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MAJORIZATION BOUNDS FOR SIGNLESS LAPLACIAN EIGENVALUES*

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Abstract. It is known that, for a simple graph G and a real number α, the *quantity* $s'_\alpha(G)$ is defined as the sum of the α-th power of non-zero singless Laplacian eigenvalues of G. In this paper, first some majorization bounds over $s'_\alpha(G)$ are presented in terms of the degree sequences, and number of vertices and edges of G. Additionally, a connection between $s'_\alpha(G)$ and the first Zagreb index, in which the Hölder's inequality plays a key role, is established. In the last part of the paper, some bounds (included Nordhauss-Gaddum type) for signless Laplacian Estrada index are presented.

Key words. Signless Laplacian matrix, Signless Laplacian-Estrada index, (First) Zagreb index, Majorization, Strictly Schur-convex.

AMS subject classifications. 05C50, 15C12, 15F10.

1. Introduction and preliminaries. Let G be a simple graph with n vertices. The *Laplacian matrix* of G is defined by $L(G) = \Delta - A$, where A and Δ are the $(0,1)$-adjacency matrix and the diagonal matrix of the vertex degrees of G, respectively. We know that *Laplacian spectrum* of G consists of the eigenvalues $\mu_1, \mu_2, \dots, \mu_n$ (arranged in non-increasing order) of $L(G)$. It is also known that $\mu_n = 0$ and the multiplicity of 0 is equal to the number of connected components of G. We may refer [26] and its citations for detailed properties of the Laplacian spectrum. On the other hand, the *signless Laplacian matrix* of G is defined by $Q(G) = \Delta + A$. We denote the eigenvalues of $Q(G)$ by $q_1 \geq q_2 \geq \cdots \geq q_n \geq 0$.

Let α be a real number. Then we denote by $s_\alpha(G)$ the sum of the α-th power of *non-zero Laplacian eigenvalues* of G. In other words,

$$s_\alpha = s_\alpha(G) = \sum_{i=1}^{t} \mu_i^\alpha, \tag{1.1}$$

where t is the number of non-zero Laplacian eigenvalues of G. In a similar manner,

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the α-th power of the *non-zero signless Laplacian eigenvalues* of G is denoted by

$$s'_{\alpha} = s'_{\alpha}(G) = \sum_{i=1}^{h} q_i^{\alpha}, \tag{1.2}$$

where h is the number of non-zero signless Laplacian eigenvalues of G. In here, the cases $\alpha = 0$ and $\alpha = 1$ are trivial since $s'_0(G) = h$ and $s'_1(G) = 2m$, where m is the number of edges of G. In the literature, the bounds over quantities s_{α} and s'_{α} have been studied largely. For instance, in [30], Zhou established some properties of s_{α} for $\alpha \neq 0$ and $\alpha \neq 1$. He also discussed further properties by taking into account $s_{\frac{1}{2}}$ and s_2. In fact, some of the results obtained in [30] are improved in [27]. Additionally, some bounds for $s_{\alpha}(G)$ related to degree sequences have been established in [31]. On the other hand, in [33], by taking G as a bipartite graph, some new bounds over $s_{\alpha}(G)$ have been given. In detail, lower and upper bounds for incidence energy, and also lower bounds for Kirchhoff index and Laplacian Estrada index have been deduced. Furthermore, Akbari et al. [1] obtained some relations between s_{α} and s'_{α} for the ranges $0 < \alpha \leq 1$, $1 < \alpha < 2$ and $2 \leq \alpha < 3$.

The *Estrada index* of a graph G with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ is defined as $EE = EE(G) = \sum_{i=1}^{n} e^{\lambda_i}$. It is very useful descriptors in a large variety of problems, including those in biochemistry and m complex networks [10, 11, 12]. (We also refer [14, 20, 29] for some recent results.) Further, in [13], the *Laplacian-spectral counterpart of the Estrada index* is defined as

$$LEE = LEE(G) = \sum_{i=1}^{n} e^{\mu_i}. \tag{1.3}$$

(One can also look at the studies [4, 8, 24, 31, 32] for more details on the theory of the Laplacian Estrada index.) The next step of LEE is termed as the *signless Laplacian Estrada index* [2] of G with n vertices which is defined as

$$SLEE = SLEE(G) = \sum_{i=1}^{n} e^{q_i}. \tag{1.4}$$

By [15, 16], since the Laplacian and signless Laplacian spectra of bipartite graphs coincide, we easily say that LEE and $SLEE$ coincide in the case of *bipartite graphs*. Therefore, since the vast majority of molecular graphs are bipartite, $SLEE$ gives nothing new outcomes relative to the previously studied LEE. On the other hand, chemically interesting case in which $SLEE$ and LEE differ are the fullerenes, fluoranthenes and other non-alternant conjugated species (see [3, 9, 21, 22, 23]).

In the next section, we present some majorization bounds for s'_{α} in terms of the degree sequences, and number of vertices and edges of a simple graph G. Moreover,

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we establish a connection between $s'_\alpha(G)$ and the first Zagreb index. In the final section, we give some bounds (included Nordhauss-Gaddum type) for $SLEE$.

2. New bounds over s'_α. In this first main section, we will give some bounds on the quantities presented in (1.1) and (1.2).

For any two non-increasing sequences

$$x = (x_1, x_2, \ldots, x_n) \quad \text{and} \quad y = (y_1, y_2, \ldots, y_n),$$

we say that x is *majorized* by y, denoted by $x \preceq y$, if

$$\sum_{i=1}^{j} x_i \leq \sum_{i=1}^{j} y_i \quad \text{for } j = 1, 2, \ldots, n-1, \quad \text{and} \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.$$

For a real-valued function f defined on a set in $\mathbf{R}^n$, if $f(x) < f(y)$ whenever $x \preceq y$ but $x \neq y$, then f is said to be *strictly Schur-convex* [25]. The following lemma makes clear the property of strictly Schur-convexity for the function f.

LEMMA 2.1 ([25]). *Let α be a real number such that $\alpha \neq 0$ and $\alpha \neq 1$.*

(i) *For $i = 1, 2, \ldots, h$, suppose $x_i \geq 0$. Then the following hold:*
- $f(x) = \sum_{i=1}^{h} x_i^\alpha$ *is strictly Schur-convex if $\alpha > 1$.*
- $f(x) = -\sum_{i=1}^{h} x_i^\alpha$ *is strictly Schur-convex if $0 < \alpha < 1$.*

(ii) *For $i = 1, 2, \ldots, h$, suppose $x_i > 0$. Then $f(x) = \sum_{i=1}^{h} x_i^\alpha$ is strictly Schur-convex if $\alpha < 0$.*

We also remind that the *degree sequence* of G is a list of the degrees of the vertices in non-increasing order which is denoted by $(d) = (d_1, d_2, \ldots, d_n)$. We let denote $\overline{(d)} = (d_1 + 1, d_2, \ldots, d_{n-1}, d_n - 1)$, where d_1 is the *maximum* vertex degree of G. We similarly denote $(\mu) = (\mu_1, \mu_2, \ldots, \mu_n)$ and $(q) = (q_1, q_2, \ldots, q_n)$ as the spectrums of the Laplacian and signless Laplacian matrices, respectively.

REMARK 2.2. In [17], Grone proved that if G has at least one edge, then $\overline{(d)} \preceq (\mu)$ is correct while $\overline{(d)} \preceq (q)$ is not. For example, if G is a connected non-bipartite graph with at least one pendant vertex, then

$$d_1 + 1 + d_2 + \cdots + d_{n-1} = 2m > q_1 + q_2 + \cdots + q_{n-1}$$

since G is non-bipartite (which implies $q_n > 0$).

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It is well known that (see, for example, [25, p. 218]) the spectrum of a positive semi definite Hermitian matrix majorizes its main diagonal (when both are rearranged in non-increasing order). As a result of this, we can give the following proposition.

PROPOSITION 2.3. *For the graph G with signless Laplacian spectrum $(q) = (q_1, q_2, \dots, q_n)$ and degree sequence $(d) = (d_1, d_2, \dots, d_n)$, it is always true that $(d) \preceq (q)$.*

We also need the following preliminary results for our main theorems in this paper.

PROPOSITION 2.4 ([1]). *Let G be a graph of order n and α be a real number. Then we have the following relations for the quantities in (1.1) and (1.2):*

i) If $0 < \alpha \leq 1$ or $2 \leq \alpha \leq 3$, then $s'_\alpha \geq s_\alpha$.
ii) If $1 \leq \alpha \leq 2$, then $s'_\alpha \leq s_\alpha$.

On the other hand, for the quantity s_α in (1.1), it has been obtained the following lower and upper bounds in [31] by considering degree sequences.

PROPOSITION 2.5 ([31]). *Let G be a connected graph with $n \geq 2$ vertices.*

(i) If $\alpha > 1$, then $s_\alpha \geq (d_1 + 1)^\alpha + \sum_{i=2}^{n-1} d_i^\alpha + (d_n - 1)^\alpha.$

(ii) If $0 < \alpha < 1$, then $s_\alpha \leq (d_1 + 1)^\alpha + \sum_{i=2}^{n-1} d_i^\alpha + (d_n - 1)^\alpha.$

Moreover, equalities hold in both (i) and (ii) if and only if $G = S_n$, where S_n is a star graph with n vertices.

After all, as the first main result of this paper, we give the new lower and upper bounds for the quantity s'_α in (1.2) depending on the degree sequence.

THEOREM 2.6. *Let G be a graph with $n \geq 2$ vertices.*

(i) If $\alpha > 1$, then $s'_\alpha \geq \sum_{i=1}^{n} d_i^\alpha.$

(ii) If $0 < \alpha < 1$, then $s'_\alpha \leq \sum_{i=1}^{n} d_i^\alpha.$

Moreover, equalities hold in both cases if and only if $(q) = (d)$.

Proof. Suppose that $\alpha > 1$. Then, by Lemma 2.1-(i), $f(x) = \sum_{i=1}^{h} x_i^\alpha$ (such that

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$x_i \geq 0$) is strictly Schur-convex which together with Proposition 2.3, implies that

$$s'_\alpha = \sum_{i=1}^{n} q_i^\alpha \geq d_1^\alpha + d_2^\alpha + \cdots + d_n^\alpha,$$

where equality holds if and only if $(q_1, q_2, \ldots, q_n) = (d_1, d_2, \ldots, d_n)$.

Now let us assume that $0 < \alpha < 1$. Then again, by Lemma 2.1-(i), $f(x) = -\sum_{i=1}^{h} x_i^\alpha$ (such that $x_i \geq 0$) is strictly Schur-convex which implies that

$$-s'_\alpha = -\sum_{i=1}^{n} q_i^\alpha \geq -\left[d_1^\alpha + d_2^\alpha + \cdots + d_n^\alpha\right],$$

or equivalently,

$$s'_\alpha = \sum_{i=1}^{n} q_i^\alpha \leq d_1^\alpha + d_2^\alpha + \cdots + d_n^\alpha.$$

It is clear that the equality holds if and only if $(q_1, q_2, \ldots, q_n) = (d_1, d_2, \ldots, d_n)$.

Hence, the result. □

Using Lemma 2.1, it is easy to see the following result.

Corollary 2.7. *Let G be a graph. Then the following inequalities hold:*

a) If $\alpha > 1$ or $\alpha < 0$, then

$$\sum_{i=1}^{n} d_i^\alpha < (d_1 + 1)^\alpha + \sum_{i=2}^{n-1} d_i^\alpha + (d_n - 1)^\alpha.$$

(We should note that for the case $\alpha < 0$, we also need the assumption $d_i > 1$ or equivalently, G has no pendant vertices.)

b) If $0 < \alpha < 1$, then

$$\sum_{i=1}^{n} d_i^\alpha > (d_1 + 1)^\alpha + \sum_{i=2}^{n-1} d_i^\alpha + (d_n - 1)^\alpha.$$

We note that Theorem 2.6 can be converted to bipartite graphs as in the following.

Theorem 2.8. *Let G be a connected bipartite graph with $n \geq 3$ vertices. If $\alpha > 1$, then*

$$s'_\alpha \geq (d_1 + 1)^\alpha + \sum_{i=2}^{n-1} d_i^\alpha + (d_n - 1)^\alpha > \sum_{i=1}^{n} d_i^\alpha.$$

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Proof. For the first inequality, by Lemma 2.1-(i), $f(x) = \sum_{i=1}^{h} x_i^{\alpha}$ is strictly Schur-convex for $x_i \geq 0$, where $i = 1, 2, \ldots, h$. Since G is bipartite, it is known that $q_n = 0$, and $Q(G)$ and $L(G)$ share the same eigenvalues. Thus, by Remark 2.2, we obtain

$$(d_1 + 1, d_2, \ldots, d_{n-1}, d_n - 1) \preceq (q_1, q_2, \ldots, q_n).$$

Hence,

$$s'_\alpha = \sum_{i=1}^{n} q_i^{\alpha} \geq (d_1 + 1)^{\alpha} + \sum_{i=2}^{n-1} d_i^{\alpha} + (d_n - 1)^{\alpha},$$

as required. We note that the second inequality follows immediately from Corollary 2.7-a). ∎

From Lemma 2.1, Remark 2.2, Proposition 2.4 and Corollary 2.7, we get the following result.

COROLLARY 2.9. *Let G be a graph. Therefore,*

a) if $0 < \alpha < 1$, then $s_\alpha < \sum_{i=1}^{n} d_i^{\alpha}$,

b) if $1 < \alpha$, then $s_\alpha > \sum_{i=1}^{n} d_i^{\alpha}$,

c) if $2 \leq \alpha \leq 3$, then $s'_\alpha \geq (d_1 + 1)^{\alpha} + \sum_{i=2}^{n-1} d_i^{\alpha} + (d_n - 1)^{\alpha}$.

Recall that the *first Zagreb index* $M_1(G)$ of the graph G is the sum of the sequares of the degrees of vertices of G. (One can find the details about this graph invariant in [19] and the refences cited therein.) This index actually has been found in many applications, specially, in chemistry [19] and received wide investigations, and lots of properties of which have been reported (see [5, 7, 18, 28] for the detials).

Now, by using the Hölder's inequality, we only establish a lower bound for the quantity s'_α given in (1.2) in terms of the first Zagreb index $M_1(G)$. By using this theorem, we may have a chance to derive lots of bounds over s'_α for a connected (molecular) graph G related to its number of vertices (atoms) and edges (bonds).

THEOREM 2.10. *Let G be a graph with n vertices and $m \geq 1$ edges. If $\alpha < 0$ or $0 < \alpha < 1$ or $\alpha > 2$, then*

$$s'_\alpha \geq \frac{(2m)^{2-\alpha}}{(M_1(G) + 2m)^{1-\alpha}}. \tag{2.1}$$

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Equality holds in (2.1) *if and only if* $q_1 = q_2 = \cdots = q_n$. *Furthermore, if* $1 < \alpha < 2$, *then the inequality in* (2.1) *is reversed.*

Proof. Let $x_1, x_2, \ldots, x_s$ be positive real numbers, and let p be a real number with $p \neq 0$, $p \neq \frac{1}{2}$, $p \neq 1$. If $p < 0$ or $p > 1$, then $\frac{2p-1}{p} > 1$. By Hölder's inequality, we have

$$\begin{aligned}\sum_{i=1}^{s} x_i^p &= \sum_{i=1}^{s} x_i^{\frac{p}{2p-1}} x_i^{\frac{2p(p-1)}{2p-1}} \\ &\leq \left[\sum_{i=1}^{s}\left(x_i^{\frac{p}{2p-1}}\right)^{\frac{2p-1}{p}}\right]^{\frac{p}{2p-1}} \left[\sum_{i=1}^{s}\left(x_i^{\frac{2p(p-1)}{2p-1}}\right)^{\frac{2p-1}{p-1}}\right]^{\frac{p-1}{2p-1}} \\ &= \left(\sum_{i=1}^{s} x_i\right)^{\frac{p}{2p-1}} \left(\sum_{i=1}^{s} x_i^{2p}\right)^{\frac{p-1}{2p-1}}.\end{aligned}$$

Shortly, we get

$$\left(\sum_{i=1}^{s} x_i\right)^{\frac{p}{2p-1}} \geq \frac{\sum_{i=1}^{s} x_i^p}{\left(\sum_{i=1}^{s} x_i^{2p}\right)^{\frac{p-1}{2p-1}}}.$$

Thus,

$$\sum_{i=1}^{s} x_i \geq \frac{\left(\sum_{i=1}^{s} x_i^p\right)^{\frac{2p-1}{p}}}{\left(\sum_{i=1}^{s} x_i^{2p}\right)^{\frac{p-1}{p}}}, \tag{2.2}$$

where the equality holds if and only if $x_1 = x_2 = \cdots = x_s$.

Now if we write $s = n$, $x_i = q_i^\alpha$ and $p = \frac{1}{\alpha}$ in (2.2), then (2.1) follows immediately when $\alpha < 0$ or $0 < \alpha < 1$ since $s_1'(G) = 2m$ and $s_2'(G) = M_1(G) + 2m$. Furthermore the equality in (2.1) holds if and only if $q_1 = q_2 = \cdots = q_n$. The proofs for the cases $\alpha > 2$ and $1 < \alpha < 2$ are similar (here, take $0 < p < \frac{1}{2}$ and $\frac{1}{2} < p < 1$, respectively). $\square$

Example 2.11. Together with some known bounds for the first Zagreb index [5, 7, 18, 28], the bound in (2.1) may directly yield a lot different bounds for s_α'. For example, let us consider the bound [5]

$$M_1(G) \leq m\left(\frac{2m}{n-1} + n - 2\right)$$

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for the first Zagreb index. This bound implies that, if $\alpha < 0$ or $0 < \alpha < 1$ (resp., $1 < \alpha < 2$), then

$$s'_\alpha \geq \;(resp., \leq)\; \frac{2m}{\left(\frac{m}{n-1} + \frac{n}{2}\right)^{1-\alpha}}.$$

EXAMPLE 2.12. Suppose that G is K_{r+1}-free with $2 \leq r \leq n-1$. Then

$$M_1(G) \leq \frac{2r-2}{r} nm$$

such that equality holds if and only if G is complete bipartite graph for $r = 2$, and a regular complete r-bipartite graph for $r \geq 3$ (cf. [28]). This yields that if $\alpha < 0$ or $0 < \alpha < 1$ (resp., $1 < \alpha < 2$), then

$$s'_\alpha > \text{(resp., } <) \frac{2m}{\left(\frac{r-1}{r}n + 1\right)^{1-\alpha}}.$$

We need the next lemma for another result over s'_α.

LEMMA 2.13. *We have*

$$\sum_{i=1}^{n} d_i^\alpha \begin{array}{ll} \nearrow & \geq \quad n^{1-\alpha}(2m)^\alpha \quad \textit{if } \alpha < 0 \textit{ or } \alpha > 1 \\ \searrow & \leq \quad n^{1-\alpha}(2m)^\alpha \quad \textit{if } 0 < \alpha < 1 \end{array}. \tag{2.3}$$

Proof. Observe that for $x > 0$, the function x^α is strictly convex if and only if $\alpha < 0$ or $\alpha > 1$. Hence, let us suppose that $\alpha < 0$ or $\alpha > 1$. Then

$$\left(\sum_{i=1}^{n} \frac{1}{n} d_i\right)^\alpha \leq \sum_{i=1}^{n} \frac{1}{n} d_i^\alpha,$$

and in other words,

$$\sum_{i=1}^{n} d_i^\alpha \geq \frac{1}{n^{\alpha-1}} \left(\sum_{i=1}^{n} d_i\right)^\alpha = n^{1-\alpha}(2m)^\alpha,$$

as required. ∎

From Theorem 2.6 and Lemma 2.13, we obtain the following consequence.

COROLLARY 2.14. *Let G be a graph with n vertices and m edges. Then for the ranges $0 < \alpha < 1$ and $\alpha > 1$, we obtain the bounds*

$$s'_\alpha \leq n\left(\frac{2m}{n}\right)^\alpha \quad \textit{and} \quad n\left(\frac{2m}{n}\right)^\alpha \leq s'_\alpha,$$

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respectively.

The following lemma will be needed for a new bound over s'_α (see Theorem 2.16 below).

LEMMA 2.15 ([6]). *Let G be a connected graph with n vertices and m edges. Then $q_1 \geq \frac{4m}{n}$ with equality if and only if G is a regular graph.*

THEOREM 2.16. *Let G be a connected graph with $n \geq 3$ vertices and m edges.*

i) If $\alpha < 0$ or $\alpha > 1$, then

$$s'_\alpha \geq \left(\frac{2m}{n}\right)^\alpha (2^\alpha + n - 2).$$

ii) If $0 < \alpha < 1$, then

$$s'_\alpha \leq \left(\frac{2m}{n}\right)^\alpha (2^\alpha + n - 2).$$

Proof. As in the proof of Lemma 2.13, it is clear that x^α (where $x > 0$) is a strictly convex function if and only if $\alpha < 0$ or $\alpha > 1$. Therefore, let us suppose that $\alpha < 0$ or $\alpha > 1$. We then have $\left(\sum_{i=2}^{n-1} \frac{1}{n-2} q_i\right)^\alpha \leq \sum_{i=2}^{n-1} \frac{1}{n-2} q_i^\alpha$, or equivalently,

$$\sum_{i=2}^{n-1} q_i^\alpha \geq \frac{1}{(n-2)^{\alpha-1}} \left(\sum_{i=2}^{n-1} q_i\right)^\alpha,$$

where the equality holds if and only if $q_2 = \cdots = q_{n-1}$. It follows that

$$s'_\alpha \geq q_1^\alpha + q_n^\alpha + \frac{1}{(n-2)^{\alpha-1}} \left(\sum_{i=2}^{n-1} q_i\right)^\alpha = q_1^\alpha + q_n^\alpha + \frac{(2m - q_1 - q_n)^\alpha}{(n-2)^{\alpha-1}}.$$

Now let us consider the function $f(x,y) = x^\alpha + y^\alpha + \frac{(2m-x-y)^\alpha}{(n-2)^{\alpha-1}}$, for $x > 0$, $y > 0$. In order to find its minimum, we have the following derivations for this function:

$$f_x = \alpha \left[x^{\alpha-1} - \frac{(2m-x-y)^{\alpha-1}}{(n-2)^{\alpha-1}}\right],\ f_y = \alpha \left[y^{\alpha-1} - \frac{(2m-x-y)^{\alpha-1}}{(n-2)^{\alpha-1}}\right],$$

$$f_{xx} = \alpha(\alpha-1) \left[x^{\alpha-2} + \frac{(2m-x-y)^{\alpha-2}}{(n-2)^{\alpha-1}}\right],\ f_{xy} = f_{yx} = \alpha(\alpha-1) \frac{(2m-x-y)^{\alpha-2}}{(n-2)^{\alpha-1}}$$

and

$$f_{yy} = \alpha(\alpha-1) \left[y^{\alpha-2} + \frac{(2m-x-y)^{\alpha-2}}{(n-2)^{\alpha-1}}\right].$$

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A simple calculation implies that

$$f_x = f_y = 0 \Longrightarrow (n-1)\,x + y = 2m \text{ and } x + (n-1)\,y = 2m \Longrightarrow x + y = \frac{4m}{n}.$$

For $x+y=\frac{4m}{n}$, we clearly get $f_{xx}>0$ and $f_{xx}f_{yy}-f_{xy}^2>0$.

From above, it is concluded that $f\,(x,y)$ has a minimum value at $x+y=\frac{4m}{n}$ and that the minimum value is $x^\alpha+\left(\frac{4m}{n}-x\right)^\alpha+\frac{\left(2m-\frac{4m}{n}\right)^\alpha}{(n-2)^{\alpha-1}}$. By Lemma 2.15, $q_1\geq\frac{4m}{n}$. Thus,

$$q_1^\alpha+\left(\frac{4m}{n}-q_1\right)^\alpha+\frac{\left(2m-\frac{4m}{n}\right)^\alpha}{(n-2)^{\alpha-1}}\geq\left(\frac{4m}{n}\right)^\alpha+\frac{\left(2m-\frac{4m}{n}\right)^\alpha}{(n-2)^{\alpha-1}}.$$

Hence, the result. □

Remark 2.17. One can easily see that the bounds in Theorem 2.16 are better than the bounds in Corollary 2.14.

One can also consider the next lemma.

Lemma 2.18 ([2]). *Let G be a graph on n vertices. Then $q_1\geq 2\,(k-1)$ where k is the chromatic number. Equality holds if and only if $G\cong K_n$ or G is the cycle C_n of odd length.*

Hence, as a consequence of Theorem 2.16 and Lemma 2.18, we get the following corollary.

Corollary 2.19. *Let G be a connected graph with $n\geq 3$ vertices and m edges.*

i) If $\alpha<0$ or $\alpha>1$, then

$$s'_\alpha\geq 2^\alpha\left[(k-1)^\alpha+\left(\frac{2m}{n}-k+1\right)^\alpha+\left(\frac{m}{n}\right)^\alpha(n-2)\right].$$

ii) If $0<\alpha<1$, then

$$s'_\alpha\leq 2^\alpha\left[(k-1)^\alpha+\left(\frac{2m}{n}-k+1\right)^\alpha+\left(\frac{m}{n}\right)^\alpha(n-2)\right].$$

3. Bounds for signless Laplacian Estrada index. Let G be a graph with n vertices. We recall that, for a non-negative integer k and the eigenvalues $q_1\geq q_2\geq\cdots\geq q_n$ of $Q\,(G)$,

$$T_r\,(G)=\sum_{i=1}^{n}q_i^r \tag{3.1}$$

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denotes the r-th *signless Laplacian spectral moment* of G. Obviously, $T_0(G) = n$ and $T_r(G) = s'_r$ for $r \geq 1$. In fact, (3.1) will also be needed in our results.

By [2], it is obvious that $SLEE$ in (1.4) can be written as

$$SLEE = \sum_{r=0}^{\infty} \frac{T_r}{r!},$$

where T_r is defined as in (3.1). In this section, we claim to convert the properties and obtained results in the previous section into the signless Laplacian Estrada index.

THEOREM 3.1. *Let G be a graph with $n \geq 2$ vertices. Then*

$$SLEE \geq \sum_{i=1}^{n} e^{d_i}.$$

Moreover, the equality holds in above if and only if $(q) = (d)$.

Proof. We know that $T_0 = n$, $T_1 = 2m$,

$$T_2 = \sum_{i=1}^{n} d_i (d_i + 1) = M_1(G) + 2m$$

and $T_r = s'_r(G)$ for $r \geq 1$. By Theorem 2.6-(i),

$$T_r \geq \sum_{i=1}^{n} d_i^r \quad \text{for} \quad r = 0, 1$$

such that equality holds for $r = 0, 1$. Thus,

$$SLEE = \sum_{r \geq 0} \frac{T_r}{r!} \geq \sum_{r \geq 0} \frac{\sum_{i=1}^{n} d_i^r}{r!} = \sum_{i=1}^{n} e^{d_i},$$

as desired. ☐

THEOREM 3.2. *Let G be a graph with $n \geq 2$ vertices and m vertices. Then*

$$SLEE \leq n + 2m - 1 - \sqrt{M_1(G) + 2m} + e^{\sqrt{M_1(G)+2m}}$$

with equality holding if and only if at most one of $q_1, q_2, \ldots, q_n$ is non-zero.

Proof. It is well known that $\sum_{i=1}^{n} q_i^2 = M_1(G) + 2m$. For an integer $r \geq 3$,

$$\begin{aligned}\left(\sum_{i=1}^{n} q_i^2\right)^r &\geq \sum_{i=1}^{n} q_i^{2r} + r \sum_{1 \leq i < j \leq n} \left(q_i^2 q_j^{2(r-1)} + q_i^{2(r-1)} q_j^2\right) \\ &\geq \sum_{i=1}^{n} q_i^{2r} + 2r \sum_{1 \leq i < j \leq n} q_i^r q_j^r \geq \left(\sum_{i=1}^{n} q_i^r\right)^2,\end{aligned}$$

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and then,

$$\sum_{i=1}^{n} q_i^r \leq \left(\sum_{i=1}^{n} q_i^2\right)^{\frac{r}{2}} = \left(M_1\left(G\right) + 2m\right)^{\frac{r}{2}}$$

with equality holding if and only if at most one of $q_1, q_2, \ldots, q_n$ is non-zero.

It is easily seen that

$$\begin{aligned} SLEE &= n + 2m + \sum_{r\geq 2} \frac{1}{r!} \sum_{i=1}^{n} q_i^r \leq n + 2m + \sum_{r\geq 2} \frac{1}{r!} \left(\sqrt{M_1\left(G\right) + 2m}\right)^r \\ &= n + 2m - 1 - \sqrt{M_1\left(G\right) + 2m} + e^{\sqrt{M_1(G)+2m}}. \quad \square \end{aligned}$$

Finally, we will give a Nordhauss-Gaddum type bound for $SLEE$.

THEOREM 3.3. *Let G be a graph with $n \geq 2$ vertices and m edges. Also let $\overline{G}$ be the complement of G. Then*

$$SLEE\left(G\right) + SLEE\left(\overline{G}\right) > 2\left[e^{n-1} + \left(n-2\right)e^{\frac{n-1}{2}}\right].$$

Proof. By the arithmetic-geometric inequality, we have

$$SLEE = e^{q_1} + e^{q_2} + \cdots + e^{q_n} \geq e^{\frac{4m}{n}} + \left(n-2\right)e^{\frac{2m}{n}} + 1$$

(cf. [2]). Let $\overline{m}$ be the number of edges of $\overline{G}$. Thus,

$$\begin{aligned} SLEE(G) + SLEE(\overline{G}) &\geq 2 + e^{\frac{4m}{n}} + e^{\frac{4\overline{m}}{n}} + \left(n-2\right)\left(e^{\frac{2m}{n}} + e^{\frac{2\overline{m}}{n}}\right) \\ &\geq 2 + 2e^{\frac{2(m+\overline{m})}{n}} + 2\left(n-2\right)e^{\frac{m+\overline{m}}{n}} \\ &= 2 + 2e^{(n-1)} + 2\left(n-2\right)e^{\frac{n-1}{2}} \\ &> 2e^{n-1} + 2\left(n-2\right)e^{\frac{n-1}{2}}. \end{aligned}$$

Hence, the result. $\square$

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