and Theorem 3.8 are as follows.

THEOREM 3.9 (Symplectic Hurwitz-Radon theorem (II)). *Let F be a field of characteristic different from* 2. *Suppose $x^2 + y^2 = -1$ has no solution in F. Then there are r matrices $A_1, \ldots, A_r \in Sp(n, F)$ satisfying* (1.1) *if and only if $r \leq 2q - 1$, where q is determined by $n = 2^q n_0$ (n_0 is odd).*

Proof. Denote $P'_F(n)$ for the largest integer r such that there exist r matrices $A_1, \ldots, A_r \in Sp(n, F)$ satisfying (1.1). Note that $Sp(n, F)$ are contained in both $GL(n, F)$ and $Sp(n, E)$ where $E = F(i)$. Hence, we have $P'_F(n) \leq R(n)$ and $P'_F(n) \leq P'_E(n) = P(n)$ (by Theorem 3.7). Note that from the formula for $R(n)$ and $P(n)$ we have

$$\begin{aligned} \min\{R(n), P(n)\} &= \begin{cases} \min\{2q, 2q-1\} & \text{if } q \equiv 0 \pmod 4 \\ \min\{2q-1, 2q+1\} & \text{if } q \equiv 1 \pmod 4 \\ \min\{2q-1, 2q\} & \text{if } q \equiv 2 \pmod 4 \\ \min\{2q+1, 2q-1\} & \text{if } q \equiv 3 \pmod 4 \end{cases} \\ &= 2q-1 \end{aligned}$$

for $n = 2^q n_0$, n_0 is odd. Therefore, we have $P'_F(n) \leq 2q - 1$. In order to show that the equality always holds, we only need to construct a $(2q - 1, 0)$-family in $Sp(n, F)$. In fact, we can easily construct such a family via a maximal (r, s)-family in $O(n/2, F)$ as follows. Let $A_1, \ldots, A_r, B_1, \ldots, B_s$ be a maximal (r, s)-family in $O(n/2, F)$. If we set

$$\mathcal{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & A_i \end{bmatrix}, \qquad (i = 1, \ldots, r), \qquad \mathcal{A}_{r+j} = \begin{bmatrix} 0 & B_j \\ -B_j & 0 \end{bmatrix} \qquad (j = 1, \ldots, s),$$

then these matrices $\mathcal{A}_1, \ldots, \mathcal{A}_{2q-1}$ form a $(2q - 1, 0)$-family in $Sp(n, F)$. $\square$

THEOREM 3.10. *Let F be a field of characteristic different from* 2. *Suppose $x^2 + y^2 = -1$ has no solution in F. Then there are s matrices $B_1, \ldots, B_s \in Sp(n, F)$*

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satisfying (1.2) *if and only if* $s \leq 2q-1$*, where* q *is determined by* $n = 2^q n_0$ *(*n_0 *is odd).*

Proof. Denote $Q'_F(n)$ for the largest integer s such that there exist r matrices $B_1, \ldots, B_s \in Sp(n,F)$ satisfying (1.2). Note that $Sp(n,F)$ are contained in both $GL(n,F)$ and $Sp(n,E)$ where $E = F(i)$. Hence, we have $Q'_F(n) \leq S(n)$ and $Q'_F(n) \leq Q'_E(n) = Q(n)$. Note that from the formula for $S(n)$ and $Q(n)$ we have $\min\{S(n), Q(n)\} = 2q-1$, for $n = 2^q n_0$, n_0 is odd. Therefore, we have $Q'_F(n) \leq 2q-1$. In order to show that the equality always holds, we only need to construct a $(0, 2q-1)$-family in $Sp(n,F)$. In fact, we can easily construct such a family via a maximal (r,s)-family in $O(n/2,F)$ as follows. Let $A_1, \ldots, A_r, B_1, \ldots, B_s$ be a maximal (r,s)-family in $O(n/2,F)$, if we set

$$\mathcal{B}_i = \begin{bmatrix} 0 & A_i \\ -A_i & 0 \end{bmatrix} \qquad (i = 1, \ldots, r), \qquad \mathcal{B}_{r+j} = \begin{bmatrix} B_j & 0 \\ 0 & B_j \end{bmatrix} \qquad (j = 1, \ldots, s)$$

then these matrices $\mathcal{B}_1, \ldots, \mathcal{B}_{2q-1}$ form a $(0, 2q-1)$-family in $Sp(n,F)$. $\square$

4. Discussions on relations to the work of D.B. Shapiro. Shapiro [13, p. 47] considered a similar problem which can be stated in in terms of matrices in the following form: For any fixed field F and positive integer n, determine the largest possible value of r for which there exist r matrices $A_1, \ldots, A_r \in M(n,F)$ and r nonzero numbers $a_1, \ldots, a_r \in F$ such that

$$\text{(4.1)} \qquad \begin{cases} A'_i J A_i = a_i J & (i = 1, \ldots, r) \\ A_i^2 = -a_i & (i = 1, \ldots, r) \\ A_i A_j = -A_j A_i & (i \neq j,\ i,j = 1, \ldots, r). \end{cases}$$

He obtained the following result: If we denote by $D(n,F)$ the largest possible value of r, then $D(n,F) = P(n)$.

One can deduce this result from Theorem 3.7 and Lemma 3.1 as follows. If F is an algebraically closed field, then the problems of Hua and Shapiro are equivalent (we can choose all a_i to be 1), by Theorem 3.7, we get the desired result. Hence, for a general field F, we must have $D(n,F) \leq P(n)$ by considering the embodying into its algebraical closure. To obtain the identity, according to Lemma 3.1, we note that a $(0,s)$-family $B_1, \ldots, B_s$ in $O(n/2,F)$ can be used to give an $(s,0)$-family in $Sp(n,F)$ as follows:

$$\mathcal{A}_i = \begin{bmatrix} 0 & B_i \\ -B_i & 0 \end{bmatrix} \qquad (i = 1, \ldots, s).$$

This $(s,0)$-family in $Sp(n,F)$ together with the following two matrices

$$\mathcal{A}_{s+1} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \qquad \text{and} \qquad \mathcal{A}_{s+2} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

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will give a solution to Shapiro's problem, with the corresponding $a_1 = \cdots = a_s = 1$ and $a_{s+1} = a_{s+2} = -1$. Thus, we can take $r = S(n/2) + 2$. However, it is easy to check that $S(n/2) + 2 = P(n)$.

5. Hua's cyclic recurrence relations and its unitary version. As had been observed by Hua, the Hurwitz-Radon theorem and its symplectic version (both for the complex number field) are closely related. In fact, in the course of determining $P(n)$, Hua found the following remarkable relations among $P(n), S(n)$ and $R(n)$:

$$P(2n) = S(n) + 2,$$
$$S(2n) = R(n) + 2.$$

However, Hua missed the number $Q(n)$ and hence missed the following

$$R(2n) = Q(n) + 2,$$
$$Q(2n) = P(n) + 2.$$

These four identities form a remarkable closed cyclic recurrence relation, which will be called *Hua's cyclic recurrence relation.*[1]

THEOREM 5.1 (Hua's cyclic recurrence relation). *Suppose F is an algebraically closed field. Let $P(n)$ (for even n) and $R(n)$ denote the largest integer r for which there exist r matrices $A_1, \ldots, A_r$ in $Sp(n, F)$ and $O(n, F)$ satisfying the Hurwitz-Radon equations* (1.1) *respectively. Let $Q(n)$ (for even n) and $S(n)$ denote the largest integer s for which there exist s matrices $B_1, \ldots, B_s$ in $Sp(n, F)$ and $O(n, F)$ satisfying the dual Hurwitz-Radon equations* (1.2) *respectively. Then we have*

$$P(2n) = S(n) + 2, \tag{5.1}$$
$$S(2n) = R(n) + 2, \tag{5.2}$$
$$R(2n) = Q(n) + 2, \tag{5.3}$$
$$Q(2n) = P(n) + 2. \tag{5.4}$$

Moreover, we have the following initial values:

$$S(n_0) = 1, \qquad R(n_0) = 0, \qquad R(2n_0) = 1, \qquad Q(2n_0) = 1. \tag{5.5}$$

Note that as a corollary, $P(n), Q(n), R(n), S(n)$ enjoy together the following intrinsic *periodicity 8 property*:

$$\lambda(16n) = \lambda(n) + 8. \tag{5.6}$$

[1]In fact, Hua made a mistake which led him to a wrong relation $R(2n) = R(n) + 1$, cf. Hua [6, p. 221, equation (75)]. The mistake appeared on p. 223 of [6], the transformation given by (86) is not an orthogonal similarity. Wong [16] had also noticed this mistake.

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Hence, Theorem 5.1 provides a simple method to determine $P(n), Q(n), R(n), S(n)$ for an algebraically closed field. This is exactly the approach of Hua.

Remark 5.2. This approach of Hurwitz-Radon theorem via Hua's cyclic recurrence relation has two advantages. Firstly, it provides a much deeper interpretation for the "periodicity 8" property (5.6) met by the Hurwitz-Radon number $R(n)$ (at least for an algebraically closed field). Secondly, this proof also provides an effective way of constructing a set of Hurwitz-Radon matrices. We will leave the construction to the interested readers.

By Theorems 2.3, 2.4, 3.7 and 3.8, it is easy to check that Hua's cyclic recurrence relation (Theorem 5.1) is precisely valid for those fields such that $-1 = x^2 + y^2$ can be solved. Of course, this remarkable relation doesn't hold for the real number field. However, from a different point of view, another proper version of the cyclic recurrence relation remains true.

The key observation is

$$O(n, \mathbb{R}) = U(n) \cap O(n, \mathbb{C})$$

rather than

$$O(n, \mathbb{R}) = GL(n, \mathbb{R}) \cap O(n, \mathbb{C}).$$

In other words, it means that the point is not realization (restricting to real matrices) but unitarization (restricting to unitary matrices). As it will be shown, if we replace the complex orthogonal group $O(n, \mathbb{C})$ by the the unitary orthogonal group $U(n) \cap O(n, \mathbb{C}) = O(n)$ (which is the realization of $O(n, \mathbb{C})$ by coincidence) and replace the complex symplectic group $Sp(n, \mathbb{C})$ by the unitary symplectic group $U(n) \cap Sp(n, \mathbb{C}) =: USp(n)$[2] simultaneously, then we obtain the following unitary cyclic recurrence relation.

Theorem 5.3 (unitary cyclic recurrence relation). *Let $\widetilde{P}(n)$ (for even n) and $\widetilde{R}(n)$ denote the largest integer r for which there exist r matrices $A_1, \ldots, A_r$ in $USp(n)$ and $O(n)$ satisfying the Hurwitz-Radon equations* (1.1) *respectively. Let $\widetilde{Q}(n)$ (for even n) and $\widetilde{S}(n)$ denote the largest integer s for which there exist s matrices $B_1, \ldots, B_s$ in $USp(n)$ and $O(n)$ satisfying the dual Hurwitz-Radon equations* (1.2) *respectively. Then we have*

$$\widetilde{P}(2n) = \widetilde{S}(n) + 2, \tag{5.7}$$

$$\widetilde{S}(2n) = \widetilde{R}(n) + 2, \tag{5.8}$$

[2] Here, we follow Hermann Weyl's notations for the symplectic group and unitary symplectic group, cf. Weyl [14]. However, the most adopted notation for unitary symplectic group is $Sp(n)$, cf. Adams [1].

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$$\widetilde{R}(2n) = \widetilde{Q}(n) + 2, \tag{5.9}$$

$$\widetilde{Q}(2n) = \widetilde{P}(n) + 2. \tag{5.10}$$

Moreover, we have the following initial values:

$$\widetilde{S}(n_0) = 1, \qquad \widetilde{R}(n_0) = 0, \qquad \widetilde{R}(2n_0) = 1, \qquad \widetilde{Q}(2n_0) = 1. \tag{5.11}$$

To prove Theorem 5.3, we need some unitary versions of Lemma 3.2 and Lemma 3.3. Here we only provide a rough interpretation that why Hua's cyclic recurrence relation still holds with respect to the restriction of the corresponding unitary subgroup. In fact, it is a good illustration of the power of unitary trick introduced first by Weyl. The unitary trick is just unitary restriction: Every matrix group is replaced by the subgroup of those elements that are unitary transformations. As it had been pointed out by Weyl [14, p. 177], its success is due to the fact that *nothing of algebraic import is lost by unitary restriction.*

We point out that $\widetilde{\lambda}(n) = \lambda(n)$ since they satisfy the same recurrence relation and have the same initial values. As a special case, we have obtained the real Hurwitz-Radon theorem as a corollary. This approach is closely related with Radon's original proof but is much simpler than his argument (however Radon considered not four but eight matrix equations, cf. Radon [11]).

Remark 5.4. As pointed by Eckmann, the unitary cyclic recurrence relation were closely related to the famous Bott periodicity theorem in algebraic topology, cf. Eckmann [4] and [5].

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REFERENCES

[1] J.F. Adams. *Lectures on Lie Groups.* University of Chicago Press, 1969.
[2] J. Dieudonné. On a problem of Hurwitz and Newman. *Duke Mathematical Journal*, 20:381–393, 1953.
[3] B. Eckmann. Gruppentheoretischer Beweis des Satzes von Hurwitz-Radon über die Komposition der quadratischen Formen. *Commentarii Mathematici Helvetici*, 15:358–366, 1942.
[4] B. Eckmann. Hurwitz-Radon matrices and periodicity modulo 8. *L'Enseignement Mathmatique*, 35:77–91, 1985.
[5] B. Eckmann. Hurwitz-Radon matrices revisited: From effective solution of the Hurwitz matrix equations to Bott periodicity. The Hilton Symposium 1993: Topics in Topology and Group Theory, *CRM Proceedings and Lecture Notes*, 6:23–35, 1994.

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[6] L.-K. Hua. Geometries of Matrices II. Study of involutions in the geometry of symmetric matrices. *Transactions of the American Mathematical Society*, 61(2):193–228, 1947.

[7] A. Hurwitz. Über die Komposition der quadratischen Formen. *Mathematische Annalen*, 88(1):1–25, 1923.

[8] T.Y. Lam. *Introduction to Quadratic Forms over Fields.* Graduate Studies in Mathematics, Vol. 67, American Mathematical Society, Providence, 2005.

[9] K.-L. Lin. Hurwitz-Radon matrix equations (in Chinese). *Mathemedia*, 36:48–63, 2012.

[10] M.A.H. Newman. Note on an algebraic theorem of Eddington & Correction. *Journal of the London Mathematical Society*, 7:93–99 & 272, 1932.

[11] J. Radon. Lineare scharen orthogonaler matrizen. *Abhandlungen aus dem Mathematischen Seminar der Universitt Hamburg*, 1:1–14, 1922.

[12] D.B. Shapiro. Spaces of similarities I: The Hurwitz problem. *Journal of Algebra*, 46:148C170, 1977.

[13] D.B. Shapiro. *Compositions of Quadratic Forms.* De Gruyter Expositions in Mathematics, Vol. 33, Berlin, 2000.

[14] H. Weyl. *The Classical Groups, Their Invariants and Representations.* Princeton University Press, 1946.

[15] J. Williamson. Sets of semi-commutative matrices, part I and part II. *Proceedings of the Edinburgh Mathematical Society (Series 2)*, 3:179–188 and 3:231–240, 1933.

[16] Y.-C. Wong. Isoclinic n-Planes in $2n$-Spaces, Clifford Parallels in Elliptic $(2n-1)$-Spaces, and Hurwitz Matrix Equations. *Memoirs of the American Mathematical Society*, 41, 1961 (second printing with corrections and minor changes, 1971).