ELA

CHARACTERIZING LIE (ξ-LIE) DERIVATIONS ON TRIANGULAR ALGEBRAS BY LOCAL ACTIONS*

XIAOFEI QI†

Abstract. Let $\mathcal{U} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra, where $\mathcal{A}$, $\mathcal{B}$ are unital algebras over a field F and $\mathcal{M}$ is a faithful $(\mathcal{A}, \mathcal{B})$-bimodule. Assume that $\xi \in F$ and $L : \mathcal{U} \to \mathcal{U}$ is a map. It is shown that, under some mild conditions, L is linear and satisfies $L([X,Y]) = [L(X),Y] + [X,L(Y)]$ for any $X, Y \in \mathcal{U}$ with $[X,Y] = XY - YX = 0$ if and only if $L(X) = \varphi(X) + ZX + f(X)$ for all A, where φ is a linear derivation, Z is a central element and f is a central valued linear map. For the case $1 \neq \xi \in F$, L is additive and satisfies $L([X,Y]_\xi) = [L(X),Y]_\xi + [X,L(Y)]_\xi$ for any $X, Y \in \mathcal{U}$ with $[X,Y]_\xi = XY - \xi YX = 0$ if and only if $L(I)$ is in the center of $\mathcal{U}$ and $L(A) = \varphi(A) + L(I)A$ for all A, where φ is an additive derivation satisfying $\varphi(\xi A) = \xi\varphi(A)$ for each A. In addition, all additive maps L satisfying $L([X,Y]_\xi) = [L(X),Y]_\xi + [X,L(Y)]_\xi$ for any $X, Y \in \mathcal{U}$ with $XY = 0$ are also characterized.

Key words. Triangular algebras, Lie derivations, Derivations, ξ-Lie derivations, Nest algebras.

AMS subject classifications. 47B47, 16W25.

1. Introduction. Let $\mathcal{A}$ be an algebra over a field F. Recall that an additive (a linear) map $\delta : \mathcal{A} \to \mathcal{A}$ is called an additive (a linear) derivation if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathcal{A}$. Regarding $\mathcal{A}$ as a Lie algebra under the Lie product $[A,B] = AB - BA$, an additive (a linear) map $L : \mathcal{A} \to \mathcal{A}$ is called an additive (a linear) Lie derivation if $L([A,B]) = [L(A),B] + [A,L(B)]$ for all $A, B \in \mathcal{A}$. The problem of whether a Lie derivation is a derivation has been studied (for example, see [2, 13] and the references therein).

For $\xi \in F$ and for $A, B \in \mathcal{A}$, A commutes with B up to a factor ξ if $AB = \xi BA$. The notion of commutativity up to a factor for pairs of operators is important and has been studied in the context of operator algebras and quantum groups [4, 11]. Motivated by this, a binary operation $[A,B]_\xi = AB - \xi BA$, called the ξ-Lie product of A and B, was introduced in [15]. An additive (a linear) map $L : \mathcal{A} \to \mathcal{A}$ is called an additive (a linear) ξ-Lie derivation if $L([A,B]_\xi) = [L(A),B]_\xi + [A,L(B)]_\xi$ for all $A, B \in \mathcal{A}$. This concept unifies several well known notions. It is clear that a ξ-Lie derivation is a derivation if $\xi = 0$; is a Lie derivation if $\xi = 1$; is a Jordan derivation

ELA

if $\xi = -1$. The structure of ξ-Lie derivations was characterized in triangular algebras and prime algebras in [15, 18] respectively.

Recently, there have been a number of papers studying conditions under which derivations of rings or operator algebras can be completely determined by the action on some elements concerning products (see [3, 14] and the references therein). For Lie derivations, some works have also been done. Let X be a Banach space with $\dim X \geq 3$ and $\mathcal{B}(X)$ the algebra of all bounded linear operators acting on X. Lu and Jing in [12] proved that, if $L : \mathcal{B}(X) \to \mathcal{B}(X)$ is a linear map satisfying $L([A, B]) = [L(A), B] + [A, L(B)]$ for any $A, B \in \mathcal{B}(X)$ with $AB = 0$ (resp. $AB = P$, where P is a fixed nontrivial idempotent), then $L = d + \tau$, where d is a derivation of $\mathcal{B}(X)$ and τ is a central valued linear map vanishing at commutators $[A, B]$ with $AB = 0$ (resp. $AB = P$). Later, this result was generalized to the maps on triangular algebras and prime rings in [10] and [17] respectively. Then, a natural problem is how to characterize all additive (linear) maps L satisfying $L([A, B]_\xi) = [L(A), B]_\xi + [A, L(B)]_\xi$ for any A, B with $AB = 0$ and for all $\xi \neq 1$.

Qi and Hou [16] gave another characterization of Lie derivations. A linear (an additive) map $L : \mathcal{A} \to \mathcal{A}$ is said to be Lie derivable at a point Z if $L([A, B]) = [L(A), B] + [A, L(B)]$ for any $A, B \in \mathcal{A}$ with $[A, B] = Z$. Clearly, this definition is not valid for some Z, for instance, for $Z = I$, as the unit I may not be a commutator $[A, B]$ in general. Qi and Hou [16] discussed such linear maps on $\mathcal{J}$-subspace lattice algebras. Thus, more generally, what is the structure of the additive (linear) maps L that are ξ-Lie derivable at zero for all possible ξ, that is, satisfy $L([A, B]_\xi) = [L(A), B]_\xi + [A, L(B)]_\xi$ for any A, B with $[A, B]_\xi = 0$?

The purpose of the present paper is to study these questions on triangular algebras and characterize all such maps on triangular algebras.

Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over a commutative ring $\mathcal{R}$, and $\mathcal{M}$ be an $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module, that is, for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$, $A\mathcal{M} = \mathcal{M}B = \{0\}$ imply $A = 0$ and $B = 0$, respectively. The $\mathcal{R}$-algebra

$$\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \left[\begin{array}{cc} A & M \\ 0 & B \end{array} \right] : A \in \mathcal{A}, M \in \mathcal{M}, B \in \mathcal{B} \right\}$$

under the usual matrix operations will be called a triangular algebra (see [5]). Denote by $\mathcal{Z}(\mathcal{U})$ the center of $\mathcal{U}$. By [5, Proposition 3],

$$\mathcal{Z}(\mathcal{U}) = \left\{ \left[\begin{array}{cc} A & 0 \\ 0 & B \end{array} \right] \in \mathcal{U} : A \in \mathcal{Z}(\mathcal{A}), B \in \mathcal{Z}(\mathcal{B}), \text{ and } AM = MB \text{ for all } M \in \mathcal{M} \right\}.$$

$$\boxed{\textbf{ELA}}$$

Define two projections $\pi_{\mathcal{A}} : \mathcal{U} \to \mathcal{A}$ and $\pi_{\mathcal{B}} : \mathcal{U} \to \mathcal{B}$ respectively by

$$\pi_{\mathcal{A}} : \left[\begin{array}{cc} A & M \\ 0 & B \end{array} \right] \mapsto A \quad \text{and} \quad \pi_{\mathcal{B}} : \left[\begin{array}{cc} A & M \\ 0 & B \end{array} \right] \mapsto B.$$

Then $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{U})) \subseteq \mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{U})) \subseteq \mathcal{Z}(\mathcal{B})$. $\mathcal{M}$ is called loyal if, for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$, $A\mathcal{M}B = \{0\}$ implies that $A = 0$ or $B = 0$. It is obvious that, if $\mathcal{M}$ is loyal, then $\mathcal{M}$ is a faithful $(\mathcal{A}, \mathcal{B})$-bimodule.

This paper is organized as follows. Let $\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra, where $\mathcal{A}$ and $\mathcal{B}$ are unital algebras over a 2-torsion free commutative ring $\mathcal{R}$ with $\frac{1}{2} \in \mathcal{R}$, and $\mathcal{M}$ is a faithful $(\mathcal{A}, \mathcal{B})$-bimodule. Assume that $L : \mathcal{U} \to \mathcal{U}$ is a map. In Section 2, we show that, if each commuting linear map on $\mathcal{A}$ or $\mathcal{B}$ is proper, $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{A}) \neq \mathcal{A}$, $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{B}) \neq \mathcal{B}$ and $\mathcal{M}$ is loyal, then every linear map L satisfies $[X, Y] = 0 \Rightarrow [L(X), Y] + [X, L(Y)] = 0$ if and only if there exist an element $Z \in \mathcal{Z}(\mathcal{U})$, a linear derivation τ on $\mathcal{U}$, and a linear map $\nu : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ such that $L(X) = ZX + \tau(X) + \nu(X)$ for all $X \in \mathcal{U}$ (Theorem 2.1). Assume that $\mathcal{A}$ and $\mathcal{B}$ are unital algebras over a field F and $\xi \in F$. Sections 3 and 4 are devoted to discuss the case $\xi \neq 1$. We prove that, an additive map L satisfies $[X, Y]_{\xi} = 0 \Rightarrow [L(X), Y]_{\xi} + [X, L(Y)]_{\xi} = 0$ if and only if $L(I) \in \mathcal{Z}(\mathcal{U})$ and, (1) $\xi = 0$, there exists an additive derivation d such that $L(X) = d(X) + L(I)X$ for all $X \in \mathcal{U}$, (2) $\xi \neq 0$, there exists an additive derivation d satisfying $d(\xi X) = \xi d(X)$ for all $X \in \mathcal{U}$ such that $L(X) = d(X) + L(I)X$ for all $X \in \mathcal{U}$ (Theorem 3.1); L satisfies $XY = 0 \Rightarrow [L(X), Y]_{\xi} + [X, L(Y)]_{\xi} = L([X, Y]_{\xi})$ for $\xi \neq 0$ if and only if there exists an additive derivation d satisfying $d(\xi I) = \xi L(I)$ such that $L(X) = d(X) + L(I)X$ for all $X \in \mathcal{U}$ (Theorem 4.1). As an application of above results, we also obtain a characterization of linear (or additive) maps ξ-Lie derivable at zero for all possible ξ on Banach space nest algebras (Theorems 2.2, 2.3, 3.2 and 4.2).

2. Linear maps Lie derivable at zero. In this section, we consider linear maps Lie derivable at zero on triangular algebra. As an application to operator algebras, we get a complete characterization of linear maps Lie derivable at zero on Banach space nest algebras.

Recall that a linear map $\Phi : \mathcal{A} \to \mathcal{A}$ is commuting if $[\Phi(A), A] = 0$ for all $A \in \mathcal{A}$ and it is proper if $\Phi(A) = \lambda A + \varphi(A)$ for some element $\lambda \in \mathcal{Z}(\mathcal{A})$ (the center of $\mathcal{A}$) and some linear map $\varphi : \mathcal{A} \to \mathcal{Z}(\mathcal{A})$. A trace of a bilinear map is a map of the form $A \mapsto g(A, A)$, where $g : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is some bilinear map.

The following is our main result in this section.

THEOREM 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over a 2-torsion free commutative ring $\mathcal{R}$ with $\frac{1}{2} \in \mathcal{R}$, and $\mathcal{M}$ be an $(\mathcal{A}, \mathcal{B})$-bimodule. Let $\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be the triangular algebra satisfying the following three conditions:*

ELA

(i) $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{A}) \neq \mathcal{A}$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{B}) \neq \mathcal{B}$,

(ii) each commuting linear map on $\mathcal{A}$ or $\mathcal{B}$ is proper,

(iii) $\mathcal{M}$ is loyal.

Assume that $L : \mathcal{U} \to \mathcal{U}$ is a linear map. Then L is Lie derivable at zero, that is, L satisfies $[L(X), Y] + [X, L(Y)] = 0$ whenever $X, Y \in \mathcal{U}$ with $[X, Y] = 0$, if and only if $L(X) = ZX + \tau(X) + \nu(X)$ for all $X \in \mathcal{U}$, where $Z \in \mathcal{Z}(\mathcal{U})$, $\tau : \mathcal{U} \to \mathcal{U}$ is a linear derivation and $\nu : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ is a linear map.

Recall that a nest $\mathcal{N}$ on a Banach space X is a chain of closed (under norm topology) subspaces of X which is closed under the formation of arbitrary closed linear span (denote by $\bigvee$) and intersection (denote by $\bigwedge$), and which includes $\{0\}$ and X. The nest algebra associated to the nest $\mathcal{N}$, denoted by $\text{Alg}\mathcal{N}$, is the weak closed operator algebra consisting of all operators that leave $\mathcal{N}$ invariant. When $\mathcal{N} \neq \{\{0\}, X\}$, we say that $\mathcal{N}$ is non-trivial. If $\mathcal{N}$ is trivial, then $\text{Alg}\mathcal{N} = \mathcal{B}(X)$. We refer the reader to [6] for the theory of nest algebras.

As an application of Theorem 2.1 to the nest algebra case, we have

THEOREM 2.2. *Let X be an infinite dimensional Banach space over the real or complex field F and let $\mathcal{N}$ be a nest on X which contains a nontrivial complemented element. Then a linear map $L : \text{Alg}\mathcal{N} \to \text{Alg}\mathcal{N}$ is Lie derivable at zero if and only if there exist a scalar $\lambda \in F$, an operator $T \in \text{Alg}\mathcal{N}$ and a linear functional $h : \text{Alg}\mathcal{N} \to F$ such that $L(A) = AT - TA + \lambda A + h(A)I$ for all $A \in \text{Alg}\mathcal{N}$.*

Proof. By the assumption on the nest, there exists a non-trivial element $E \in \mathcal{N}$ which is complemented in X. So $\text{Alg}\mathcal{N}$ can be viewed as a triangular algebra

$$\text{Alg}\mathcal{N} = \begin{bmatrix} \text{Alg}(E\mathcal{N}E) & E\text{Alg}\mathcal{N}E^{\perp} \\ 0 & \text{Alg}(E^{\perp}\mathcal{N}E^{\perp}) \end{bmatrix},$$

where $E^{\perp} = I - E$. Note that $\mathcal{Z}(\text{Alg}\mathcal{N}) = FI$. It is obvious that $\pi_{\mathcal{A}}(\mathcal{Z}(\text{Alg}\mathcal{N})) = \mathcal{Z}(\text{Alg}(E\mathcal{N}E)) = FI_{\text{ran}E} \neq \text{Alg}(E\mathcal{N}E)$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\text{Alg}\mathcal{N})) = \mathcal{Z}(\text{Alg}(E^{\perp}\mathcal{N}E^{\perp})) = FI_{\text{ker}E} \neq \text{Alg}(E^{\perp}\mathcal{N}E^{\perp})$. Moreover, every linear commuting map on $\text{Alg}(E\mathcal{N}E)$ and $\text{Alg}(E^{\perp}\mathcal{N}E^{\perp})$ is proper (see [5]). We claim that $E\text{Alg}\mathcal{N}E^{\perp}$ is also loyal. Indeed, for nonzero operators $A \in \text{Alg}(E\mathcal{N}E) = E\text{Alg}\mathcal{N}E$ and $B \in \text{Alg}(E^{\perp}\mathcal{N}E^{\perp}) = E^{\perp}\text{Alg}\mathcal{N}E^{\perp}$, there exist $x \in E$ and $f \in E^{\perp}$ such that $Ax \neq 0$ and $Bf \neq 0$. Let $M = x \otimes Bf$. It is clear that $M \in E\text{Alg}\mathcal{N}E^{\perp}$. However, $AMBf = \|Bf\|^2 Ax \neq 0$, which implies that $E\text{Alg}\mathcal{N}E^{\perp}$ is loyal. Thus, this nest algebra meets all hypotheses of Theorem 2.1. Therefore, L is Lie derivable at zero if and only if there exist a scalar λ, a linear derivation $\tau : \text{Alg}\mathcal{N} \to \text{Alg}\mathcal{N}$ and a linear functional $h : \text{Alg}\mathcal{N} \to F$ such that $L(A) = \tau(A) + \lambda A + h(A)I$ for all $A \in \text{Alg}\mathcal{N}$.

Furthermore, note that every linear derivation of a nest algebra on a Banach space is continuous (see [9, Theorem 2.2]) and every continuous linear derivation of

ELA

a nest algebra on a Banach space is inner (see [20]). Hence, there exists an operator $T \in \text{Alg}\mathcal{N}$ such that $\tau(A) = AT - TA$ for all $A \in \text{Alg}\mathcal{N}$. □

For the finite dimensional case, it is clear that every nest algebra on a finite dimensional space is isomorphic to an upper triangular block matrix algebra. Let $\mathcal{M}_n(F)$ denote the algebra of all $n \times n$ matrices over F. Recall that an upper triangular block matrix algebra $\mathcal{T} = \mathcal{T}(n_1, n_2, \ldots, n_k)$ is a subalgebra of $\mathcal{M}_n(F)$ consisting of all $n \times n$ matrices of the form

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ 0 & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{kk} \end{bmatrix},$$

where $\{n_1, n_2, \cdots, n_k\}$ is a finite sequence of positive integers satisfying $n_1 + n_2 + \cdots + n_k = n$ and $A_{ij} \in \mathcal{M}_{n_i \times n_j}(F)$, the space of all $n_i \times n_j$ matrices over F.

By Theorem 2.1, the following result is immediate.

THEOREM 2.3. *Let F be the real or complex field and $n > 2$ be a positive integer. Let $\mathcal{T} = \mathcal{T}(n_1, n_2, \ldots, n_k) \subseteq \mathcal{M}_n(F)$ be an upper triangular block matrix algebra. Then a linear map $L : \mathcal{T} \to \mathcal{T}$ is Lie derivable at zero if and only if there exist a scalar λ, a matrix $T \in \mathcal{T}$ and a linear functional $h : \mathcal{T} \to F$ such that $L(A) = AT - TA + \lambda A + h(A)I$ for all $A \in \mathcal{T}$.*

We remark that, the condition $n > 2$ in Theorem 2.3 can not be deleted. In fact, we have the following result for 2×2 upper triangular matrix algebras.

PROPOSITION 2.4. *Let $\mathcal{T}_2(F)$ be the algebra of all 2×2 upper triangular matrices over a field F and let $L : \mathcal{T}_2(F) \to \mathcal{T}_2(F)$ be a linear map. Then L is Lie derivable at zero if and only if $L(I) = \lambda I$ for some $\lambda \in F$.*

Proof. We need only check the "if" part. For any $A \in \mathcal{T}_2(F)$, if $A = \alpha I \in FI$, then $[A, B] = 0$ for all $B \in \mathcal{T}_2(F)$. Since L is linear, we have

$$[L(A), B] + [A, L(B)] = [\alpha L(I), B] + [\alpha I, L(B)] = [\alpha \lambda I, B] + [\alpha I, L(B)] = 0.$$

Now assume that $A \notin FI$. Note that, if $[A, B] = AB - BA = 0$, then $B = \mu(B)A + \nu(B)I$ for some $\mu(B), \nu(B) \in F$. In fact, one can easily check that, for $A = (a_{ij}) \notin FI$ and $B = (b_{ij})$, $AB = BA$ implies that $B = b_{12}a_{12}^{-1}A + (b_{22} - b_{12}a_{12}^{-1}a_{22})I$ if $a_{12} \neq 0$; $B = \frac{b_{11} - b_{22}}{a_{11} - a_{22}}A + (b_{11} - \frac{b_{11} - b_{22}}{a_{11} - a_{22}}a_{11})I$ if $a_{12} = 0$. It follows that

$$\begin{aligned}[L(A), B] + [A, L(B)] &= [L(A), \mu(B)A + \nu(B)I] + [A, \mu(B)L(A) + \nu(B)\lambda I] \\ &= \mu(B)[L(A), A] + \mu(B)[A, L(A)] = 0.\end{aligned}$$

Thus, L is Lie derivable at zero. $\square$

Therefore, unlike the linear Lie derivations (every linear Lie derivation on $\mathcal{T}_2(F)$ has a standard form), the linear maps on $\mathcal{T}_2(F)$ Lie derivable at zero behave wildly and are not always of the form stated in Theorem 2.3. To illustrate this, we give a simple example here.

Let $\mathcal{T}_2(F)$ be the algebra of all 2×2 upper triangular matrices over the real or complex field F. We define a map $L : \mathcal{T}_2(F) \to \mathcal{T}_2(F)$ by $L(A) = \begin{pmatrix} 0 & a_{12} \\ 0 & a_{12} \end{pmatrix}$ for each $A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \in \mathcal{T}_2(F)$. We will check that L is a linear map Lie derivable at zero, but there do not exist $\lambda \in F$, $T \in \mathcal{T}_2(F)$ and linear functional f on $\mathcal{T}_2(F)$ such that $L(A) = TA - AT + \lambda A + f(A)I$ holds for all $A \in \mathcal{T}_2(F)$.

It is clear that L is linear and $L(I) = 0$. By Proposition 2.4, L is Lie derivable at zero. Suppose, on the contrary, that there exist $\lambda \in F$, an operator $T \in \mathcal{T}_2(F)$ and a linear functional f on $\mathcal{T}_2(F)$ such that $L(A) = TA - AT + \lambda A + f(A)I$ for all $A \in \mathcal{T}_2(F)$. Take $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. By the definition of L, we have $0 = L(A) = TA - AT + \lambda A + f(A)I$. Through a simple calculation, one can get $\lambda = 0$ and $T = \begin{pmatrix} t_{11} & 0 \\ 0 & t_{22} \end{pmatrix}$ for some $t_{11}, t_{22} \in F$.

Now for any $A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$, we have

$$\begin{pmatrix} 0 & a_{12} \\ 0 & a_{12} \end{pmatrix} = \begin{pmatrix} t_{11} & 0 \\ 0 & t_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} t_{11} & 0 \\ 0 & t_{22} \end{pmatrix}$$
$$+ \begin{pmatrix} f(A) & 0 \\ 0 & f(A) \end{pmatrix} = \begin{pmatrix} f(A) & (t_{11} - t_{22})a_{12} \\ 0 & f(A) \end{pmatrix}.$$

This leads to $f(A) = 0$ and $f(A) = a_{12}$, a contradiction.

Now we are at a position to give the proof of Theorem 2.1. The following two lemmas are needed.

LEMMA 2.5. ([1, Theorem 3.1]) *Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over a 2-torsion free commutative ring $\mathcal{R}$, and let $\mathcal{M}$ be a loyal $(\mathcal{A}, \mathcal{B})$-bimodule. Let $\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be the triangular algebra. If $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{A}) \neq \mathcal{A}$, $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{B}) \neq \mathcal{B}$ and each commuting linear map on $\mathcal{A}$ or $\mathcal{B}$ is proper, then each commuting trace $q : \mathcal{U} \to \mathcal{U}$ of a bilinear map has the form $q(A) = \lambda A^2 + \mu(A) + \nu(A)$ for all $A \in \mathcal{U}$, where $\lambda \in \mathcal{Z}(\mathcal{U})$, $\mu : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ is a linear map and $\nu : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ is a trace of some bilinear map.*

LEMMA 2.6. ([1, Lemma 2.7]) *Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over a 2-torsion*

ELA

free commutative ring $\mathcal{R}$, and $\mathcal{M}$ be a faithful $(\mathcal{A}, \mathcal{B})$-bimodule. Let $\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be the triangular algebra. Then $\mathcal{U}$ satisfies the polynomial identity $[[X^2, Y], [X, Y]]$ if and only if both $\mathcal{A}$ and $\mathcal{B}$ are commutative.

Proof of Theorem 2.1. The "if" part is obvious and we only need to show the "only if" part. We remark that our proof approach is similar to that of [19, Theorem 2.1(2)]. For completeness, we give an outline of its proof here.

By the assumption on L, we have

(2.1) $\qquad [L(X), Y] + [X, L(Y)] = 0 \quad$ for all $\ X, Y \in \mathcal{U} \ $ with $\ [X, Y] = 0$.

Let $Y = X^2$ in Eq.(2.1), one gets $[L(X), X^2] + [X, L(X^2)] = 0$, that is, $L(X)X^2 - X^2L(X) + XL(X^2) - L(X^2)X = 0$. This yields

(2.2) $\qquad\qquad [L(X^2) - L(X)X - XL(X), X] = 0 \quad$ for all $X \in \mathcal{U}$.

For any $X, Y \in \mathcal{U}$, let $\delta(X, Y) = L(XY) - L(X)Y - XL(Y)$. It is obvious that $\delta : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ is a bilinear map, and by Eq.(2.2), $\delta(X, X)$ is a trace of the bilinear map δ. Thus, by Lemma 2.5, there exist an element $X_0 \in \mathcal{Z}(\mathcal{U})$, a $\mathcal{R}$-linear map $\mu : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ and a trace of some bilinear map $\nu : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ such that $\delta(X, X) = X_0X^2 + \mu(X)X + \nu(X)$, that is,

(2.3) $\quad L(X^2) - L(X)X - XL(X) = X_0X^2 + \mu(X)X + \nu(X) \ $ for $\ $ all $\ X \in \mathcal{U}$.

Define a map $\tau : \mathcal{U} \to \mathcal{U}$ by

(2.4) $\qquad\qquad \tau(X) = L(X) + \frac{1}{2}\mu(X) + X_0X \ $ for $\ $ all $\ X \in \mathcal{U}$.

Combining Eq.(2.3) with Eq.(2.4), on the one hand, we have

$$\begin{aligned} \tau(X^2) &= \ L(X^2) + \tfrac{1}{2}\mu(X^2) + X_0X^2 \\ &= \ L(X)X + XL(X) + \mu(X)X + \nu(X) + \tfrac{1}{2}\mu(X^2) + 2X_0X^2. \end{aligned}$$

On the other hand,

$$\tau(X)X + X\tau(X) = L(X)X + XL(X) + \mu(X)X + 2X_0X^2.$$

Comparing the above two equations obtains

(2.5) $\qquad\qquad \tau(X^2) - \tau(X)X - X\tau(X) = \nu(X) + \frac{1}{2}\mu(X^2) \in \mathcal{Z}(\mathcal{U})$.

Now let us define a map $\epsilon : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ by

$$\epsilon(X, Y) = \tau(XY + YX) - \tau(X)Y - X\tau(Y) - \tau(Y)X - Y\tau(X) \text{ for all } \ X, Y \in \mathcal{U}.$$

ELA

Obviously, ϵ is bilinear and satisfies $\epsilon(X, Y) = \epsilon(Y, X)$ for all $X, Y \in \mathcal{U}$. Replacing X by $X + Y$ in Eq.(2.5), we have $\tau(XY + YX) - \tau(X)Y - X\tau(Y) - \tau(Y)X - Y\tau(X) \in \mathcal{Z}(\mathcal{U})$. Hence, ϵ in fact maps from $\mathcal{U} \times \mathcal{U}$ into $\mathcal{Z}(\mathcal{U})$. In order to show that τ is a Jordan derivation, one must prove that $\epsilon(X, Y) = 0$ for all $X, Y \in \mathcal{U}$. In fact, by calculating $\tau(X^2(XY + YX) + (XY + YX)X^2)$ and $\tau((X^2Y + YX^2)X + X(X^2Y + YX^2))$, and noting that $X^2(XY + YX) + (XY + YX)X^2 = (X^2Y + YX^2)X + X(X^2Y + YX^2)$, one can obtain

$$(2.6) \qquad \epsilon(X, Y)X^2 - \epsilon(X^2, Y)X \in \mathcal{Z}(\mathcal{U}) \quad \text{for all} \ \ X, Y \in \mathcal{U},$$

which implies that

$$(2.7) \qquad \epsilon(X, Y)[[X^2, W], [X, W]] = 0 \quad \text{for all} \ \ X, Y, W \in \mathcal{U}.$$

By Lemma 2.5, there exist two elements $X_1, W_1 \in \mathcal{U}$ such that

$$[[X_1^2, W_1], [X_1, W_1]] \neq 0.$$

Hence, we may take $X_1 = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $W_1 = \begin{pmatrix} A_2 & M \\ 0 & 0 \end{pmatrix}$ with $A_1[A_1, A_2]A_1 \neq 0$ in Eq.(2.7). Let $\epsilon(X_1, Y) = \begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix} \in \mathcal{Z}(\mathcal{U})$. Then $A' \in \mathcal{Z}(\mathcal{A})$, $B' \in \mathcal{Z}(\mathcal{B})$ and $A'M = MB'$ for all $M \in \mathcal{M}$. It follows from Eq.(2.7) that $A'A_1[A_1, A_2]A_1M = 0$ holds for all $M \in \mathcal{M}$. Since $\mathcal{M}$ is a faithful left $\mathcal{A}$-module, we get $A'A_1[A_1, A_2]A_1 = 0$, and so

$$0 = A'A_1[A_1, A_2]A_1M = A_1[A_1, A_2]A_1A'M = A_1[A_1, A_2]A_1MB'.$$

This implies that $B' = 0$ as $\mathcal{M}$ is loyal and $A_1[A_1, A_2]A_1 \neq 0$. Furthermore, $A' = 0$ and $\epsilon(X_1, Y) = 0$.

Replacing X by $Y + X_1$ and $Y - X_1$ in Eq.(2.6), respectively, one gets

$$\epsilon(Y, Y)X_1^2 + \epsilon(Y, Y)(X_1Y + YX_1) - \epsilon(X_1Y + YX_1, Y)X_1$$
$$-\epsilon(X_1Y + YX_1, Y)Y - \epsilon(Y^2, Y)X_1 - \epsilon(X_1^2, Y)Y \in \mathcal{Z}(\mathcal{U})$$

and

$$\epsilon(Y, Y)X_1^2 - \epsilon(Y, Y)(X_1Y + YX_1) - \epsilon(X_1Y + YX_1, Y)X_1$$
$$+\epsilon(X_1Y + YX_1, Y)Y + \epsilon(Y^2, Y)X_1 - \epsilon(X_1^2, Y)Y \in \mathcal{Z}(\mathcal{U}).$$

Comparing the above two equations achieves

$$(2.8) \ \ \epsilon(Y, Y)X_1^2 - \epsilon(X_1Y + YX_1, Y)X_1 - \epsilon(X_1^2, Y)Y \in \mathcal{Z}(\mathcal{U}) \quad \text{for all} \ \ Y \in \mathcal{U}.$$

$$\boxed{\textbf{ELA}}$$

824 X.F. Qi

We claim that $\epsilon(X_1^2, Y) = 0$. Indeed, since $\epsilon(X_1, Y) = 0$, we have $\epsilon(X_1^2, Y)X_1 \in \mathcal{Z}(\mathcal{U})$. It follows that $\epsilon(X_1^2, Y) = 0$ since $X_1 \notin \mathcal{Z}(\mathcal{U})$. Thus, Eq.(2.8) yields $\epsilon(Y, Y)X_1^2 - \epsilon(X_1Y + YX_1, Y)X_1 \in \mathcal{Z}(\mathcal{U})$ for all $Y \in \mathcal{U}$. It follows that

$$\epsilon(Y, Y)[[X_1^2, W_1], [X_1, W_1]] = 0,$$

which implies $\epsilon(Y, Y) = 0$ for all $Y \in \mathcal{U}$. Since ϵ is symmetric, it is easy to prove that $\epsilon(X, Y) = 0$ for all $X, Y \in \mathcal{U}$, that is, τ is a Jordan derivation. Now, by [21], τ is a derivation. Using the definition of τ, we get that $L(A) = \tau(A) + ZA + \nu(A)$, where $\nu = -\frac{1}{2}\mu(A)$ and $Z = -X_0$. □

In fact, by observing the proof of Theorem 2.1, one can easily obtain characterizations of the linear maps Lie triple derivable at zero and Lie triple derivations on Banach space nest algebras.

Recall that a linear map L on an algebra $\mathcal{A}$ is called a Lie triple derivation if $[[L(A), B], C] + [[A, L(B)], C] + [[A, B], L(C)] = L([[A, B], C])$ for all $A, B, C \in \mathcal{A}$; is called Lie triple derivable at some $Z \in \mathcal{A}$ if $[[L(A), B], C] + [[A, L(B)], C] + [[A, B], L(C)] = 0$ for any $A, B, C \in \mathcal{A}$ with $[[A, B], C] = 0$.

THEOREM 2.7. *Let X be a Banach space with $\dim X \geq 3$ and let $\mathcal{N}$ be a nest on X containing a nontrivial complemented element. Assume that $L : \mathrm{Alg}\mathcal{N} \to \mathrm{Alg}\mathcal{N}$ is a linear map. Then L is Lie triple derivable at zero point if and only if there exist a scalar λ, an operator $T \in \mathrm{Alg}\mathcal{N}$ and a linear functional $h : \mathrm{Alg}\mathcal{N} \to F$ such that $L(A) = AT - TA + \lambda A + h(A)I$ for all $A \in \mathrm{Alg}\mathcal{N}$.*

Proof. Obviously, we only need to check the "only if" part. Since L is Lie triple derivable at zero point, for any $A, B, C \in \mathrm{Alg}\mathcal{N}$ with $[[A, B], C] = 0$, we have

(2.9) $$[[L(A), B], C] + [[A, L(B)], C] + [[A, B], L(C)] = 0.$$

Let $B = A^2$ in Eq.(2.9), one gets $[[L(A), A^2], C] + [[A, L(A^2)], C] = 0$, that is,

$$[L(A)A^2 - A^2L(A) + AL(A^2) - L(A^2)A, C] = 0 \quad \text{for all } C \in \mathrm{Alg}\mathcal{N}.$$

This yields $(L(A^2) - L(A)A - AL(A))A - A(L(A^2) - L(A)A - AL(A)) \in FI$, that is, $[L(A^2) - L(A)A - AL(A), A] = \lambda I$ for some scalar λ. Note that I can not be a commutator. It follows that

$$[L(A^2) - L(A)A - AL(A), A] = 0 \quad \text{holds for all } A \in \mathrm{Alg}\mathcal{N}.$$

Now by a similar argument to that of Theorem 2.1, one can obtain the desired result. □

THEOREM 2.8. *Let X be a Banach space over the real or complex field F with $\dim X \geq 3$ and let $\mathcal{N}$ be a nest on X containing a nontrivial complemented element.*

$$\boxed{\text{ELA}}$$

Assume that $L : \mathrm{Alg}\mathcal{N} \to \mathrm{Alg}\mathcal{N}$ is a linear map. Then L is a Lie triple derivation if and only if there exist an operator $T \in \mathrm{Alg}\mathcal{N}$ and a linear functional h of $\mathrm{Alg}\mathcal{N}$ satisfying $h([[A,B],C]) = 0$ for any $A, B, C \in \mathrm{Alg}\mathcal{N}$ such that $L(A) = AT - TA + h(A)I$ for all $A \in \mathrm{Alg}\mathcal{N}$.

Proof. Still, we only need to check the "only if" part. If L is a Lie triple derivation, then L is Lie triple derivable at zero. By Theorem 2.7, there exists a scalar λ, an operator $T \in \mathrm{Alg}\mathcal{N}$ and a linear functional h of $\mathrm{Alg}\mathcal{N}$ such that $L(A) = AT - TA + \lambda A + h(A)I$ for all $A \in \mathrm{Alg}\mathcal{N}$. For completing the proof, we have to check that $\lambda = 0$ and $h([[A,B],C]) = 0$ for all $A, B, C \in \mathrm{Alg}\mathcal{N}$. Note that derivations are Lie triple derivations. By the definition of Lie triple derivation, it follows from the above relation that $2\lambda[[A,B],C] = h([[A,B],C]) \in FI$. If $\lambda \neq 0$, then we have $[[A,B],C] \in FI$ for all $A, B, C \in \mathrm{Alg}\mathcal{N}$. Again, using the fact that I can not be a commutator, we get $[A,B] = 0$ for all $A, B \in \mathrm{Alg}\mathcal{N}$, which is impossible. Thus, $\lambda = 0$, and so $h([[A,B],C]) = 0$ for all $A, B, C \in \mathrm{Alg}\mathcal{N}$. $\square$

3. Additive maps ξ-Lie derivable at zero. In this section, we consider additive maps ξ-Lie derivable at zero on triangular algebras for the case $\xi \neq 1$. The following is our main result in this section.

THEOREM 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over a field F, and $\mathcal{M}$ be an $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. Let $\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be the triangular algebra and $1 \neq \xi \in F$. Assume that $L : \mathcal{U} \to \mathcal{U}$ is an additive map. If $\mathcal{U}$ satisfies $P\mathcal{Z}(\mathcal{U})P = \mathcal{Z}(P\mathcal{U}P)$ and $Q\mathcal{Z}(\mathcal{U})Q = \mathcal{Z}(Q\mathcal{U}Q)$, then L is ξ-Lie derivable at zero, that is, L satisfies $[L(X), Y]_\xi + [X, L(Y)]_\xi = 0$ for any $X, Y \in \mathcal{U}$ with $[X,Y]_\xi = 0$ if and only if $L(I) \in \mathcal{Z}(\mathcal{U})$ and one of the following statements holds:*

(1) if $\xi = 0$, then there exists an additive derivation $d : \mathcal{U} \to \mathcal{U}$ such that $L(X) = d(X) + L(I)X$ for all $X \in \mathcal{U}$, that is, L is an additive generalized derivation;

(2) if $\xi \neq 0$, then there exists an additive derivation $d : \mathcal{U} \to \mathcal{U}$ satisfying $d(\xi X) = \xi d(X)$ for each $X \in \mathcal{U}$ such that $L(X) = d(X) + L(I)X$ for all $X \in \mathcal{U}$.

For additive maps ξ-Lie derivable at zero point on infinite dimensional Banach space nest algebras, we have the following finer characterization.

THEOREM 3.2. *Let $\mathcal{N}$ be a nest on an infinite dimensional Banach space X over the real or complex field F, and let $\mathrm{Alg}\mathcal{N}$ be the associated nest algebra. Assume that $\xi \in F$ with $\xi \neq 1$ and $L : \mathrm{Alg}\mathcal{N} \to \mathrm{Alg}\mathcal{N}$ is an additive map. If there exists a non-trivial element in $\mathcal{N}$ which is complemented in X, then L is ξ-Lie derivable at zero if and only if there exist a scalar $\lambda \in F$ and an operator $T \in \mathrm{Alg}\mathcal{N}$ such that $L(A) = AT - TA + \lambda A$ for all $A \in \mathrm{Alg}\mathcal{N}$.*

Proof. Obviously, we need to check the "only if" part. Assume that $L : \mathrm{Alg}\mathcal{N} \to$

$$\boxed{\textbf{ELA}}$$

$\mathrm{Alg}\mathcal{N}$ is an additive map ξ-Lie derivable at zero point. By Theorem 3.1, $L(I) = \lambda I \in \mathcal{Z}(\mathrm{Alg}\mathcal{N}) = FI$ and there exists an additive derivation d such that $L(A) = d(A) + \lambda A$ for all $A \in \mathrm{Alg}\mathcal{N}$, where d satisfies $d(\xi X) = \xi d(X)$ whenever $\xi \neq 0$. Note that, by [7, 8], all additive derivations on infinite dimensional Banach space nest algebras are linear. Hence, by [9, 20], there exists $T \in \mathrm{Alg}\mathcal{N}$ such that $d(A) = AT - TA$ for all $A \in \mathrm{Alg}\mathcal{N}$. ☐

To prove Theorem 3.1, the following lemma is needed, which was proved in [15].

LEMMA 3.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras over any commutative ring $\mathcal{R}$, and $\mathcal{M}$ be an $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. Let $\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be the triangular algebra. Assume that $A_0 \in \mathcal{A}$ and $B_0 \in \mathcal{B}$. If $A_0 M = M B_0$ for all $M \in \mathcal{M}$, then $A_0 \in \mathcal{Z}(\mathcal{A})$ and $B_0 \in \mathcal{Z}(\mathcal{B})$. Furthermore, $\begin{bmatrix} A_0 & 0 \\ 0 & B_0 \end{bmatrix} \in \mathcal{Z}(\mathcal{U})$.*

Proof of Theorem 3.1. Denote by I, $I_\mathcal{A}$ and $I_\mathcal{B}$ the units of $\mathcal{U}$, $\mathcal{A}$ and $\mathcal{B}$, respectively. Write $P = \begin{bmatrix} I_\mathcal{A} & 0 \\ 0 & 0 \end{bmatrix}$ and $Q = I - P = \begin{bmatrix} 0 & 0 \\ 0 & I_\mathcal{B} \end{bmatrix}$. Then $\mathcal{U} = PUP + PUQ + QUQ$.

The "if" part is obvious. We will prove the "only if" part by checking several claims.

Claim 1. *For any $X \in \mathcal{U}$, we have $QL(PXP)Q = 0$, $PL(QXQ)P = 0$, $PL(PXP)Q = -PXPL(Q)Q$ and $PL(QXQ)Q = -PL(P)QXQ$.*

Take any $X, Y \in \mathcal{U}$. Since $[PXP, QYQ]_\xi = 0$, we have

$$[L(PXP), QYQ]_\xi + [PXP, L(QYQ)]_\xi = 0,$$

that is,

$$PL(PXP)QYQ + QL(PXP)QYQ - \xi QYQL(PXP)Q$$
$$+ PXPL(QYQ)P + PXPL(QYQ)Q - \xi PL(QYQ)PXP = 0.$$

It follows that

(3.1) $$PXPL(QYQ)P - \xi PL(QYQ)PXP = 0,$$

(3.2) $$PL(PXP)QYQ + PXPL(QYQ)Q = 0$$

and

(3.3) $$QL(PXP)QYQ - \xi QYQL(PXP)Q = 0.$$

$$\boxed{\textbf{ELA}}$$

Let $X = P$ and $Y = Q$ in Eqs.(3.1) and (3.3), respectively, and noting that $\xi \neq 1$, one gets $PL(QYQ)P = 0$ and $QL(PXP)Q = 0$; let $X = P$ and $Y = Q$ in Eq.(3.2), respectively, one gets $PL(QYQ)Q = -PL(P)QYQ$ and $PL(PXP)Q = -PXPL(Q)Q$. The claim holds.

By Claim 1, for P and Q, we have

$$(3.4) \qquad QL(P)Q = 0, \quad PL(Q)P = 0 \quad \text{and} \quad PL(Q)Q = -PL(P)Q.$$

Claim 2. *For any $X \in \mathcal{U}$, $L(PXQ) \in PUQ$. Moreover, if $\xi \neq 0$, we have $\xi PL(P)PXQ = PXQL(\xi Q)Q$ and $PL(\xi P)PXQ = \xi PXQL(Q)Q$.*

For any $X \in \mathcal{U}$, since $[PXQ, P + \xi Q]_\xi = 0$, by the definition of L and Claim 1, we have

$$\begin{aligned} 0 = & \; [L(PXQ), P + \xi Q]_\xi + [PXQ, L(P + \xi Q)]_\xi \\ = & \; PL(PXQ)P + \xi QL(PXQ)Q - \xi PL(PXQ)P \\ & -\xi^2 QL(PXQ)Q + PXQL(\xi Q)Q - \xi PL(P)PXQ, \end{aligned}$$

which implies

$$PXQL(\xi Q)Q = \xi PL(P)PXQ$$

and

$$(3.5) \qquad\qquad PL(PXQ)P = \xi PL(PXQ)P.$$

Note that $\xi \neq 1$ and F is a field. Eq.(3.5) yields

$$(3.6) \qquad\qquad PL(PXQ)P = 0 \quad \text{for all} \quad X \in \mathcal{U}.$$

Similarly, by using of the equation $[\xi P + Q, PXQ]_\xi = 0$, one can get $[L(\xi P + Q), PXQ]_\xi + [\xi P + Q, L(PXQ)]_\xi = 0$. It follows from Claim 1 and Eq.(3.6) that

$$PL(\xi P)PXQ = \xi PXQL(Q)Q$$

and

$$(3.7) \qquad\qquad QL(PXQ)Q = 0.$$

Combining Eqs.(3.6) and (3.7) obtains $L(PXQ) \in PUQ$.

Claim 3. *Assume that $\xi \neq 0$. Then $PL(\xi P)P = \xi PL(P)P$; and for any $X, Y \in \mathcal{U}$, we have $PL(PXPYP)P = PL(PXP)PYP + PXPL(PYP)P - PL(P)PXPYP$.*

For any $X, Z \in \mathcal{U}$, since $[PXP + PXPZQ, PZQ - Q]_\xi = 0$, we have

$$[L(PXP + PXPZQ), PZQ - Q]_\xi + [PXP + PXPZQ, L(PZQ - Q)]_\xi = 0.$$

ELA

By Claims 1-2, the above equation can be reduced to

$$(3.8) \quad PL(PXPZQ)Q = PL(PXP)PZQ + PXPL(PZQ)Q - PXPZQL(Q)Q,$$

that is,

$$(3.9) \quad \begin{aligned} &PL(PXP)PZQ \\ =\ & PL(PXPZQ)Q - PXPL(PZQ)Q + PXPZQL(Q)Q \\ =\ & PL(PXPZQ)Q - PXPL(PZQ)Q + \xi^{-1}PL(\xi P)PXPZQ \end{aligned}$$

for all $X, Z \in \mathcal{U}$. Here ξ^{-1} exists as $\xi \neq 0$ and F is a field. Now, taking any $Y \in \mathcal{U}$, by Eqs.(3.8)-(3.9) and Claim 2, we get

$$\begin{aligned} &PL(PXPYP)PZQ \\ =\ & PL(PXPYPZQ)Q - PXPYPL(PZQ)Q + \xi^{-1}PL(\xi P)PXPYPZQ \\ =\ & PL(PXP)PYPZQ + PXPL(PYPZQ)Q - \xi^{-1}PL(\xi P)PXPYPZQ \\ & -PXPYPL(PZQ)Q + \xi^{-1}PL(\xi P)PXPYPZQ \\ =\ & PL(PXP)PYPZQ + PXPL(PYPZQ)Q - PXPYPL(PZQ)Q \\ =\ & PL(PXP)PYPZQ + PXP(PL(PYPZQ)Q - PYPL(PZQ)Q) \\ =\ & PL(PXP)PYPZQ + PXP(PL(PYP)PZQ - PYPZQL(Q)Q) \\ =\ & PL(PXP)PYPZQ + PXPL(PYP)PZQ - PXPYPZQL(Q)Q \\ =\ & PL(PXP)PYPZQ + PXPL(PYP)PZQ - \xi^{-1}PL(\xi P)PXPYPZQ. \end{aligned}$$

That is,

$$\begin{aligned} &PL(PXPYP)PZQ \\ =\ & \{PL(PXP)PYP + PXPL(PYP)P - \xi^{-1}PL(\xi P)PXPYP\}PZQ \end{aligned}$$

holds for all $X, Y, Z \in \mathcal{U}$. Note that $\mathcal{M}$ is a faithful left $\mathcal{A}$-module. It follows that

$$(3.10) \quad \begin{aligned} &PL(PXPYP)P \\ =\ & PL(PXP)PYP + PXPL(PYP)P - \xi^{-1}PL(\xi P)PXPYP \end{aligned}$$

holds for all $X, Y \in \mathcal{U}$. Taking $X = \xi P$ and $Y = P$ in Eq.(3.10), one obtains $PL(\xi P)P = PL(\xi P)P + \xi PL(P)P - PL(\xi P)P$, which implies

$$\xi PL(P)P = PL(\xi P)P.$$

This and Eq.(3.10) imply

$$PL(PXPYP)P = PL(PXP)PYP + PXPL(PYP)P - PL(P)PXPYP$$

for all $X, Y \in \mathcal{U}$. The claim is true.

Claim 4. *Assume that $\xi \neq 0$. Then $QL(\xi Q)Q = \xi QL(Q)Q$; and for any $X, Y \in \mathcal{U}$, we have $QL(QXQYQ)Q = QL(QXQ)QYQ + QXQL(QYQ)Q - QXQYQL(Q)Q$.*

ELA

Take any $X, Y, Z \in \mathcal{U}$ and let $\xi \neq 0$. Note that $[P+PZQ, PZQXQ-QXQ]_\xi = 0$. By a similar argument to that of Claim 3, one can check that

$$(3.11) \qquad \begin{aligned} &PL(PZQXQ)Q \\ =\ &PL(PZQ)QXQ + PZQL(QXQ)Q - PL(P)PZQXQ, \end{aligned}$$

$QL(\xi Q)Q = \xi QL(Q)Q$ and

$$QL(QXQYQ)Q = QL(QXQ)QYQ + QXQL(QYQ)Q - QXQYQL(Q)Q$$

hold for all $X, Y, Z \in \mathcal{U}$. The claim is true.

Now combining Claims 2-3 (or Claims 2 and 4), we have proved that, if $\xi \neq 0$, then

$$(3.12) \qquad PL(P)PXQ = PXQL(Q)Q \quad \text{for all} \quad X \in \mathcal{U}.$$

Claim 5. *If $\xi = 0$, then $PL(P)PXQ = PXQL(Q)Q$ for all $X \in \mathcal{U}$.*

For any $X \in \mathcal{U}$, since $(P+PXQ)(Q-PXQ) = 0$, we have $(L(P)+L(PXQ))(Q-PXQ) + (P+PXQ)(L(Q)-L(PXQ)) = 0$. By Claims 1-2, one can easily check that $PL(P)PXQ = PXQL(Q)Q$ for each $X \in \mathcal{U}$.

By Claim 5 and Eq.(3.12), one has proved that, for any $\xi \neq 1$, we have

$$PL(P)PXQ = PXQL(Q)Q \quad \text{for all} \quad X \in \mathcal{U}.$$

It follows from Lemma 3.3 that $PL(P)P \in \mathcal{Z}(P\mathcal{U}P) = P\mathcal{Z}(\mathcal{U})P$, $QL(Q)Q \in \mathcal{Z}(Q\mathcal{U}Q) = Q\mathcal{Z}(\mathcal{U})Q$ and $PL(P)P + QL(Q)Q \in \mathcal{Z}(\mathcal{U})$. Note that $QL(P)Q = PL(Q)P = 0$ (Eq.(3.4)). We get

$$(3.13) \qquad L(I) = PL(P)P + QL(Q)Q \in \mathcal{Z}(\mathcal{U}).$$

Claim 6. *If $\xi = 0$, then for any $X, Y \in \mathcal{U}$, the following equations hold:*

(i) $PL(PXPZQ)Q = PL(PXP)PZQ + PXPL(PZQ)Q - PXPZQL(Q)Q$.
(ii) $PL(PXPYP)P = PL(PXP)PYP + PXPL(PYP)P - PL(P)PXPYP$.
(iii) $PL(PZQXQ)Q = PL(PZQ)QXQ + PZQL(QXQ)Q - PL(P)PZQXQ$.
(iv) $QL(QXQYQ)Q = QL(QXQ)QYQ + QXQL(QYQ)Q - QXQYQL(Q)Q$.

In fact, noting that $(PXP+PXPZQ)(PZQ-Q) = 0$ and $(P+PZQ)(PZQXQ-QXQ) = 0$ for all $X, Z \in \mathcal{U}$, the proof is similar to that of Claims 3-4. So we omit it here.

Claim 7. *There exists an additive derivation $d : \mathcal{U} \to \mathcal{U}$ such that $L(X) = d(X) + L(I)X$ for all $X \in \mathcal{U}$.*

$$\boxed{\textbf{ELA}}$$

830   X.F. Qi

We first show that, for any $X, Y \in \mathcal{U}$, $L(XY) = L(X)Y + XL(Y) - L(I)XY$, that is, L is a generalized derivation. In fact, take any $X = PXP + PXQ + QXQ$, $Y = PYP + PYQ + QYQ \in \mathcal{U}$. Then by the additivity of L, Claims 1-6 and Eqs.(3.8), (3.11), (3.13), one obtains

$$
\begin{aligned}
&L(XY)\\
=\ &L(PXPYP) + L(PXPYQ) + L(PXQYQ) + L(QXQYQ)\\
=\ &PL(PXPYP)P + PL(PXPYP)Q + PL(PXPYQ)Q\\
&+PL(PXQYQ)Q + PL(QXQYQ)Q + QL(QXQYQ)Q\\
=\ &PL(PXP)PYP + PXPL(PYP)P - PL(P)PXPYP - PXPYPL(Q)Q\\
&+PL(PXP)PYQ + PXPL(PYQ)Q - PXPYQL(Q)Q\\
&+PL(PXQ)QYQ + PXQL(QYQ)Q - PL(P)PXQYQ - PL(P)QXQYQ\\
&+QL(QXQ)QYQ + QXQL(QYQ)Q - QXQYQL(Q)Q\\
=\ &\{PL(PXP)PYP + PL(PXP)PYQ + PL(PXP)QYQ\\
&+PL(PXQ)QYQ + PL(QXQ)QYQ + QL(QXQ)QYQ\}\\
&+\{PXPL(PYP)P + PXPL(PYP)Q + PXPL(PYQ)Q\\
&+PXPL(QYQ)Q + PXQL(QYQ)Q + QXQL(QYQ)Q\}\\
&-\{PL(P)PXPYP + PXPYPL(Q)Q + PXPYQL(Q)Q\\
&+PL(P)PXQYQ + PL(P)QXQYQ + QXQYQL(Q)Q\}\\
&-\{PL(PXP)QYQ + PL(QXQ)QYQ + PXPL(PYP)Q + PXPL(QYQ)Q\}\\
=\ &L(X)Y + XL(Y) - \{PL(P)PXPYP + PXPYPL(Q)Q\\
&+PL(P)PXPYQ + PL(P)PXQYQ + PL(P)QXQYQ + QL(Q)QXQYQ\}\\
&+\{PXPL(Q)QYQ + PL(P)QXQYQ + PXPYPL(Q)Q + PXPL(P)QYQ\}\\
=\ &L(X)Y + XL(Y) - PL(P)P(PXPYP + PXPYQ + PXQYQ)\\
&-QL(Q)QXQYQ + PXP(PL(Q)Q + PL(P)Q)QYQ\\
=\ &L(X)Y + XL(Y)\\
&-(PL(P)P + QL(Q)Q)(PXPYP + PXPYQ + PXQYQ + QXQYQ)\\
=\ &L(X)Y + XL(Y) - L(I)XY.
\end{aligned}
$$

Now define a map $d : \mathcal{U} \to \mathcal{U}$ by $d(X) = L(X) - L(I)X$ for all $X \in \mathcal{U}$. Obviously, d is additive. Moreover, since $L(I) \in \mathcal{Z}(\mathcal{U})$ (Eq.(3.13)), it is easy to check that d is a derivation, that is, d satisfies $d(XY) = d(X)Y + Xd(Y)$ for all $X, Y \in \mathcal{U}$. Hence, there exists an additive derivation $d : \mathcal{U} \to \mathcal{U}$ such that $L(X) = d(X) + L(I)X$ for all $X \in \mathcal{U}$.

Claim 8. *If $\xi \neq 0$, then $d(\xi X) = \xi d(X)$ for all $X \in \mathcal{U}$. Therefore, the theorem is true.*

We first show $d(\xi I) = 0$. In fact, by Claim 1, we have

$$(3.14) \qquad QL(\xi P)Q = PL(\xi Q)P = 0 \quad \text{and} \quad PL(\xi P)Q + PL(\xi Q)Q = 0.$$

By Claims 3-4, one gets $PL(\xi P)P = \xi PL(P)P$ and $QL(\xi Q)Q = \xi QL(Q)Q$. Com-

bining these two equations with Eqs.(3.13)-(3.14), one obtains

$$(3.15) \qquad \xi L(I) = \xi QL(Q)Q + \xi PL(P)P = QL(\xi Q)Q + P(\xi P)P = L(\xi I).$$

In addition, by Claim 7, we have $L(\xi I) = d(\xi I) + L(I)(\xi I) = d(\xi I) + \xi L(I)$. This equation and Eq.(3.15) yield $d(\xi I) = 0$.

Now, since d is a derivation, for any $X \in \mathcal{U}$, we have $d(\xi X) = d(\xi I)X + (\xi I)d(X) = \xi d(X)$. The claim holds.

The proof of the theorem is complete. ☐

4. Characterizing additive ξ-Lie derivations by acting on zero products.
In this section, we give another characterization of ξ-Lie derivations on triangular algebras by acting on zero products. Since the cases $\xi = 1$ and $\xi = 0$ are considered in [10] and Theorem 3.1 respectively, we only deal with the case $\xi \neq 0, 1$. Our main result is the following.

THEOREM 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over a field F, and $\mathcal{M}$ be an $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. Let $\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be the triangular algebra and $\xi \in F$ with $\xi \neq 0, 1$. Assume that $L : \mathcal{U} \to \mathcal{U}$ is an additive map. If $\mathcal{U}$ satisfies $P\mathcal{Z}(\mathcal{U})P = \mathcal{Z}(P\mathcal{U}P)$ and $Q\mathcal{Z}(\mathcal{U})Q = \mathcal{Z}(Q\mathcal{U}Q)$, then L satisfies $[L(X), Y]_\xi + [X, L(Y)]_\xi = L([X, Y]_\xi)$ for any $X, Y \in \mathcal{U}$ with $XY = 0$ if and only if $L(I) \in \mathcal{Z}(\mathcal{U})$ and there exists an additive derivation $d : \mathcal{U} \to \mathcal{U}$ with $d(\xi I) = \xi L(I)$ such that $L(X) = d(X) + L(I)X$ for all $X \in \mathcal{U}$.*

Applying Theorem 4.1 to Banach space nest algebras case and noting that all additive derivations on infinite dimensional Banach space nest algebras are inner, we have the following result.

THEOREM 4.2. *Let $\mathcal{N}$ be a nest on an infinite dimensional Banach space X over the real or complex field F, and let $\mathrm{Alg}\mathcal{N}$ be the associated nest algebra. Assume that $\xi \in F$ with $\xi \neq 0, 1$ and $L : \mathrm{Alg}\mathcal{N} \to \mathrm{Alg}\mathcal{N}$ is an additive map. If there exists a non-trivial element in $\mathcal{N}$ which is complemented in X, then L satisfies $[L(A), B]_\xi + [A, L(B)]_\xi = L([A, B]_\xi)$ for any $A, B \in \mathrm{Alg}\mathcal{N}$ with $AB = 0$ if and only if there exists an operator $T \in \mathrm{Alg}\mathcal{N}$ such that $L(A) = AT - TA$ for all $A \in \mathrm{Alg}\mathcal{N}$.*

Proof of Theorem 4.1. We use the same symbol as that in Theorem 3.1. Still, we only need to prove the "only if" part.

Claim 1. *For any $X \in \mathcal{U}$, we have $QL(PXP)Q = 0$, $PL(QXQ)P = 0$, $PL(PXP)Q = -PXPL(Q)Q$ and $PL(QXQ)Q = -PL(P)QXQ$.*

For any $X, Y \in \mathcal{U}$, since $PXPQYQ = 0$, we have

$$[L(PXP), QYQ]_\xi + [PXP, L(QYQ)]_\xi = 0.$$

ELA

832   X.F. Qi

Now, by the same argument as that of Claim 1 in Theorem 3.1, one can check that
the claim is true.

By Claim 1, for P and Q, we have

(4.1) $QL(P)Q = 0,\ PL(Q)P = 0$ and $PL(Q)Q = -PL(P)Q.$

Claim 2. *For any $X \in \mathcal{U}$, we have $L(PXQ) \in P\mathcal{U}Q$ and $PL(P)PXQ = PXQL(Q)Q$.*

Taking any $X \in \mathcal{U}$, since $PXQP = 0$, we get $L([PXQ, P]_\xi) = [L(PXQ), P]_\xi + [PXQ, L(P)]_\xi$, that is,

(4.2) $L(-\xi PXQ) = L(PXQ)P - \xi PL(PXQ) + PXQL(P) - \xi L(P)PXQ.$

Since $QPXQ = 0$, we get

(4.3) $L(-\xi PXQ) = L(Q)PXQ - \xi PXQL(Q) + QL(PXQ) - \xi L(PXQ)Q.$

Eqs.(4.2) and (4.3) imply

(4.4)
$$\begin{aligned} & L(PXQ)P - \xi PL(PXQ) + PXQL(P) - \xi L(P)PXQ \\ = {} & L(Q)PXQ - \xi PXQL(Q) + QL(PXQ) - \xi L(PXQ)Q. \end{aligned}$$

Multiplying by P from both sides in Eq.(4.4), one obtains

$$PL(PXQ)P = \xi PL(PXQ)P.$$

Note that $\xi \neq 1$ and F is a field. It follows that $PL(PXQ)P = 0$. Similarly, multiplying by Q from both sides in Eq.(4.4), one can obtain $QL(PXQ)Q = 0$. Hence,

$$L(PXQ) = PL(PXQ)Q \in P\mathcal{U}Q.$$

Multiplying by P and Q from the left and the right respectively in Eq.(4.4), by Eq.(4.1), one gets $\xi PL(P)PXQ = \xi PXQL(Q)Q$, which implies $PL(P)PXQ = PXQL(Q)Q$ as $\xi \neq 0$. The claim holds.

Claim 3. $L(I) \in \mathcal{Z}(\mathcal{U})$.

By Claim 2, $PL(P)PXQ = PXQL(Q)Q$ for all $X \in \mathcal{U}$. It follows from Lemma 3.3 that $PL(P)P \in \mathcal{Z}(P\mathcal{U}P) = P\mathcal{Z}(\mathcal{U})P$, $QL(Q)Q \in \mathcal{Z}(Q\mathcal{U}Q) = Q\mathcal{Z}(\mathcal{U})Q$ and $PL(P)P + QL(Q)Q \in \mathcal{Z}(\mathcal{U})$. By Eq.(4.1), we get $L(I) = PL(P)P + QL(Q)Q \in \mathcal{Z}(\mathcal{U})$.

Claim 4. *For any $X, Y \in \mathcal{U}$, we have*

$$PL(PXPYP)P = PL(PXP)PYP + PXPL(PYP)P - PL(P)PXPYP.$$

$$\boxed{\textbf{ELA}}$$

Taking any $X, Z \in \mathcal{U}$, since $(PXP + PXPZQ)(PZQ - Q) = 0$, we have

$$[L(PXP + PXPZQ), PZQ - Q]_\xi + [PXP + PXPZQ, L(PZQ - Q)]_\xi = 0.$$

By Claims 1-2, the above equation reduces to

(4.5) $PL(PXPZQ)Q = PL(PXP)PZQ + PXPL(PZQ)Q - PXPZQL(Q)Q,$

that is,

$$
\begin{aligned}
(4.6) \quad PL(PXP)PZQ = \;& PL(PXPZQ)Q - PXPL(PZQ)Q + PXPZQL(Q)Q \\
= \;& PL(PXPZQ)Q - PXPL(PZQ)Q + PL(P)PXPZQ
\end{aligned}
$$

for all $X, Z \in \mathcal{U}$. Now, taking any $Y \in \mathcal{U}$, by Eqs.(4.5)-(4.6) and Claim 2, one can easily check that

$$PL(PXPYP)PZQ = \{PL(PXP)PYP + PXPL(PYP)P - PL(P)PXPYP\}PZQ$$

holds for all $X, Y, Z \in \mathcal{U}$. Note that $\mathcal{M}$ is a faithful left $\mathcal{A}$-module. It follows that

$$PL(PXPYP)P = PL(PXP)PYP + PXPL(PYP)P - PL(P)PXPYP$$

holds for all $X, Y \in \mathcal{U}$. The claim is true.

Similarly, one can show that the following Claim 5 is true.

Claim 5. *For any $X, Y \in \mathcal{U}$, we have*

$$QL(QXQYQ)Q = QL(QXQ)QYQ + QXQL(QYQ)Q - QXQYQL(Q)Q.$$

Now by a similar argument to that of Claim 7 in Theorem 3.1, one can show the following claim.

Claim 6. *There exists an additive derivation $d : \mathcal{U} \to \mathcal{U}$ such that $L(X) = d(X) + L(I)X$ for all $X \in \mathcal{U}$.*

Claim 7. $d(\xi I) = \xi L(I)$.

We first show $d(\xi I) \in \mathcal{Z}(\mathcal{U})$. In fact, since d is a derivation, for any $X \in \mathcal{U}$, we have

$$d(\xi X) = d(\xi I)X + (\xi I)d(X) = d(X)(\xi I) + Xd(\xi I).$$

It follows that $d(\xi I)X = Xd(\xi I)$ for all $X \in \mathcal{U}$, which implies $d(\xi I) \in \mathcal{Z}(\mathcal{U})$.

ELA

834   X.F. Qi

Next, for any $X \in \mathcal{U}$, since $PXQP = 0$, by the assumption on L and Claim 6, we get

$$
\begin{aligned}
d(-\xi PXQ) - \xi L(I)PXQ = L(-\xi PXQ) &= L([PXQ, P]_\xi) \\
&= [L(PXQ), P]_\xi + [PXQ, L(P)]_\xi \\
&= [d(PXQ) + L(I)PXQ, P]_\xi + [PXQ, d(P) + L(I)P]_\xi \\
&= -\xi d(PXQ) - 2\xi L(I)PXQ.
\end{aligned}
$$

That is, $d(\xi PXQ) = \xi d(PXQ) + \xi L(I)PXQ$ for all $X \in \mathcal{U}$. Note that $d(\xi PXQ) = d(\xi I)PXQ + \xi d(PXQ)$. It follows that $(d(\xi I) - \xi L(I))PXQ = 0$ for all $X \in \mathcal{U}$, which implies $d(\xi I) = \xi L(I)$ as $\mathcal{M}$ is a faithful $(\mathcal{A}, \mathcal{B})$-bimodule and $d(\xi I) - \xi L(I) \in \mathcal{Z}(\mathcal{U})$.

The proof of the theorem is complete. ☐

Acknowledgment. The author thanks the referee(s) for helpful comments and suggestions.

<div align="center">REFERENCES</div>

[1] D. Benkovič and D. Eremita. Commuting traces and commutativity preserving maps on triangular algebras. *J. Algebra*, 280:797–824, 2004.

[2] M. Brešar. Commuting traces of biadditive mappings, commutativity-preserving mappings and Lie mappings. *Trans. Amer. Math. Soc.*, 335:525–546, 1993.

[3] M. Brešar. Characterizing homomorphisms, derivations, and multipliers in rings with idempotents. *Proc. Roy. Soc. Edinburgh Sect. A*, 137:9–21, 2007.

[4] J.A. Brooke, P. Busch, and D.B. Pearson. Commutativity up to a factor of bounded operators in complex Hilbert spaces. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., A*, 458:109–118, 2002.

[5] W.S. Cheung. Commuting maps of triangular algebras. *J. London Math. Soc.*, 63:117–127, 2001.

[6] K.R. Davidson. *Nest Algebras*. Pitman Research Notes in Mathematics, Vol. 191, Longman, London, New York, 1988.

[7] D.G. Han. Additive derivations of nest algebras. *Proc. Amer. Math. Soc.*, 119:1165–1169, 1993.

[8] D.G. Han. Continuity and linearity of additive derivations of nest algebras on Banach spaces. *Chinese Ann. Math., Ser. B*, 17:227–236, 1996.

[9] C.J. Hou and D.G. Han. Derivations and isomorphisms of certain reflexive operator algebras (in Chinese). *Acta Math. Sinica*, 14:105–112, 1998.

[10] P. Ji and W. Qi. Characterizations of Lie derivations of triangular algebras. *Linear Algebra Appl.*, 435:1137–1146, 2011.

[11] C. Kassel. *Quantum Groups*. Springer-Verlag, New York, 1995.

[12] F. Lu and W. Jing. Characterizations of Lie derivations of $\mathcal{B}(X)$. *Linear Algebra Appl.*, 432:89–99, 2009.

[13] M. Mathieu and A.R. Villena. The structure of Lie derivations on C^*-algebras. *J. Funct. Anal.*, 202:504–525, 2003.

[14] Z.D. Pan. Derivable maps and derivational points. *Linear Algebra Appl.*, 436(11):4251–4260, 2012.

[15] X.F. Qi and J.C. Hou. Additive Lie (ξ-Lie) derivations and generalized Lie (ξ-Lie) derivations on nest algebras. *Linear Algebra Appl.*, 431:843–854, 2009.

$$\boxed{\textbf{ELA}}$$

[16] X.F. Qi and J.C. Hou. Linear maps Lie derivable at zero point on $\mathcal{J}$-subspace lattice algebras. *Studia Math.*, 197:157–169, 2010.

[17] X.F. Qi and J.C. Hou. Characterization of Lie derivations on prime rings. *Comm. Algebra*, 39(10):3824–3835, 2011.

[18] X.F. Qi and J.C. Hou. Additive Lie (ξ-Lie) derivations and generalized Lie (ξ-Lie) derivations on prime algebras. *Acta Math. Sin. (English Series)*, 29:383-392, 2013.

[19] X.F. Qi, J.L. Cui, and J.C. Hou. Characterizing additive ξ-Lie derivations of prime algebras by ξ-Lie zero products. *Linear Algebra Appl.*, 434:669–682, 2011.

[20] M. Spivac. Derivations of nest algebras on Banach spaces. *Israel. J. Math.*, 50(2):193–200, 1985.

[21] J.H. Zhang and W.Y. Yu. Jordan derivations of triangular algebras. *Linear Algebra Appl.*, 419:251–255, 2006.