Minimum rank, maximum nullity, and zero forcing number of simple digraphs

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MINIMUM RANK, MAXIMUM NULLITY, AND ZERO FORCING NUMBER OF SIMPLE DIGRAPHS*

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Abstract. A simple digraph describes the off-diagonal zero-nonzero pattern of a family of (not necessarily symmetric) matrices. Minimum rank of a simple digraph is the minimum rank of this family of matrices; maximum nullity is defined analogously. The simple digraph zero forcing number is an upper bound for maximum nullity. Cut-vertex reduction formulas for minimum rank and zero forcing number for simple digraphs are established. The effect of deletion of a vertex on minimum rank or zero forcing number is analyzed, and simple digraphs having very low or very high zero forcing number are characterized.

Key words. Zero forcing number, Maximum nullity, Minimum rank, Simple directed graph, Simple digraph.

AMS subject classifications. 05C50, 15A03.

1. Introduction. Extensive work has been done on problems related to finding the minimum rank among the family of real symmetric matrices whose off-diagonal zero-nonzero pattern is described by a given simple graph G (see [7] for a current survey). The problem of determining the minimum rank of matrices whose off-diagonal zero-nonzero pattern is described by a digraph Γ (where loops constrain the diagonal entries of the matrix) was studied in [3].

A similar problem, in which the diagonal entries of the matrices are free, was introduced in [8]. For a square matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, the off-diagonal zero-

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nonzero pattern of the entries describes a simple digraph (a directed graph without loops) $\Gamma(A) = (V, E)$, where the set of vertices is $V = \{1, 2, \ldots, n\}$ and the set of arcs is $E = \{(i, j) : a_{ij} \neq 0, i \neq j\}$. Note that the value of the diagonal entries of A does not affect $\Gamma(A)$. Conversely, given any simple digraph Γ (along with an ordering of the vertices), we may associate with Γ a family of matrices

$$\mathcal{M}(\Gamma) = \{A \in \mathbb{R}^{n\times n} : \Gamma(A) = \Gamma\}.$$

The *minimum rank* of a digraph Γ is

$$\mathrm{mr}(\Gamma) = \min\{\mathrm{rank}\, A : A \in \mathcal{M}(\Gamma)\},$$

and the *maximum nullity* of Γ is

$$\mathrm{M}(\Gamma) = \max\{\mathrm{null}\ A : A \in \mathcal{M}(\Gamma)\},$$

where it is clear that $\mathrm{mr}(\Gamma) + \mathrm{M}(\Gamma) = n$.

For a simple digraph $\Gamma = (V, E)$ having $v, u \in V$ and $(v, u) \in E$, u is an *out-neighbor* of v and v is an *in-neighbor* of u. The *out-degree* of v, denoted by $\deg^+(v)$, is the number of out-neighbors of v in Γ, and similarly for *in-degree*, denoted by $\deg^-(v)$. We define $\delta^+(\Gamma) = \min\{\deg^+(v) : v \in V\}$. In other words, $\delta^+(\Gamma)$ is the smallest out-degree amongst all vertices of Γ. We similarly define $\delta^-(\Gamma) = \min\{\deg^-(v) : v \in V\}$.

LEMMA 1.1. [5] *Let Y be an $n \times n$ zero-nonzero pattern such that each row (or column) of Y has at least r nonzero entries. Then there exists a matrix $A \in \mathbb{R}^{n\times n}$ whose zero-nonzero pattern is Y and* $\mathrm{rank}\, A \leq n - r + 1$.

For a simple digraph Γ, the minimum number of entries allowed to be nonzero in a row of $A \in \mathcal{M}(\Gamma)$ is $\delta^+(\Gamma) + 1$ and in a column of A is $\delta^-(\Gamma) + 1$. Therefore, we have the following corollary.

COROLLARY 1.2. *For any simple digraph* Γ, $\max\{\delta^+(\Gamma), \delta^-(\Gamma)\} \leq \mathrm{M}(\Gamma)$.

The notions of zero forcing sets and zero forcing number $\mathrm{Z}(G)$ for simple graphs was introduced in [1]. We define zero forcing for simple digraphs as in [8]. If Γ is a simple digraph with each vertex colored either white or blue[1], u is a blue vertex of Γ, and exactly one out-neighbor v of u is white, then change the color of v to blue (this is the *color change rule*). In this situation, we say that u *forces* v and write $u \to v$. Given a coloring of Γ, the *final coloring* is the result of applying the color change rule until no more changes are possible. A *zero forcing set* for Γ is a subset of vertices B such that, if initially the vertices of B are colored blue and the remaining vertices are white, the final coloring of Γ is all blue. The *zero forcing number* $\mathrm{Z}(\Gamma)$ is the minimum of $|B|$ over all zero forcing sets $B \subseteq V(\Gamma)$.

[1]Much of the earlier literature uses the color black rather than blue.

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If Γ is a simple digraph, the *underlying* simple graph of Γ, $\widehat{\Gamma}$, is the graph having the same vertex set, and $\{i,j\} \in E(\widehat{\Gamma})$ exactly when at least one of $(i,j),(j,i) \in E(\Gamma)$. An undirected (simple) graph G is *connected* if there is path in G from any vertex of G to any other vertex of G; a simple digraph is connected if its underlying graph is connected. The *components* of simple digraph are the subdigraphs induced by the vertices of the (connected) components its underlying graph. Since minimum rank, maximum nullity, and zero forcing number all sum over components, for the most part we work with connected simple digraphs.

As in the case for undirected graphs [1], the zero forcing number for simple directed graphs gives a bound for the maximum nullity: If Γ is a simple digraph, then $\mathrm{M}(\Gamma) \le \mathrm{Z}(\Gamma)$ [8]. One of the earliest families of simple graphs for which the minimum rank can be easily computed is trees, and when zero forcing was introduced it was shown that for any (simple, undirected) tree T, $\mathrm{M}(T) = \mathrm{Z}(T)$ [1]. In [8], it was shown that $\mathrm{M}(T) = \mathrm{Z}(T)$ for simple ditree (a ditree is a directed graph whose underlying graph has no cycles).

One could define the zero-forcing number using in-neighbors instead of out-neighbors. Using the in-neighbor definition of zero forcing would be equivalent to finding $\mathrm{Z}(\Gamma^{\mathrm{T}})$, where Γ^{T} is obtained from Γ by reversing all the arcs. Since using the in-neighbor definition of zero forcing does not give any additional advantages (Proposition 1.5 below), we will use the out-neighbor definition. Note that $A \in \mathcal{M}(\Gamma)$ if and only if $A^{\mathrm{T}} \in \mathcal{M}(\Gamma^{\mathrm{T}})$. Therefore, we have the following:

Observation 1.3. If Γ is a simple digraph, then $\mathrm{mr}(\Gamma^{\mathrm{T}}) = \mathrm{mr}(\Gamma)$.

For a given zero forcing set B for Γ, we construct the final coloring, listing the forces in the order in which they were performed. This list is a *chronological list of forces*. Note that B need not have a unique chronological list of forces, even though the final coloring is unique. The *order* of a chronological list of forces $\mathcal{F}$, denoted $|\mathcal{F}|$ is the number of forces performed. Suppose Γ is a simple digraph and $\mathcal{F}$ is a chronological list of forces of a zero forcing set B. A *forcing chain* is an ordered set of vertices $(w_1, w_2, \ldots, w_k)$ where $w_j \to w_{j+1}$ is a force in $\mathcal{F}$ for $1 \le j \le k-1$. A *maximal forcing chain* is a forcing chain that is not a proper subset of another forcing chain. The following lemma will be used.

Lemma 1.4. [8] *Suppose Γ is a simple digraph and $\mathcal{F}$ is a chronological list of forces of a zero forcing set B. Then, every maximal forcing chain is a path that starts with a vertex in B.*

The proof that $\mathrm{Z}(\Gamma^{\mathrm{T}}) = \mathrm{Z}(\Gamma)$ (and thus, that it does not matter whether we use the out-neighbor or in-neighbor definition of zero forcing number) uses the terminus and reversal of a chronological list of forces; these concepts are defined for simple

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graphs in [2, 9]. Let Γ be a simple digraph, let B be a minimum zero forcing set of Γ, and let $\mathcal{F}$ be a chronological list of forces of B. The *terminus* of $\mathcal{F}$, denoted $\mathrm{Term}(\mathcal{F})$, is the set of vertices that do not perform a force in $\mathcal{F}$, i.e., the vertices that appear as the last vertex in a maximal zero forcing chain of $\mathcal{F}$. The *reverse chronological list of forces* of $\mathcal{F}$, denoted $\mathrm{Rev}(\mathcal{F})$, is the result of reversing each individual force in $\mathcal{F}$, and also reversing the order in which the forces are performed. Clearly $|\,\mathrm{Term}(\mathcal{F})| = |B|$ = the number of maximal forcing chains of $\mathcal{F}$. One can show by induction on $|\mathcal{F}|$ that $\mathrm{Term}(\mathcal{F})$ is a zero forcing set for Γ^{T} with chronological list of forces $\mathrm{Rev}(\mathcal{F})$; the proof is similar to [2, Theorem 2.6] and is omitted.

PROPOSITION 1.5. *Suppose* Γ *is a simple digraph,* B *is a minimum zero forcing set of* Γ*, and* $\mathcal{F}$ *is a chronological list of forces of* B*. Then* $\mathrm{Term}(\mathcal{F})$ *is a zero forcing set for* Γ^{T} *with chronological list of forces* $\mathrm{Rev}(\mathcal{F})$*. Hence,* $\mathrm{Z}(\Gamma^{\mathrm{T}}) = \mathrm{Z}(\Gamma)$.

Although there are many similarities between zero forcing for simple graphs and zero forcing for simple digraphs, there are some fundamental differences. In [2], it is shown that if G is a simple graph with no isolated vertices, then for every vertex v of G there is some minimum zero forcing set B of G such that $v \notin B$. That is not the case for simple digraphs.

OBSERVATION 1.6. Let Γ be a simple digraph. If v is a vertex with $\deg^{-} v = 0$ then v is in every zero forcing set of Γ.

The next example shows that having in-degree zero, although sufficient for inclusion in the intersection of the minimum zero forcing sets, is not necessary.

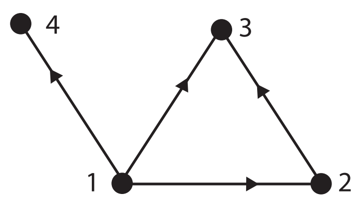

FIG. 1.1. *Simple digraph with a unique minimum zero forcing set.*

EXAMPLE 1.7. Let Γ be the digraph shown in Figure 1.1. Since $\deg^{-}(1) = 0$, vertex 1 is in every zero forcing set. But unless another vertex is in the set, no forces can be performed, so $\mathrm{Z}(\Gamma) \geq 2$. Since $\{1, 2\}$ is a zero forcing set, $\mathrm{Z}(\Gamma) = 2$. In fact, $\{1, 2\}$ is the unique minimum zero forcing set, because neither $\{1, 3\}$ nor $\{1, 4\}$ is a zero forcing set. Observe that $\deg^{-}(2) = 1$, but vertex 2 is in the unique minimum zero forcing set.

In Sections 2 and 3, we analyze the effect that deletion of a vertex or an arc has on the minimum rank and the zero forcing number, respectively. In those sections we also establish cut-vertex reduction formulas for minimum rank and zero forcing

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number. In Section 4, we characterize simple digraphs whose zero forcing number is very low or very high, and relate this to extreme values for minimum rank and maximum nullity.

2. Vertex spread and cut-vertex reduction for minimum rank. The terminology for spreads in the literature is that 'rank spread' means the spread of minimum rank when deleting a vertex, whereas the spread of minimum rank when deleting an edge is called 'rank edge spread,' and we follow this convention.

2.1. Rank spread. The effect of the deletion of a vertex v in a simple undirected graph G on minimum rank is studied in [4], where the rank spread of G at v is defined to be $\mathrm{r}_v(G) = \mathrm{mr}(G) - \mathrm{mr}(G-v)$. Similarly, we define the *rank spread of* Γ *at* v to be $\mathrm{r}_v(\Gamma) = \mathrm{mr}(\Gamma) - \mathrm{mr}(\Gamma - v)$.

OBSERVATION 2.1. If $\Gamma = (V, E)$ is a simple digraph and $v \in V$, then $0 \leq \mathrm{r}_v(\Gamma) \leq 2$.

We will often partition an $n \times n$ matrix A into a 2×2 block matrix as

$$A = \begin{bmatrix} a & \mathbf{w}^{\mathrm{T}} \\ \mathbf{z} & A' \end{bmatrix}, \quad \text{where } a \in \mathbb{R},\ \mathbf{z}, \mathbf{w} \in \mathbb{R}^{n-1},\ A' \in \mathbb{R}^{n-1 \times n-1}. \tag{2.1}$$

If $A \in \mathcal{M}(\Gamma)$, then $A' \in \mathcal{M}(\Gamma - v)$, where v is the first vertex in the ordering of the vertices of Γ and $\Gamma - v$ inherits a vertex ordering from Γ. We use this partitioned form several times throughout this paper.

DEFINITION 2.2. Let $\Gamma = (V, E)$ be a simple digraph, v be a vertex of Γ, and let $(v_1, \ldots, v_{n-1})$ be an ordering of the vertices of $\Gamma - v$. A vector $\mathbf{z} = [z_j] \in \mathbb{R}^{n-1}$ has the *in-pattern* (respectively, *out-pattern*) *of* v when $z_j \neq 0$ if and only if $(v_j, v) \in E$ (respectively, $(v, v_j) \in E$) for all $j = 1, \ldots, n-1$.

DEFINITION 2.3. Let Γ be a simple digraph and v be a vertex of Γ, and choose an ordering on the vertices of $\Gamma - v$. We define two properties that Γ may satisfy:

(C) There exists a matrix $A' \in \mathcal{M}(\Gamma - v)$ with $\operatorname{rank} A' = \mathrm{mr}(\Gamma - v)$ and a vector $\mathbf{z}$ in range A' that has the in-pattern of v.
(R) There exists a matrix $A' \in \mathcal{M}(\Gamma - v)$ with $\operatorname{rank} A' = \mathrm{mr}(\Gamma - v)$ and a vector $\mathbf{w}$ in range ${A'}^{\mathrm{T}}$ that has the out-pattern of v.

The *spread type* of Γ at vertex v, denoted by $\mathrm{type}_v(\Gamma)$, is the subset of $\{\mathrm{C}, \mathrm{R}\}$ such that $\mathrm{C} \in \mathrm{type}_v(\Gamma)$ if and only if Γ satisfies condition (C), and similarly for (R).

THEOREM 2.4. *Let* Γ *be a simple digraph and let* v *be a vertex of* Γ. *To simplify the notation, we assume* v *is the first vertex in the ordering of the vertices of* Γ. *Then*

(1) The following are equivalent:
(a) $\mathrm{r}_v(\Gamma) = 0$.

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(b) There exists a matrix $A' \in \mathcal{M}(\Gamma - v)$ *and vectors* $\mathbf{z} \in \operatorname{range} A'$ *and* $\mathbf{w} \in \operatorname{range} {A'}^{\mathrm{T}}$ *such that* $\operatorname{rank} A' = \operatorname{mr}(\Gamma - v)$*,* $\mathbf{z}$ *has the in-pattern of* v*, and* $\mathbf{w}$ *has the out-pattern of* v*.*

(c) There exists $A \in \mathcal{M}(\Gamma)$ *having the form* $A = \begin{bmatrix} \mathbf{y}^{\mathrm{T}} A'\mathbf{x} & \mathbf{y}^{\mathrm{T}} A' \\ A'\mathbf{x} & A' \end{bmatrix}$*, where* $\operatorname{rank} A' = \operatorname{mr}(\Gamma - v)$*.*

In this case, $\operatorname{type}_v(\Gamma) = \{\mathrm{C}, \mathrm{R}\}$*.*

(2) $\mathrm{r}_v(\Gamma) = 1$ *if and only if one of the following is true.*

(i) $\operatorname{type}_v(\Gamma) = \{\mathrm{C}\}$*. In this case, there exists* $A \in \mathcal{M}(\Gamma)$ *of the form* $A = \begin{bmatrix} a & \mathbf{w}^{\mathrm{T}} \\ A'\mathbf{x} & A' \end{bmatrix}$ *and* $\operatorname{rank} A' = \operatorname{mr}(\Gamma - v)$*.*

(ii) $\operatorname{type}_v(\Gamma) = \{\mathrm{R}\}$*. In this case, there exists* $A \in \mathcal{M}(\Gamma)$ *of the form* $A = \begin{bmatrix} a & \mathbf{y}^{\mathrm{T}} A' \\ \mathbf{z} & A' \end{bmatrix}$ *and* $\operatorname{rank} A' = \operatorname{mr}(\Gamma - v)$*.*

(iii) $\operatorname{type}_v(\Gamma) = \{\mathrm{C}, \mathrm{R}\}$ *and* $\mathrm{r}_v(\Gamma) \neq 0$*. In this case, there is a matrix* A' *realizing property (C) and a different* A' *realizing property (R), but no one* A' *allows both the in-pattern of* v *for* $\mathbf{z} \in \operatorname{range} A'$ *and the out-pattern of* v *for* $\mathbf{w} \in \operatorname{range} {A'}^{\mathrm{T}}$*.*

(iv) $\operatorname{type}_v(\Gamma) = \emptyset$ *and there exists a matrix* $A \in \mathcal{M}(\Gamma)$ *of the form* $A = \begin{bmatrix} a & \mathbf{w}^{\mathrm{T}} \\ \mathbf{z} & A' \end{bmatrix}$ *such that* $\operatorname{rank} A' = \operatorname{mr}(\Gamma - v) + 1$*,* $\mathbf{z} \in \operatorname{range} A'$*, and* $\mathbf{w} \in \operatorname{range} {A'}^{\mathrm{T}}$*. In this case,* $A = \begin{bmatrix} \mathbf{y}^{\mathrm{T}} A'\mathbf{x} & \mathbf{y}^{\mathrm{T}} A' \\ A'\mathbf{x} & A' \end{bmatrix}$ *for some* $\mathbf{x}, \mathbf{y}$*.*

(3) $\mathrm{r}_v(\Gamma) = 2$ *if and only if* $\operatorname{type}_v(\Gamma) = \emptyset$ *and there does not exist a matrix* $A \in \mathcal{M}(\Gamma)$ *of the form* $A = \begin{bmatrix} a & \mathbf{w}^{\mathrm{T}} \\ \mathbf{z} & A' \end{bmatrix}$ *such that* $\operatorname{rank} A' = \operatorname{mr}(\Gamma - v) + 1$*,* $\mathbf{z} \in \operatorname{range} A'$*, and* $\mathbf{w} \in \operatorname{range} {A'}^{\mathrm{T}}$*. Equivalently, for* $A = \begin{bmatrix} a & \mathbf{w}^{\mathrm{T}} \\ \mathbf{z} & A' \end{bmatrix} \in \mathcal{M}(\Gamma)$*, (I)* $\operatorname{rank} A' \geq \operatorname{mr}(\Gamma - v) + 2$*, or (II)* $\operatorname{rank} A' = \operatorname{mr}(\Gamma - v) + 1$ *and (*$\mathbf{z} \notin \operatorname{range} A'$ *or* $\mathbf{w} \notin \operatorname{range} {A'}^{\mathrm{T}}$*), or (III)* $\operatorname{rank} A' = \operatorname{mr}(\Gamma - v)$ *and* $\mathbf{z} \notin \operatorname{range} A'$ *and* $\mathbf{w} \notin \operatorname{range} {A'}^{\mathrm{T}}$*.*

Proof.

(1) Since $\mathbf{z} \in \operatorname{range} A'$ if and only if there exists $\mathbf{x}$ such that $\mathbf{z} = A'\mathbf{x}$, conditions (b) and (c) are equivalent. Condition (c) implies (a) because of the structure of the matrix A. Suppose $\mathrm{r}_v(\Gamma) = 0$. Choose A such that $\operatorname{rank} A = \operatorname{mr}(\Gamma) = \operatorname{mr}(\Gamma - v)$ and partition A in the form (2.1). Since $\operatorname{mr}(\Gamma - v) \leq \operatorname{rank} A' \leq \operatorname{rank} A = \operatorname{mr}(\Gamma - v)$, $\operatorname{rank} A' = \operatorname{rank} A$. Therefore, $\mathbf{z} \in \operatorname{range} A'$ and $\mathbf{w} \in \operatorname{range} {A'}^{\mathrm{T}}$, so condition (b) is satisfied. The characterization of the type is clear, as condition (b) implies $\operatorname{type}_v(\Gamma) = \{\mathrm{C}, \mathrm{R}\}$.

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(2) For Subcases (i) and (ii), the characterization of the form is immediate from the type hypothesis. In Subcase (iii), the assertions that separate matrices realize conditions (C) and (R) follows type hypothesis together with the rank spread nonzero, using Case (1).
Since $\mathrm{r}_v(\Gamma) = 0$ requires $\mathrm{type}_v(\Gamma) = \{\mathrm{C}, \mathrm{R}\}$, in all four Subcases (i) – (iv), $\mathrm{r}_v(\Gamma) > 0$. Since each subcase allows the construction of a matrix $A \in \mathcal{M}(\Gamma)$ with $\operatorname{rank} A = \mathrm{mr}(\Gamma - v) + 1$, each Subcase (i) – (iv) implies $\mathrm{r}_v(\Gamma) = 1$.
Suppose $\mathrm{r}_v(\Gamma) = 1$. Since $\mathrm{type}_v(\Gamma) \subseteq \{\mathrm{C}, \mathrm{R}\}$, we have one of Subcases (i), (ii), (iii), or $\mathrm{type}_v(\Gamma) = \emptyset$. Suppose $\mathrm{type}_v(\Gamma) = \emptyset$. Let $A \in \mathcal{M}(\Gamma)$ with $\operatorname{rank} A = \mathrm{mr}(\Gamma)$, and partition A in the form (2.1). Since $\mathrm{type}_v(\Gamma) = \emptyset$, $\mathrm{mr}(\Gamma - v) + 1 \le \operatorname{rank} A'$, and $\mathrm{r}_v(\Gamma) = 1$ implies $\operatorname{rank} A' \le \mathrm{mr}(\Gamma - v) + 1$, so $\operatorname{rank} A' = \mathrm{mr}(\Gamma - v) + 1$. Thus, $\operatorname{rank} A = \operatorname{rank} A'$, and necessarily A has the specified form.

(3) Since $\mathrm{r}_v(\Gamma) \le 2$, the characterization of $\mathrm{r}_v(\Gamma) = 2$ follows from (1) and (2), and the equivalent characterization is clear. $\square$

The following three examples show all four possibilities for $\mathrm{type}_v(\Gamma)$ may occur if $\mathrm{r}_v(\Gamma) = 1$.

FIG. 2.1. *An example that demonstrates spread type* $\emptyset$ *for rank spread* 1.

EXAMPLE 2.5. $\mathrm{r}_v(\Gamma) = 1$ and $\mathrm{type}_v(\Gamma) = \emptyset$: Let Γ be the simple digraph shown in Figure 2.1 and consider the vertex labeled v. It is easy to see that $\mathrm{mr}(\Gamma) = 1$ and $\mathrm{mr}(\Gamma - v) = 0$, so $\mathrm{r}_v(\Gamma) = 1$. Partition A in the form (2.1) with the first row and column corresponding to v. Since $\mathrm{mr}(\Gamma - v) = 0$, $A' = [0]$ for any matrix A' such that $\operatorname{rank} A' = \mathrm{mr}(\Gamma - v)$. A vector $A'\mathbf{x}$ has the in-pattern of v if and only if its one entry is nonzero. Thus, Γ does not satisfy condition (C) and $\mathrm{C} \notin \mathrm{type}_v(\Gamma)$. Similarly $\mathrm{R} \notin \mathrm{type}_v(\Gamma)$, and $\mathrm{type}_v(\Gamma) = \emptyset$.

FIG. 2.2. *An example that demonstrates spread types* $\{\mathrm{R}\}$ *and* $\{\mathrm{C}\}$ *for rank spread* 1.

EXAMPLE 2.6. $\mathrm{r}_v(\Gamma) = 1$ and ($\mathrm{type}_v(\Gamma) = \{\mathrm{C}\}$ or $\mathrm{type}_v(\Gamma) = \{\mathrm{R}\}$): Let Γ be the simple digraph as shown in Figure 2.2. It is easy to see that $\mathrm{mr}(\Gamma) = 1$ and $\mathrm{mr}(\Gamma - v) = 0$, so $\mathrm{r}_v(\Gamma) = 1$. Partition A in the form (2.1) with the first row and column corresponding to v. Since $\mathrm{mr}(\Gamma - v) = 0$, $\operatorname{rank} A' = \mathrm{mr}(\Gamma - v)$ implies $A' = [0]$. Then for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^1$, $\mathbf{x}A' = 0$ has the in-pattern of v and $\mathbf{y}^{\mathrm{T}} A' = 0$ does not have the out-pattern of v. Therefore, only (C) holds for v so $\mathrm{type}_v(\Gamma) = \{\mathrm{C}\}$. Similar reasoning shows $\mathrm{type}_u(\Gamma) = \{\mathrm{R}\}$.

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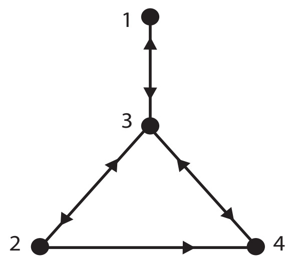

FIG. 2.3. *An example that demonstrates spread type* $\{\mathrm{C}, \mathrm{R}\}$ *for rank spread* 1.

EXAMPLE 2.7. $\mathrm{r}_v(\Gamma) = 1$ and $\mathrm{type}_v(\Gamma) = \{\mathrm{C}, \mathrm{R}\}$:

Let Γ be the simple digraph shown in Figure 2.3 with vertices in numerical order, and consider the vertex $v = 1$. It is straightforward to check that $\mathrm{mr}(\Gamma - v) = 2$. The nonzero pattern of Γ is $\begin{bmatrix} ? & 0 & * & 0 \\ 0 & ? & * & * \\ * & * & ? & * \\ 0 & 0 & * & ? \end{bmatrix}$, where $*$ entries must be nonzero and diagonal entries (labeled ?) may take any real value. Let $A'_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $A'_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Clearly, $A'_j \in \mathcal{M}(\Gamma - v)$ and $\mathrm{rank}\, A'_j = 2 = \mathrm{mr}(\Gamma - v)$ for $j = 1, 2$. Matrix A'_1 shows $\mathrm{C} \in \mathrm{type}_v(\Gamma)$ and A'_2 shows $\mathrm{R} \in \mathrm{type}_v(\Gamma)$, so $\mathrm{type}_v(\Gamma) = \{\mathrm{C}, \mathrm{R}\}$.

If we show $A \in \mathcal{M}(\Gamma)$ implies $\mathrm{rank}\, A \geq 3$, then $\mathrm{r}_v(\Gamma) = 1$. If $A \in \mathcal{M}(\Gamma)$, then A has the form $A = \begin{bmatrix} x & 0 & a & 0 \\ 0 & y & b & c \\ d & e & z & f \\ 0 & 0 & g & w \end{bmatrix}$ for some $a, b, c, d, e, f, g \neq 0$ and $x, y, z, w \in \mathbb{R}$. By considering the last two columns, we see that rows one and two are linearly independent since $a, c \neq 0$. We assume that $\mathrm{rank}\, A = 2$ to derive a contradiction. In this case, row three must be a linear combination of rows one and two. Therefore, since $d, e \neq 0$, we must have $x, y \neq 0$. Since row four must also be a linear combination of rows one and two, we must have $g = w = 0$, which contradicts the fact that $g \neq 0$. This contradiction proves $\mathrm{rank}\, A \geq 3$ and thus completes the argument.

2.2. Cut-vertex reduction. In a simple digraph Γ, we say that a vertex v is a *cut-vertex* if the underlying undirected graph of Γ is connected but becomes disconnected when v is removed. Let $V(\Gamma - v) = \dot\cup_{j=1}^{h} W_j$ be a partition of the

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vertices, with each W_j being the vertices of one or more components of $\Gamma - v$ (so there are no edges between vertices in W_j and W_k for $j \neq k$). Denote by Γ_j the subgraph induced by $W_j \cup \{v\}$. We use this notation throughout when a cut-vertex is involved. Clearly $\mathrm{mr}(\Gamma - v) = \sum_{j=1}^{h} \mathrm{mr}(\Gamma_j - v)$.

In [4], the rank spread for cut-vertices in a simple undirected graph G is characterized. The next theorem characterizes the rank spread of cut-vertices in simple directed graphs.

THEOREM 2.8. *Let $\Gamma = (V, E)$ be a simple digraph and v be a cut-vertex of Γ. Let $V(\Gamma - v) = \dot{\cup}_{j=1}^{h} W_j$ and let Γ_j be the subgraph induced by $W_j \cup \{v\}$. Then*

(1) $\mathrm{r}_v(\Gamma) = 0$ *if and only if* $\mathrm{r}_v(\Gamma_j) = 0$ *for all* j.

(2) $\mathrm{r}_v(\Gamma) = 1$ *if and only if* *(a)* $\mathrm{r}_v(\Gamma_j) \leq 1$ *for all* $j, \mathrm{r}_v(\Gamma_k) = 1$ *for some* k, *and*

$$\bigcap_{j=1}^{h} \mathrm{type}_v(\Gamma_j) \neq \emptyset,$$

or (b) $\mathrm{r}_v(\Gamma_k) = 1$ *for some* k *and* $\mathrm{r}_v(\Gamma_j) = 0$ *for all* $j \neq k$.

(3) $\mathrm{r}_v(\Gamma) = 2$ *if and only if* *(i)* $\mathrm{r}_v(\Gamma_k) = 2$ *for some* k,

or (ii) $\mathrm{r}_v(\Gamma_k) = \mathrm{r}_v(\Gamma_\ell) = 1$ *and* $\mathrm{type}_v(\Gamma_k) \cap \mathrm{type}_v(\Gamma_\ell) = \emptyset$ *for some* $k \neq \ell$.

Proof. By ordering the vertices so that v is the first vertex, the vertices of W_1 are next, then the vertices of W_2, etc., a matrix $A \in \mathcal{M}(\Gamma)$ can be written in the form

$$A = \begin{bmatrix} a & \mathbf{w}^{\mathrm{T}} \\ \mathbf{z} & A' \end{bmatrix} = \begin{bmatrix} a & \mathbf{w}_1^{\mathrm{T}} & \cdots & \mathbf{w}_h^{\mathrm{T}} \\ \mathbf{z}_1 & A_1' & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{z}_h & \mathbf{0} & \cdots & A_h' \end{bmatrix}, \tag{2.2}$$

where $A_j' \in \mathcal{M}(\Gamma_j - v)$, $j = 1, \ldots, h$. It suffices to prove Cases (1) and (2).

Case 1: Suppose $\mathrm{r}_v(\Gamma_j) = 0$ for all $j = 1, \ldots, h$. By Theorem 2.4, we can find matrices $A_j = \begin{bmatrix} a_j & \mathbf{w}_j^{\mathrm{T}} \\ \mathbf{z}_j & A_j' \end{bmatrix} \in \mathcal{M}(\Gamma_j)$ with $\mathrm{rank}\, A_j' = \mathrm{mr}(\Gamma_j - v)$, $\mathbf{z}_j \in \mathrm{range}\, A_j'$ and $\mathbf{w}_j \in \mathrm{range}\, {A_j'}^{\mathrm{T}}$. We can then construct a matrix as in (2.2) with $a = \sum_{j=1}^{h} a_j$. Thus, $\mathbf{z} \in \mathrm{range}\, A'$, $\mathbf{w} \in \mathrm{range}\, {A'}^{\mathrm{T}}$, and

$$\mathrm{rank}\, A' = \sum_{j=1}^{h} \mathrm{rank}\, A_j' = \sum_{j=1}^{h} \mathrm{mr}(\Gamma_j - v) = \mathrm{mr}(\Gamma - v).$$

Therefore, by Theorem 2.4, we conclude $\mathrm{r}_v(\Gamma) = 0$. Conversely, suppose $\mathrm{r}_v(\Gamma) = 0$. By Theorem 2.4, there exists a matrix A of the form (2.2) such that $\mathrm{rank}\, A' = \mathrm{mr}(\Gamma - v)$, $\mathbf{z} \in \mathrm{range}\, A'$, and $\mathbf{w} \in \mathrm{range}\, {A'}^{\mathrm{T}}$. Therefore, for each j, $\mathbf{z}_j \in \mathrm{range}\, A_j'$,

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$\mathbf{w}_j \in \operatorname{range} A_j'^{\mathrm{T}}$. Furthermore, $\sum_{j=1}^h \operatorname{rank} A_j' = \sum_{j=1}^h \operatorname{mr}(\Gamma_j - v)$ and since $A_j' \in \mathcal{M}(\Gamma_j)$, $\operatorname{rank} A_j' \geq \operatorname{mr}(\Gamma_j - v)$ for all j. Thus, $\operatorname{rank} A_j' = \operatorname{mr}(\Gamma_j - v)$ for all j. Applying Theorem 2.4 again, we have $\mathrm{r}_v(\Gamma_j) = 0$ for each $j = 1, \ldots, h$.

Case 2: To show that (a) or (b) implies $\mathrm{r}_v(\Gamma) = 1$, in each case we construct a matrix $A \in \mathcal{M}(\Gamma)$ of rank at most $\operatorname{mr}(\Gamma - v) + 1$, so $\mathrm{r}_v(\Gamma) \leq 1$. Since for (a) or (b) there exists some k such that $\mathrm{r}_v(\Gamma_k) = 1$, $\mathrm{r}_v(\Gamma) \neq 0$ by Case 1, and so $\mathrm{r}_v(\Gamma) = 1$. Suppose first that (a) is true. Without loss of generality, suppose $\mathrm{C} \in \bigcap_{j=1}^h \operatorname{type}_v(\Gamma_j)$. Then for each j, there exists a matrix $A_j = \begin{bmatrix} a_j & \mathbf{w}_j^{\mathrm{T}} \\ \mathbf{z}_j & A_j' \end{bmatrix} \in \mathcal{M}(\Gamma_j)$ with $\operatorname{rank} A_j' = \operatorname{mr}(\Gamma_j - v)$, and $\mathbf{z}_j \in \operatorname{range} A_j'$. Then we can construct a matrix of the form (2.2), where $A' = A_1' \oplus \cdots \oplus A_h'$, $\mathbf{z} = [\mathbf{z}_1^{\mathrm{T}},\ \ldots,\ \mathbf{z}_h^{\mathrm{T}}]^{\mathrm{T}} \in \operatorname{range} A'$, $\mathbf{w} = [\mathbf{w}_1^{\mathrm{T}},\ \ldots,\ \mathbf{w}_h^{\mathrm{T}}]^{\mathrm{T}}$, and $a \in \mathbb{R}$. Clearly $\operatorname{mr}(\Gamma) \leq \operatorname{rank} A \leq \operatorname{rank} A' + 1 = \sum_{j=1}^h \operatorname{rank} A_j' + 1 = \sum_{j=1}^h \operatorname{mr}(\Gamma_j - v) + 1 = \operatorname{mr}(\Gamma - v) + 1$. Now suppose (b) is true. Then for $j = 1, \ldots, h$ we can find matrices $A_j = \begin{bmatrix} a_j & \mathbf{w}_j^{\mathrm{T}} \\ \mathbf{z}_j & A_j' \end{bmatrix} \in \mathcal{M}(\Gamma_j)$ such that $\operatorname{rank} A_j = \operatorname{mr}(\Gamma_j) = \operatorname{mr}(\Gamma_j - v)$ for $j \neq k$, and $\operatorname{rank} A_k = \operatorname{mr}(\Gamma_k) = \operatorname{mr}(\Gamma_k - v) + 1$. Again we can construct a matrix of the form (2.2) with $A' = A_1' \oplus \cdots \oplus A_h'$, $\mathbf{z} = [\mathbf{z}_1^{\mathrm{T}},\ \ldots,\ \mathbf{z}_h^{\mathrm{T}}]^{\mathrm{T}}$, $\mathbf{w} = [\mathbf{w}_1^{\mathrm{T}},\ \ldots,\ \mathbf{w}_h^{\mathrm{T}}]^{\mathrm{T}}$, and $a = \sum_{j=1}^h a_j$. Thus,

$$\operatorname{mr}(\Gamma) \leq \operatorname{rank} A \leq \sum_{j=1}^h \operatorname{rank} A_j = \sum_{j=1}^h \operatorname{mr}(\Gamma_j) = \sum_{j=1}^h \operatorname{mr}(\Gamma_j - v) + 1 = \operatorname{mr}(\Gamma - v) + 1.$$

Conversely, suppose $\mathrm{r}_v(\Gamma) = 1$. First we show that for all j, $\mathrm{r}_v(\Gamma_j) \leq 1$. So suppose there exists a subgraph Γ_k such that $\mathrm{r}_v(\Gamma_k) = 2$. Let A be of the form (2.2) such that $\operatorname{rank} A = \operatorname{mr}(\Gamma) = \operatorname{mr}(\Gamma - v) + 1$. Since $\mathrm{r}_v(\Gamma_k) = 2$, $\operatorname{rank} A_k \geq \operatorname{mr}(\Gamma_k) = \operatorname{mr}(\Gamma_k - v) + 2$. Then by Theorem 2.4, (I) $\operatorname{rank} A_k' \geq \operatorname{mr}(\Gamma_k - v) + 2$, or (II) $\operatorname{rank} A_k' = \operatorname{mr}(\Gamma_k - v) + 1$ and ($\mathbf{z}_k \notin \operatorname{range} A_k'$ or $\mathbf{w}_k \notin \operatorname{range} {A_k'}^{\mathrm{T}}$), or (III) $\operatorname{rank} A_k' = \operatorname{mr}(\Gamma_k - v)$ and $\mathbf{z}_k \notin \operatorname{range} A_k'$ and $\mathbf{w}_k \notin \operatorname{range} {A_k'}^{\mathrm{T}}$.

In case (I),

$$\operatorname{mr}(\Gamma) = \operatorname{rank} A \geq \sum_{j=1}^h \operatorname{rank} A_j' \geq \sum_{j=1}^h \operatorname{mr}(\Gamma_j - v) + 2 = \operatorname{mr}(\Gamma - v) + 2,$$

and in case (II),

$$\operatorname{mr}(\Gamma) = \operatorname{rank} A \geq \sum_{j=1}^h \operatorname{rank} A_j' + 1 \geq \sum_{j=1}^h \operatorname{mr}(\Gamma_j - v) + 1 + 1 = \operatorname{mr}(\Gamma - v) + 2,$$

both contradicting $\mathrm{r}_v(\Gamma) = 1$. In case (III), $\mathbf{z} \notin \operatorname{range} A'$ and $\mathbf{w} \notin \operatorname{range} {A'}^{\mathrm{T}}$ contradicting Theorem 2.4. Therefore, for all j, $\mathrm{r}_v(\Gamma_j) \leq 1$ and by Case 1, there exists k such that $\mathrm{r}_v(\Gamma_k) = 1$.

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We have the following four subcases:

(i) Suppose $\mathrm{type}_v(\Gamma) = \{\mathrm{C}\}$. By Theorem 2.4 there exists a matrix $A \in \mathcal{M}(\Gamma)$ of the form (2.1) such that $A' \in \mathcal{M}(\Gamma - v)$, $\operatorname{rank} A' = \mathrm{mr}(\Gamma - v)$ and $\mathbf{z} \in \operatorname{range} A'$. Partition A as in (2.2). Since $\sum_{j=1}^{h} \mathrm{mr}(\Gamma_j - v) = \mathrm{mr}(\Gamma - v) = \operatorname{rank} A' = \sum_{j=1}^{h} \operatorname{rank} A'_j$ and $\operatorname{rank} A'_j \geq \mathrm{mr}(\Gamma_j - v)$ for all j, $\operatorname{rank} A'_j = \mathrm{mr}(\Gamma_j - v)$ for all $j = 1, \ldots, h$. Since $\mathbf{z} \in \operatorname{range} A'$, $\mathbf{z}_j \in \operatorname{range} A'_j$. Therefore, $\{\mathrm{C}\} \subseteq \mathrm{type}_v(\Gamma_j)$ for all j, which implies that $\bigcap_{j=1}^{h} \mathrm{type}_v(\Gamma_j) \neq \emptyset$.

(ii) $\mathrm{type}_v(\Gamma) = \{\mathrm{R}\}$, follows similarly to (i).

(iii) Suppose $\mathrm{type}_v(\Gamma) = \{\mathrm{R}, \mathrm{C}\}$. From the previous cases, it follows that $\bigcap_{j=1}^{h} \mathrm{type}_v(\Gamma_j) \neq \emptyset$.

(iv) Suppose $\mathrm{type}_v(\Gamma) = \emptyset$. By Theorem 2.4, there exists a matrix $A \in \mathcal{M}(\Gamma)$ of the form (2.1) such that $\operatorname{rank} A' = \mathrm{mr}(\Gamma - v) + 1$, $\mathbf{z} \in \operatorname{range} A'$, and $\mathbf{w} \in \operatorname{range} A'^{\mathrm{T}}$. Partition A as in (2.2), so $\mathbf{z}_j \in \operatorname{range} A'_j$ and $\mathbf{w}_j \in \operatorname{range} {A'_j}^{\mathrm{T}}$ for all $j = 1, \ldots, h$. Since $\mathrm{mr}(\Gamma - v) + 1 = \mathrm{mr}(\Gamma) = \operatorname{rank} A' = \sum_{j=1}^{h} \operatorname{rank} A'_j$ and $\operatorname{rank} A'_j \geq \mathrm{mr}(\Gamma_j - v)$ for all j, there exists k such that $\operatorname{rank} A'_k = \mathrm{mr}(\Gamma_k - v) + 1$ and for all $j \neq k$, $\operatorname{rank} A'_j = \mathrm{mr}(\Gamma_j - v)$. Then by Theorem 2.4, $\mathrm{r}_v(\Gamma_j) = 0$ for all $j \neq k$. Since $\mathrm{r}_v(\Gamma) = 1$, $\mathrm{r}_v(\Gamma_k) = 1$. $\square$

Corollary 2.9. *Let $\Gamma = (V, E)$ be a simple digraph and v be a cut-vertex of Γ. Let $V(\Gamma - v) = \dot{\cup}_{j=1}^{h} W_j$ and let Γ_j be the subdigraph induced by $W_j \cup \{v\}$. If $\mathrm{r}_v(\Gamma_1) = 0$, then*

$$\mathrm{r}_v(\Gamma) = \mathrm{r}_v(\Gamma - W_1) \textit{ and } \mathrm{mr}(\Gamma) = \mathrm{mr}(\Gamma_1) + \mathrm{mr}(\Gamma - W_1).$$

2.3. Cut-arc reduction. Suppose that Γ_1 and Γ_2 are simple digraphs and let v_1 and v_2 be vertices of Γ_1 and Γ_2, respectively. If we connect Γ_1 and Γ_2 by adding the arc $e = (v_1, v_2)$, the resulting simple digraph Γ is the arc sum of Γ_1 and Γ_2, and is denoted by $\Gamma = \Gamma_1 +_e \Gamma_2$. A simple digraph $\Gamma = \Gamma_1 +_e \Gamma_2$ where $e = (v_1, v_2)$ clearly has cut-vertices v_1 and v_2, and cut-vertex reduction can be applied. In this section, we summarize the results of doing so, in terms of the minimum ranks of Γ_1 and Γ_2.

Lemma 2.10. *Let Γ be a digraph, v be a vertex of Γ, and u be a vertex not in Γ. For the digraph Γ' obtained by appending the vertex u and the arc (v, u) to Γ,*

$$\mathrm{mr}(\Gamma') = \begin{cases} \mathrm{mr}(\Gamma) & \textit{if} \quad \mathrm{r}_v(\Gamma) = 2, \textit{ or} \\ & \qquad \mathrm{r}_v(\Gamma) = 1 \textit{ and } \mathrm{C} \in \mathrm{type}_v(\Gamma); \\ \mathrm{mr}(\Gamma) + 1 & \textit{otherwise.} \end{cases} \tag{2.3}$$

In case Γ' is obtained by appending the vertex u and the arc (u, v) to Γ, in (2.3) the condition $\mathrm{C} \in \mathrm{type}_v(\Gamma)$ is replaced by $\mathrm{R} \in \mathrm{type}_v(\Gamma)$.

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Proof. We apply Theorem 2.8 to cut-vertex v with partition $W_1 = V(\Gamma) - v$ and $W_2 = \{u\}$, so $\Gamma_1 = \Gamma$ and Γ_2 is the single-arc subdigraph of Γ induced by $\{v, u\}$. Observe that $\mathrm{r}_v(\Gamma_2) = 1$ and $\mathrm{type}_v(\Gamma_2) = \{\mathrm{C}\}$.

If $\mathrm{r}_v(\Gamma) = 2$, then $\mathrm{r}_v(\Gamma') = 2$, so $\mathrm{mr}(\Gamma') = 2+\mathrm{mr}(\Gamma'-v) = 2+\mathrm{mr}(\Gamma-v) = \mathrm{mr}(\Gamma)$. If $\mathrm{r}_v(\Gamma) = 1$ and $\mathrm{C} \in \mathrm{type}_v(\Gamma)$, then $\mathrm{type}(\Gamma) \cap \mathrm{type}(\Gamma_2) = \{\mathrm{C}\}$, so $\mathrm{r}_v(\Gamma') = 1$. Thus, $\mathrm{mr}(\Gamma') = 1 + \mathrm{mr}(\Gamma' - v) = 1 + \mathrm{mr}(\Gamma - v) = \mathrm{mr}(\Gamma)$.

If $\mathrm{r}_v(\Gamma) = 1$ and $\mathrm{C} \notin \mathrm{type}_v(\Gamma)$, then $\mathrm{type}(\Gamma) \cap \mathrm{type}(\Gamma_2) = \emptyset$, so $\mathrm{r}_v(\Gamma') = 2$. Thus, $\mathrm{mr}(\Gamma') = 2 + \mathrm{mr}(\Gamma' - v) = 2 + \mathrm{mr}(\Gamma - v) = 1 + \mathrm{mr}(\Gamma)$. If $\mathrm{r}_v(\Gamma) = 0$, then $\mathrm{r}_v(\Gamma') = 1$, so $\mathrm{mr}(\Gamma') = 1 + \mathrm{mr}(\Gamma' - v) = 1 + \mathrm{mr}(\Gamma - v) = 1 + \mathrm{mr}(\Gamma)$. ☐

THEOREM 2.11. *Let $\Gamma = \Gamma_1 +_e \Gamma_2$ where $e = (v_1, v_2)$. Then,*

$$\mathrm{mr}(\Gamma) = \begin{cases} \mathrm{mr}(\Gamma_1) + \mathrm{mr}(\Gamma_2) & \textit{if} \quad \mathrm{r}_{v_i}(\Gamma_i) = 2 \textit{ for some } i, \textit{ or} \\ & \qquad \mathrm{r}_{v_1}(\Gamma_1) = 1 \textit{ and } \mathrm{C} \in \mathrm{type}_{v_1}(\Gamma_1), \textit{ or} \\ & \qquad \mathrm{r}_{v_2}(\Gamma_2) = 1 \textit{ and } \mathrm{R} \in \mathrm{type}_{v_2}(\Gamma_2); \\ \mathrm{mr}(\Gamma_1) + \mathrm{mr}(\Gamma_2) + 1 & \textit{otherwise.} \end{cases}$$

Proof. Let Γ_1' be the digraph induced by $V(\Gamma_1) \cup \{v_2\}$ and Γ_2' be the digraph induced by $V(\Gamma_2) \cup \{v_1\}$. We apply Theorem 2.8 and Corollary 2.9 with v_1 or v_2 as the cut-vertex.

If $\mathrm{r}_{v_1}(\Gamma_1) = 2$ or if $\mathrm{r}_{v_1}(\Gamma_1) = 1$ and $\mathrm{C} \in \mathrm{type}_{v_1}(\Gamma_1)$, then $\mathrm{mr}(\Gamma_1') = \mathrm{mr}(\Gamma_1)$ by Lemma 2.10, so $\mathrm{r}_{v_2}(\Gamma_1') = \mathrm{mr}(\Gamma_1') - \mathrm{mr}(\Gamma_1) = 0$. Therefore, by Corollary 2.9 applied to cut-vertex v_2 with first component Γ_1, $\mathrm{mr}(\Gamma) = \mathrm{mr}(\Gamma_1') + \mathrm{mr}(\Gamma_2) = \mathrm{mr}(\Gamma_1) + \mathrm{mr}(\Gamma_2)$. The case $\mathrm{r}_{v_2}(\Gamma_2) = 2$ or $\mathrm{r}_{v_2}(\Gamma_2) = 1$ and $\mathrm{R} \in \mathrm{type}_{v_2}(\Gamma_2)$ is similar.

Now suppose $\mathrm{r}_{v_2}(\Gamma_2) = 0$, and $\mathrm{r}_{v_1}(\Gamma_1) = 0$ or $\mathrm{r}_{v_1}(\Gamma_1) = 1$ with $\mathrm{C} \notin \mathrm{type}_{v_1}(\Gamma_1)$. Then $\mathrm{mr}(\Gamma_1') = \mathrm{mr}(\Gamma_1) + 1$ by Lemma 2.10, so $\mathrm{r}_{v_2}(\Gamma_1') = \mathrm{mr}(\Gamma_1') - \mathrm{mr}(\Gamma_1) = 1$. Therefore, by Corollary 2.9 applied to cut-vertex v_2 with first component $\Gamma_2 - v_2$, $\mathrm{mr}(\Gamma) = \mathrm{mr}(\Gamma_2) + \mathrm{mr}(\Gamma_1') = \mathrm{mr}(\Gamma_2) + \mathrm{mr}(\Gamma_1) + 1$. The case $\mathrm{r}_{v_1}(\Gamma_1) = 0$, $\mathrm{r}_{v_2}(\Gamma_2) = 1$, and $\mathrm{R} \notin \mathrm{type}_{v_2}(\Gamma_2)$ is similar.

Finally, we consider the case where $\mathrm{r}_{v_1}(\Gamma_1) = 1$, $\mathrm{r}_{v_2}(\Gamma_2) = 1$, $\mathrm{C} \notin \mathrm{type}_{v_1}(\Gamma_1)$, and $\mathrm{R} \notin \mathrm{type}_{v_2}(\Gamma_2)$. By Lemma 2.10, $\mathrm{mr}(\Gamma_2') = \mathrm{mr}(\Gamma_2) + 1$. Thus, $\mathrm{r}_{v_1}(\Gamma_2') = 1$; since for any matrix realizing Γ_2, an all zero column can be used for v_1, necessarily $\mathrm{R} \notin \mathrm{type}_{v_1}(\Gamma_2')$. Since $\mathrm{r}_{v_1}(\Gamma_1) = 1$ and $\mathrm{C} \notin \mathrm{type}_{v_1}(\Gamma_1)$, by Theorem 2.8 applied to cut-vertex v_1, $\mathrm{r}_{v_1}(\Gamma) = 2$. Then

$$\begin{aligned} \mathrm{mr}(\Gamma) &= 1 + \mathrm{mr}(\Gamma_1 - v_1) + \mathrm{mr}(\Gamma_2' - v_1) + 1 \\ &= \mathrm{r}_{v_1}(\Gamma_1) + \mathrm{mr}(\Gamma_1 - v_1) + \mathrm{mr}(\Gamma_2) + 1 = \mathrm{mr}(\Gamma_1) + \mathrm{mr}(\Gamma_2) + 1. \quad ☐ \end{aligned}$$

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3. Vertex spread, cut-vertex reduction, and arc spread for zero forcing. In this section, we examine the effect of deleting a vertex or an arc on zero forcing number, and obtain a cut-vertex reduction formula for $\mathrm{Z}(\Gamma)$.

3.1. Zero spread. The effect that the deletion of a vertex v in a simple undirected graph G has on zero forcing number is studied in [6], where the zero spread of G at v is defined to be $\mathrm{z}_v(G) = \mathrm{Z}(G) - \mathrm{Z}(G - v)$. Similarly, we define the *zero spread* of Γ at v to be $\mathrm{z}_v(\Gamma) = \mathrm{Z}(\Gamma) - \mathrm{Z}(\Gamma - v)$. Since $\mathrm{Z}(\Gamma) = \mathrm{Z}(\Gamma^{\mathrm{T}})$ and $(\Gamma - v)^{\mathrm{T}} = \Gamma^{\mathrm{T}} - v$, $\mathrm{z}_v(\Gamma) = \mathrm{z}_v(\Gamma^{\mathrm{T}})$.

Many of the results about vertex spread for simple graphs extend to simple digraphs. Since the proofs of the next four results for simple digraphs are similar to the proofs for simple graphs ([6, Theorem 2.3], [6, Theorem 2.7], [6, Theorem 2.8], and [6, Theorem 2.12]), we omit them.

Proposition 3.1. *For every simple digraph Γ and vertex v of Γ, $-1 \le \mathrm{z}_v(\Gamma) \le 1$.*

Proposition 3.2. *Let $\Gamma = (V, E)$ be a simple digraph and $v \in V$. Then $\mathrm{z}_v(\Gamma) = 1$ if and only if there exists a minimum zero forcing set B of Γ that contains v and a chronological list of forces $\mathcal{F}$ of B such that v does not perform a force.*

Proposition 3.3. *Let $\Gamma = (V, E)$ be a simple digraph and $v \in V$. If $\mathrm{z}_v(\Gamma) = -1$, then $v \notin B$ for all minimum zero forcing sets B of Γ. Equivalently, if $v \in B$ for some minimum zero forcing set B of Γ, then $\mathrm{z}_v(\Gamma) \ge 0$.*

Corollary 3.4. *There does not exist a simple digraph $\Gamma = (V, E)$ such that $\mathrm{z}_v(\Gamma) = -1$ for every $v \in V$.*

Since Proposition 3.2 is an equivalence, it is natural to ask whether the same is true for Proposition 3.3. That is, if v is never in a minimum zero forcing set of Γ, does $\mathrm{z}_v(\Gamma) = -1$? The next example provides a negative answer.

Example 3.5. Let Γ be the simple digraph on two vertices v and u with the one arc (v, u), shown in Figure 2.2. Clearly $\mathrm{Z}(\Gamma) = 1$ and $\mathrm{Z}(\Gamma - u) = 1$, so $z_u(\Gamma) = 0$. However, u can never be in minimum zero forcing set. Indeed, $\{v\}$ is the unique minimum zero forcing set of Γ.

If $\deg^- v = 0$, then v is in every zero forcing set, so $\mathrm{z}_v(\Gamma) \ge 0$, and $\mathrm{z}_v(\Gamma) = 1$ if and only if there is a minimum zero forcing set B and chronological list of forces $\mathcal{F}$ for B in which v does not perform a force. The analogous characterization is also true for vertices with no out-neighbor, as can be seen by considering Γ^T: If $\deg^+_\Gamma v = 0$, then $\mathrm{z}_v(\Gamma) = \mathrm{z}_v(\Gamma^{\mathrm{T}}) \ge 0$ since $\deg^-_{\Gamma^{\mathrm{T}}} v = 0$. Since $\deg^+_\Gamma v = 0$, v can never perform a force, and $\mathrm{z}_v(\Gamma) = 1$ if and only if there is a minimum zero forcing set B containing v.

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3.2. Cut-vertex reduction for zero spread. Throughout this section, Γ is a simple digraph with a cut-vertex v, $W_j \subseteq V(\Gamma)$ is the set of vertices of the jth component of $\Gamma - v$, $j = 1, \ldots, h$, and Γ_j is the subgraph induced by $\{v\} \cup W_j$. We begin our analysis with two basic results. The proofs are similar to the proofs of [10, Lemma 3.1 and Corollary 3.2] and [10, Lemma 3.3 and Corollary 3.4], and we omit the proofs.

Lemma 3.6. *Let Γ be a simple digraph with a cut-vertex v. Then* $\mathrm{Z}(\Gamma) \geq \sum_{j=1}^{h} \mathrm{Z}(\Gamma_j) - h + 1$ *and* $\mathrm{z}_v(\Gamma) \geq \sum_{j=1}^{h} \mathrm{z}_v(\Gamma_j) - h + 1$.

Lemma 3.7. *Let Γ be a simple digraph with a cut-vertex v. Then* $\mathrm{Z}(\Gamma) \leq \min_{1\leq k\leq h}\{\mathrm{Z}(\Gamma_k) + \sum_{j=1,i\neq k}^{h} \mathrm{Z}(\Gamma_j - v)\}$ *and* $\mathrm{z}_v(\Gamma) \leq \min_{1\leq k\leq h} \mathrm{z}_v(\Gamma_k)$.

Although there are similarities between the proofs of the simple digraph cut-vertex reduction theorem (Theorem 3.8 below) and Row's cut-vertex reduction theorem for simple graphs [10, Theorem 3.8], there are also differences caused by the orientation. Let Γ be a simple digraph. A vertex v is *initial* if there exists a minimum zero forcing set B such that $v \in B$. A vertex v is *terminal* if there exists a minimum zero forcing set B and a chronological list of forces $\mathcal{F}$ for B in which v does not perform a force.

Theorem 3.8. *Let Γ be a simple digraph with a cut-vertex v. For $j = 1, \ldots, h$, let $W_j \subseteq V(\Gamma)$ be the vertices of the jth component of $\Gamma - v$ and let Γ_j be the subgraph induced by $\{v\} \cup W_j$. Let $m = \min_{1\leq j\leq h} \mathrm{z}_v(\Gamma_j)$. Then*

$$\mathrm{z}_v(\Gamma) = \begin{cases} 1 & \textit{if and only if} \quad m = 1; \\ -1 & \textit{if and only if} \quad m = -1 \textit{ or} \\ & \qquad (m = 0 \textit{ and there exist } \ell \neq k \textit{ where } v \textit{ is initial in} \\ & \qquad \Gamma_\ell \textit{ and terminal in } \Gamma_k \textit{ and } \mathrm{z}_v(\Gamma_\ell) = \mathrm{z}_v(\Gamma_k) = 0); \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We establish the characterizations for $\mathrm{z}_v(\Gamma) = 1$ and $\mathrm{z}_v(\Gamma) = -1$. Recall that by Proposition 3.1, $-1 \leq \mathrm{z}_v(\Gamma) \leq 1$ and $-1 \leq \mathrm{z}_v(\Gamma_j) \leq 1$ for $j = 1, \ldots, h$. If $\mathrm{z}_v(\Gamma) = 1$, then for all j, $\mathrm{z}_v(\Gamma_j) \geq 1$ by Lemma 3.7, so $\mathrm{z}_v(\Gamma_j) = 1$ for all j, and thus, $m = 1$. If $m = 1$, then $\mathrm{z}_v(\Gamma) \geq 1$ by Lemma 3.6, so $\mathrm{z}_v(\Gamma) = 1$.

Suppose $\mathrm{z}_v(\Gamma) = -1$, so $m \leq 0$ by the above. If $m = -1$, we are done, so we assume $m = 0$. Let B be a minimum zero forcing set for Γ and define $B_j = B \cap V(\Gamma_j)$. Since $\mathrm{z}_v(\Gamma) = -1$, $v \notin B$ by Proposition 3.3, and thus, $|B| = \sum_{j=1}^{h} |B_j|$. Consider a process by which B forces Γ. Since $v \notin B$, v is forced by some vertex u. Without loss of generality we may assume $u \in B_1$. Since $m = 0$, $\mathrm{Z}(\Gamma_1 - v) \leq \mathrm{Z}(\Gamma_1) \leq |B_1|$. If for all $j \geq 2$, v does not force any vertex of Γ_j, then B_j is a zero forcing set for $\Gamma_j - v$ and $\mathrm{Z}(\Gamma_j - v) \leq |B_j|$ for all $j \geq 2$. Thus, if v does not perform a force in some Γ_j with $j \geq 2$, then $\mathrm{Z}(\Gamma) = |B| = \sum_{j=1}^{h} |B_j| \geq \sum_{j=1}^{h} \mathrm{Z}(\Gamma_j - v)$, contradicting

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$\mathrm{z}_v(\Gamma) = -1$. Without loss of generality we may assume v forces w for some $w \in V(\Gamma_2)$. Furthermore $\widetilde{B}_2 := B_2 \cup \{v\}$ is a zero forcing set for Γ_2. Since v can perform at most one force, B_j is a zero forcing set for $\Gamma_j - v$ for $j = 3, \ldots, h$. Thus,

$$\begin{aligned}
-1 = \mathrm{Z}(\Gamma) - \sum_{j=1}^{h} \mathrm{Z}(\Gamma_j - v) &= \sum_{j=1}^{h} |B_j| - \left(\mathrm{Z}(\Gamma_1 - v) + \mathrm{Z}(\Gamma_2 - v) + \sum_{j=3}^{h} |B_j| \right) \\
&= |B_1| - \mathrm{Z}(\Gamma_1 - v) + |B_2| - \mathrm{Z}(\Gamma_2 - v) = |B_1| - \mathrm{Z}(\Gamma_1 - v) + |\widetilde{B}_2| - 1 - \mathrm{Z}(\Gamma_2 - v) \\
&\geq \mathrm{Z}(\Gamma_1) - \mathrm{Z}(\Gamma_1 - v) + \mathrm{Z}(\Gamma_2) - \mathrm{Z}(\Gamma_2 - v) - 1 = \mathrm{z}_v(\Gamma_1) + \mathrm{z}_v(\Gamma_2) - 1 \geq -1,
\end{aligned}$$

since $m = 0$ implies $\mathrm{z}_v(\Gamma_1), \mathrm{z}_v(\Gamma_2) \geq 0$. We must have equality throughout, implying $\mathrm{z}_v(\Gamma_1) = \mathrm{z}_v(\Gamma_2) = 0$ and B_1 and $\widetilde{B}_2$ are minimum zero forcing sets for Γ_1 and Γ_2, respectively.

For the converse, if $m = -1$ then $\mathrm{z}_v(\Gamma) \leq -1$ by Lemma 3.7, so $\mathrm{z}_v(\Gamma) = -1$. So suppose $m = 0$, and without loss of generality, $\mathrm{z}_v(\Gamma_1) = 0$, $\mathrm{z}_v(\Gamma_2) = 0$, v is terminal in Γ_1 with minimum zero forcing set B_1, and $v \in B_2$ where B_2 is a minimum zero forcing set for Γ_2. For $j = 3, \ldots, h$, choose minimum zero forcing sets B_j for $\Gamma_j - v$. Then $B = B_1 \cup (B_2 \setminus \{v\}) \cup B_3 \cup \cdots \cup B_h$ is a zero forcing set for Γ and since $\mathrm{z}_v(\Gamma_1) = 0 = \mathrm{z}_v(\Gamma_2)$,

$$|B| = \sum_{j=1}^{h} |B_j| - 1 = \sum_{j=1}^{h} \mathrm{Z}(\Gamma_j - v) - 1 = \mathrm{Z}(\Gamma - v) - 1.$$

Thus, $\mathrm{z}_v(\Gamma) = -1$. $\square$

3.3. Zero arc spread. The effect of the deletion of an edge e in a simple undirected graph G on zero forcing is studied in [6], where the zero edge spread of G at e is defined to be $\mathrm{z}_e(G) = \mathrm{Z}(G) - \mathrm{Z}(G - e)$. Here we explore a similar series of questions about arc deletion in simple digraphs. For a simple digraph $\Gamma = (V, E)$ and arc $e \in E$, the *zero arc spread* of Γ at e is defined to be $\mathrm{z}_e(\Gamma) = \mathrm{Z}(\Gamma) - \mathrm{Z}(\Gamma - e)$. The proofs of propositions 3.9 and 3.10 below are omitted, as they follow [6, Theorem 2.17] and [6, Theorem 2.21], respectively.

PROPOSITION 3.9. *For every simple digraph Γ and arc e of Γ, $-1 \leq \mathrm{z}_e(\Gamma) \leq 1$.*

PROPOSITION 3.10. *Let $\Gamma = (V, E)$ be a simple digraph and $e \in E$. If $\mathrm{z}_e(\Gamma) = -1$, then for every minimum zero forcing set B of Γ, for every chronological list of forces $\mathcal{F}$ of B, a force is performed along e in $\mathcal{F}$. Equivalently, if there is some chronological list of forces $\mathcal{F}$ such that no force is performed along e in $\mathcal{F}$, then $\mathrm{z}_e(\Gamma) \geq 0$.*

The statement of the next proposition is similar to [6, Theorem 2.23], but a modification of the proof is needed because the proof in [6] relies on the ability to

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exclude any vertex from a minimum zero forcing set.

PROPOSITION 3.11. *Let* $\Gamma = (V, E)$ *be a simple digraph and* $e \in E$*. If* $\mathrm{z}_e(\Gamma) = 1$*, then there exists a minimum zero forcing set* B *and chronological list of forces* $\mathcal{F}$ *for* B *such that no force is performed along* e *in* $\mathcal{F}$*. Equivalently, if for every minimum zero forcing set* B *of* Γ *and for every chronological list of forces* $\mathcal{F}$ *of* B*, a force is performed along* e *in* $\mathcal{F}$*, then* $\mathrm{z}_e(\Gamma) \leq 0$*.*

Proof. We prove the second statement. Let $e = (u, w)$ be an arc such that a force is performed along e in every chronological list of forces for every minimum zero forcing set of Γ. Observe first that w is not in any minimum zero forcing set for Γ. Let B be a minimum zero forcing set for $\Gamma - e$. If $w \in B$, then B is a zero forcing set for Γ, so $\mathrm{Z}(\Gamma) \leq |B| = \mathrm{Z}(\Gamma - e)$. If $w \notin B$, then $B \cup \{w\}$ is a zero forcing set for Γ. Note that $B \cup \{w\}$ cannot be an minimum zero forcing set for Γ since $w \in B \cup \{w\}$ and w is not in any minimum zero forcing set for Γ. Then, since $B \cup \{w\}$ is not minimum, $\mathrm{Z}(\Gamma) \leq |B| = \mathrm{Z}(\Gamma - e)$. Thus, in either case, $\mathrm{z}_e(\Gamma) \leq 0$. ∎

The converse of Proposition 3.11 is false, as the next example shows. A path $(v_1, \ldots, v_k)$ in a simple digraph $\Gamma = (V, E)$ is *Hessenberg* if E does not contain any arc of the form (v_i, v_j) with $j > i + 1$ [8]. An arc of the form (v_i, v_j) with $j < i$ is called a *back-arc* of the Hessenberg path.

EXAMPLE 3.12. Any back-arc e in a Hessenberg path on vertices $v_1, \ldots, v_n$ is an arc such that $\mathrm{z}_e(\Gamma) = 0$. In the chronological list of forces $v_i \to v_{i+1}, i = 1, \ldots, n-1$, where $B = \{v_1\}$, no force is performed along e.

As in [6, Theorem 2.25], bounds for the zero forcing number of a simple digraph interact with the notion of transmission of zero forcing across a boundary. For a simple digraph $\Gamma = (V, G)$ and subset $W \subset V$, $\partial(W)$ denotes the number of arcs in E with one endpoint in W and one endpoint outside W, regardless of direction. The proof is omitted.

PROPOSITION 3.13. *For any simple digraph* $\Gamma = (V, E)$ *with* $W \subseteq V$*,*

$$\mathrm{Z}(\Gamma) \geq \mathrm{Z}(\Gamma[W]) + \mathrm{Z}(\Gamma[\overline{W}]) - \partial(W).$$

4. Extreme minimum rank, maximum nullity and zero forcing number. In this section, we seek to describe the simple digraphs Γ for which $\mathrm{Z}(\Gamma)$, $\mathrm{M}(\Gamma)$, or $\mathrm{mr}(\Gamma)$ are very low or very high. We begin with the case where $\mathrm{Z}(\Gamma)$ and $\mathrm{M}(\Gamma)$ are very low (so $\mathrm{mr}(\Gamma)$ is very high).

LEMMA 4.1. [8] *Suppose* Γ *is a simple digraph and* $\mathcal{F}$ *is a chronological list of forces of a zero forcing set* B*. A maximal forcing chain is a Hessenberg path.*

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This lemma, along with Lemma 1.4 makes it easy to characterize the simple digraphs Γ such that $\mathrm{Z}(\Gamma) = 1$.

OBSERVATION 4.2. [8] $\mathrm{Z}(\Gamma) = 1$ if and only if Γ is a Hessenberg path. In this case, $\mathrm{M}(\Gamma) = \mathrm{Z}(\Gamma) = 1$ and $\mathrm{mr}(\Gamma) = |\Gamma| - 1$.

However, $\mathrm{M}(\Gamma) = 1$ does not necessarily imply that $\mathrm{Z}(\Gamma) = 1$, as the following example shows.

EXAMPLE 4.3. Let Γ be the graph in Figure 2.3. In Example 2.7, it was shown that $\mathrm{mr}(\Gamma) = 3$, and thus, $\mathrm{M}(\Gamma) = 1$. However, one can quickly check that $\mathrm{Z}(\Gamma) = 2$, as no one blue vertex has an all-blue final coloring.

We now characterize the simple digraphs Γ for which $\mathrm{Z}(\Gamma) = 2$. A simple digraph Γ is a *digraph of two parallel Hessenberg paths* if Γ is not itself a Hessenberg path, and $V(\Gamma) = \{u_1, \ldots, u_r, v_1, \ldots, v_s\}$ (where $r, s \neq 0$), $(u_1, \ldots, u_r)$ and $(v_1, \ldots, v_s)$ are Hessenberg paths, and there do not exist i, j, k, ℓ such that $i < j$, $k < \ell$, $(u_k, v_j) \in E(\Gamma)$, and $(v_i, u_\ell) \in E(\Gamma)$.

THEOREM 4.4. $\mathrm{Z}(\Gamma) = 2$ *if and only if* Γ *is a digraph of two parallel Hessenberg paths.*

Proof. Suppose $\mathrm{Z}(\Gamma) = 2$, $B = \{u_1, v_1\}$ is a minimum zero forcing set, and $\mathcal{F}$ is a chronological list of forces for B. One maximal forcing chain of $\mathcal{F}$ starts with u_1 and another starts with v_1. Let $(u_1, u_2, \ldots, u_r)$ and $(v_1, v_2, \ldots, v_s)$ denote these two chains, which are the only two maximal forcing chains. By Lemma 4.1, the subgraphs induced on each chain must be a Hessenberg path. Now, suppose $(u_k, v_j), (v_i, u_\ell) \in E(\Gamma)$ where $i < j$ and $k < \ell$. Proceed with the forcing until the first of the two forces $u_k \to u_{k+1}$ and $v_i \to v_{i+1}$ appears in the chronological list. Since $(u_k, v_j), (u_k, u_{k+1}), (v_i, u_\ell), (v_i, v_{i+1}) \in E(\Gamma)$ and $v_j, u_{k+1}, u_\ell, v_{i+1}$ are currently white, neither $u_k \to u_{k+1}$ nor $v_i \to v_{i+1}$ can occur, contradicting the fact that B is a zero forcing set.

Now suppose Γ is a digraph of two parallel Hessenberg paths, where the two paths are $(u_1, \ldots, u_r)$ and $(v_1, \ldots, v_s)$. We claim that $\{u_1, v_1\}$ is a zero forcing set for Γ, and thus, $\mathrm{Z}(\Gamma) \leq 2$. We color u_1 and v_1 blue. Starting with $k = 1$, perform the forces $u_k \to u_{k+1}$ until we reach a value k for which we cannot perform the force $u_k \to u_{k+1}$ (or until all of the vertices $u_1, \ldots, u_r$ are blue). Unless all the vertices $u_1, \ldots, u_r$ are blue, there is an index $j > 1$ such that $(u_k, v_j) \in E(\Gamma)$. Next, we perform the forces $v_i \to v_{i+1}$ until we reach a value of i for which we cannot perform the force $v_i \to v_{i+1}$ (or until all of $v_1, \ldots, v_s$ are blue). Forces can be performed at least until $v_{j-1} \to v_j$, because $(v_1, \ldots, v_s)$ is a Hessenberg paths and there cannot exist $i < j$ and $k < \ell$ such that $(v_i, u_\ell) \in E(\Gamma)$. At this point, v_j is blue and we return to u_k and continue forcing with $u_k \to u_{k+1}$, etc. Thus, $\{u_1, v_1\}$ is a zero forcing set. Since Γ is not itself

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a Hessenberg path, $\mathrm{Z}(\Gamma) = 2$. ☐

Finally, we consider the case where $\mathrm{Z}(\Gamma)$ and $\mathrm{M}(\Gamma)$ are very high (so $\mathrm{mr}(\Gamma)$ is very low).

PROPOSITION 4.5. *Let $\Gamma = (V, E)$ be a digraph of order n. The following are equivalent:*

(1) $\mathrm{mr}(\Gamma) = 1$ *(or equivalently,* $\mathrm{M}(\Gamma) = n - 1$*).*
(2) $\mathrm{Z}(\Gamma) = n - 1$.
(3) $E \neq \emptyset$, *and*
$(\deg^+ u > 0 \;\&\; \deg^- v > 0) \Rightarrow (u, v) \in E$.

Proof. (1) $\Rightarrow$ (2): Suppose $\mathrm{M}(\Gamma) = n - 1$. Thus, Γ has an arc, so $\mathrm{Z}(\Gamma) \leq n - 1$, but also $n - 1 = \mathrm{M}(\Gamma) \leq \mathrm{Z}(\Gamma)$.

(2) $\Rightarrow$ (3): Suppose Γ does not satisfy (3). If $E = \emptyset$ then $\mathrm{Z}(\Gamma) = n$. So $E \neq \emptyset$, and thus, there exist vertices u and v such that $\deg^+ u > 0, \deg^- v > 0$, and $(u, v) \notin E$. Since $\deg^+ u > 0, \deg^- v > 0$, there exist vertices x and y (not necessarily distinct) such that $(u, x) \in E$ and $(y, v) \in E$. The set $B := V \setminus \{v, x\}$ is a zero forcing set for Γ because $u \to x$ and then $y \to v$. Thus, $\mathrm{Z}(\Gamma) \leq n - 2$. So $(\deg^+ u > 0 \;\&\; \deg^- v > 0) \Rightarrow (u, v) \in E$.

(3) $\Rightarrow$ (1): Suppose Γ satisfies (3). Since $E \neq \emptyset$, $\mathrm{mr}(\Gamma) > 0$. Define $A = [a_{uv}] \in \mathcal{M}(\Gamma)$ by

$$a_{uv} = \begin{cases} 1 & \text{if } (u, v) \in E, \text{ or } (v = u, \deg^+ u > 0 \text{ and } \deg^- u > 0); \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\operatorname{rank} A = 1$ so $\mathrm{mr}(\Gamma) = 1$. ☐

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