Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 26, pp. 742-753, October 2013

ELA

REFINEMENTS OF THE OPERATOR JENSEN–MERCER INEQUALITY*

MOHSEN KIAN† AND MOHAMMAD SAL MOSLEHIAN‡

Abstract. A Hermite–Hadamard–Mercer type inequality is presented and then generalized to Hilbert space operators. It is shown that $f\left(M+m-\sum_{i=1}^n x_iA_i\right) \le f(M)+f(m)-\sum_{i=1}^n f(x_i)A_i$, where f is a convex function on an interval $[m,M]$ containing 0, $x_i \in [m,M]$, $i=1,\dots,n$, and A_i are positive operators acting on a finite dimensional Hilbert space whose sum is equal to the identity operator. A Jensen–Mercer operator type inequality for separately operator convex functions is also presented.

Key words. Jensen–Mercer inequality, Operator convex, Jensen inequality, Hermite–Hadamard inequality, Jointly operator convex.

AMS subject classifications. 47A63, 47A64.

1. Introduction. The well-known Jensen inequality for the convex functions states that if f is a convex function on an interval $[m,M]$, then

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \le \sum_{i=1}^n \lambda_i f(x_i) \tag{1.1}$$

for all $x_i \in [m,M]$ and all $\lambda_i \in [0,1]$ $(i=1,\dots,n)$ with $\sum_{i=1}^n \lambda_i = 1$. The Hermite–Hadamard inequality asserts that if f is a convex function on $[m,M]$, then

$$f\left(\frac{m+M}{2}\right) \le \frac{1}{M-m}\int_m^M f(x)dx \le \frac{f(m)+f(M)}{2}.$$

For more information, see [5, 15] and references therein. Mercer [12] established a variant of the Jensen inequality (1.1) as follows.

Theorem 1.1. *If f is a convex function on $[m,M]$, then*

$$f\left(M+m-\sum_{i=1}^n \lambda_i x_i\right) \le f(M)+f(m)-\sum_{i=1}^n \lambda_i f(x_i) \tag{1.2}$$

*Received by the editors on December 2, 2012. Accepted for publication on October 6, 2013. Handling Editor: Harm Bart.

†Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, PO Box 1339, Bojnord 94531, Iran; Tusi Mathematical Research Group (TMRG), PO Box 1113, Mashhad 91775, Iran (kian@member.ams.org, kian@ub.ac.ir).

‡Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, PO Box 1159, Mashhad 91775, Iran (moslehian@um.ac.ir, moslehian@member.ams.org).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 26, pp. 742-753, October 2013

for all $x_i \in [m, M]$ *and* $\lambda_i \in [0, 1]$ $(i = 1, \dots, n)$ *with* $\sum_{i=1}^n \lambda_i = 1$.

Inequality (1.2) is known as the Jensen–Mercer inequality. Recently, inequality (1.2) has been generalized; see [1, 2, 16].

Let $\mathbb{B}(\mathscr{H})$ be the algebra of all bounded linear operators on a Hilbert space $\mathscr{H}$ and I denote the identity operator. An operator A is positive (denoted by $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all vectors $x \in \mathscr{H}$. If, in addition, A is invertible, then it is strictly positive (denoted by $A > 0$). By $A \geq B$ we mean that $A - B$ is positive, while $A > B$ means that $A - B$ is strictly positive. An operator C is an isometry if $C^*C = I$, a contraction if $C^*C \leq I$ and an expansive operator if $C^*C \geq I$. A linear map Φ on $\mathbb{B}(\mathscr{H})$ is positive if $\Phi(A) \geq 0$ for each $A \geq 0$ and is strictly positive if $\Phi(A) > 0$ for each $A > 0$. A positive linear map Φ is strictly positive if and only if $\Phi(I) > 0$ [8].

A continuous function f defined on an interval J is said to be operator convex if

$$f(\lambda A + (1 - \lambda)B) \leq \lambda f(A) + (1 - \lambda)f(B)$$

for all $\lambda \in [0, 1]$ and all self-adjoint operators A, B with spectra in J.

Hansen and Pedersen (see [6, Theorem 1.9]) presented an operator version of the Jensen inequality for an operator convex function by establishing that if f is operator convex on J, then

$$f(C^*AC) \leq C^*f(A)C \tag{1.3}$$

for any self-adjoint operator A with spectrum in J and any isometry C. Several versions of the Jensen operator inequality can be found in [6, 8]. Among them, we are interested in the following generalization of (1.3):

$$f\left(\sum_{i=1}^n \Phi_i(A_i)\right) \leq \sum_{i=1}^n \Phi_i(f(A_i)),$$

in which the operators A_i $(i = 1, \dots, n)$ are self-adjoint with spectra in J and $\Phi_1, \dots, \Phi_n$ are positive linear maps on $\mathbb{B}(\mathscr{H})$ with $\sum_{i=1}^n \Phi_i(I) = I$ [14]. Recently, J. Mićić, Z. Pavić and J. Pečarić [13] obtained a Jensen inequality for operators without the assumption of operator convexity.

Regarding the possible operator extensions of (1.2), there are some interesting works.

Let f be a continuous convex function on $[m, M]$, $A_1, \dots, A_n$ be self-adjoint operators with spectra in $[m, M]$ and $\Phi_1, \dots, \Phi_n$ be positive linear maps with $\sum_{i=1}^n \Phi_i(I) = I$. The following operator version of the Mercer inequality (1.2) was proved in [10]:

$$f\left(M + m - \sum_{i=1}^n \Phi_i(A_i)\right) \leq f(m) + f(M) - \sum_{i=1}^n \Phi_i(f(A_i)). \tag{1.4}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 26, pp. 742-753, October 2013

ELA

Furthermore, some refinements, applications and reformulations of inequality (1.4) for some other types of functions have been obtained in [3, 7, 9, 11].

The function $f : [m, M] \times [m', M'] \longrightarrow \mathbb{R}$ is separately operator convex if the functions g_t and h_s defined by $g_t(s) = f(t, s) = h_s(t)$ are operator convex, respectively, on $[m', M']$ and $[m, M]$ for each $t \in [m, M]$ and $s \in [m', M']$.

In Section 2, we present a Hermite–Hadamard–Mercer type inequality and then generalize it for Hilbert space operators. In Section 3, we obtain another variant of inequality (1.4). Also we give a Jensen–Mercer operator type inequality for separately operator convex functions.

2. Hermite–Hadamard–Mercer type inequalities. In this section, we present a Hermite–Hadamard type inequality using the Mercer inequality (1.2) and then give its operator extension.

THEOREM 2.1. *Let f be a convex function on $[m, M]$. Then*

$$\begin{aligned} f\left(M + m - \frac{x+y}{2}\right) &\le f(M) + f(m) - \int_0^1 f(tx + (1-t)y)dt \\ &\le f(M) + f(m) - f\left(\frac{x+y}{2}\right), \end{aligned} \tag{2.1}$$

and

$$f\left(M + m - \frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(M+m-t)dt \le f(M) + f(m) - \frac{f(x)+f(y)}{2} \tag{2.2}$$

for all $x, y \in [m, M]$.

Proof. It follows from the Jensen–Mercer inequality that

$$f\left(M + m - \frac{a+b}{2}\right) \le f(M) + f(m) - \frac{f(a)+f(b)}{2} \tag{2.3}$$

for each $a, b \in [m, M]$. Let $t \in [0, 1]$ and $x, y \in [m, M]$. Replacing a and b respectively by $tx + (1-t)y$ and $(1-t)x + ty$ in (2.3), we obtain

$$f\left(M + m - \frac{x+y}{2}\right) \le f(M) + f(m) - \frac{f(tx + (1-t)y) + f((1-t)x + ty)}{2}. \tag{2.4}$$

By integrating both sides of (2.4), we get

$$\begin{aligned} f\left(M + m - \frac{x+y}{2}\right) &\le f(M) + f(m) \\ &\quad - \frac{1}{2}\int_0^1 (f(tx + (1-t)y) + f((1-t)x + ty))dt. \end{aligned} \tag{2.5}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 26, pp. 742-753, October 2013

Due to

$$\int_0^1 f(tx+(1-t)y)dt = \int_0^1 f((1-t)x+ty)dt = \frac{1}{y-x}\int_x^y f(t)dt, \tag{2.6}$$

inequality (2.5) gives rise to the first inequality of (2.1). The second inequality of (2.1) follows directly from the Hermite–Hadamard inequality.

Next we prove inequality (2.2). The Hermite–Hadamard inequality implies that

$$\begin{aligned}\int_0^1 f(M+m-(tx+(1-t)y))dt &= \int_0^1 f(t(M+m-x)+(1-t)(M+m-y))dt\\ &\geq f\left(\frac{M+m-x+M+m-y}{2}\right)\\ &= f\left(M+m-\frac{x+y}{2}\right). \end{aligned}\tag{2.7}$$

On the other hand, the Mercer inequality gives

$$f(M+m-(tx+(1-t)y) \leq f(M)+f(m)-(tf(x)+(1-t)f(y)). \tag{2.8}$$

Integrating both sides of (2.8) we get

$$\begin{aligned}\int_0^1 f(M+m-(tx+(1-t)y))dt &\leq f(M)+f(m)-\int_0^1 (tf(x)+(1-t)f(y))dt\\ &= f(M)+f(m)-\frac{f(x)+f(y)}{2}.\end{aligned}\tag{2.9}$$

Inequality (2.2) now follows immediately from inequalities (2.6), (2.7) and (2.9). □

The following operator version of inequality (2.2) holds true.

THEOREM 2.2. *If f is convex on $[m, M]$, then*

$$\int_0^1 f(M+m-(t\Phi(A)+(1-t)\Phi(B)))dt \leq f(M)+f(m)-\frac{\Phi(f(A))+\Phi(f(B))}{2}, \tag{2.10}$$

$$f\left(m+M-\frac{\Phi(A)+\Phi(B)}{2}\right) \leq f(M)+f(m)-\int_0^1 f(t\Phi(A)+(1-t)\Phi(B))dt \tag{2.11}$$

for all self-adjoint operators A, B with spectra in $[m, M]$ and a unital positive linear map Φ. Furthermore, if f is operator convex, then

$$\begin{aligned}f\left(M+m-\frac{\Phi(A)+\Phi(B)}{2}\right) &\leq \int_0^1 f(M+m-(t\Phi(A)+(1-t)\Phi(B)))dt\\ &\leq f(M)+f(m)-\frac{\Phi(f(A))+\Phi(f(B))}{2}.\end{aligned}\tag{2.12}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 26, pp. 742-753, October 2013

ELA

Proof. First note that since f is continuous, the vector valued integrals such as (2.10) exist for all self-adjoint operators A and B with spectra in $[m, M]$. We have

$$\begin{aligned}
&\int_0^1 f(M + m - (t\Phi(A) + (1-t)\Phi(B)))dt \\
&\leq \int_0^1 [f(M) + f(m) - t\Phi(f(A)) - (1-t)\Phi(f(B))]dt \\
&\qquad\qquad (\text{ by the Jensen–Mercer operator inequality (1.4)}) \\
&= f(M) + f(m) - \frac{\Phi(f(A)) + \Phi(f(B))}{2},
\end{aligned}$$

which is the desired inequality (2.10). Moreover, using the Jensen–Mercer operator inequality, we get

$$\begin{aligned}
&f\left(m + M - \frac{\Phi(A) + \Phi(B)}{2}\right) \\
&= f\left(m + M - \frac{(t\Phi(A) + (1-t)\Phi(B)) + ((1-t)\Phi(A) + t\Phi(B))}{2}\right) \\
&\leq f(m) + f(M) - \frac{f(t\Phi(A) + (1-t)\Phi(B)) + f((1-t)\Phi(A) + t\Phi(B))}{2}.
\end{aligned}$$

Integrating from both sides of the later inequality leads us to (2.11). If f is operator convex, then

$$\begin{aligned}
&f\left(M + m - \frac{\Phi(A) + \Phi(B)}{2}\right) \\
&= f\left(\frac{(M + m - t\Phi(A) - (1-t)\Phi(B)) + (M + m - t\Phi(B) - (1-t)\Phi(A))}{2}\right) \\
&\leq \frac{1}{2}\left[f(M + m - t\Phi(A) - (1-t)\Phi(B)) + f(M + m - t\Phi(B) - (1-t)\Phi(A))\right]. \qquad (2.13)
\end{aligned}$$

Integrating from both sides of inequality (2.13) we get the first inequality of (2.12). The second one is clear. ▯

EXAMPLE 2.3. If $f : J \to \mathbb{R}$ is a convex function, then the inequality

$$f\left(\frac{A+B}{2}\right) \leq \int_0^1 f(tA + (1-t)B)dt \leq \frac{f(A) + f(B)}{2}$$

may not hold in general [15]. To see this, consider the convex function $f(t) = t^4$ which appears in some counter-examples, starting with a work of M.-D. Choi [4], and Hermitian matrices

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Nevertheless, inequalities (2.10) and (2.11) are valid for all $m, M \in J$ provided that the spectra of A and B are contained in $[m, M]$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 26, pp. 742-753, October 2013

ELA

3. A variant of the Jensen–Mercer inequality for operators. We use an idea from [17] to obtain one of our main results.

THEOREM 3.1. *Let* $A_i \quad (i = 1, \ldots, n)$ *be positive operators acting on a finite dimensional Hilbert space* $\mathscr{H}$ *with* $\sum_{i=1}^n A_i = I$. *If* f *is convex on an interval* $[m, M]$ *containing* 0, *then*

$$f\left(M + m - \sum_{i=1}^n x_i A_i\right) \leq f(M) + f(m) - \sum_{i=1}^n f(x_i) A_i \tag{3.1}$$

for all $x_1, \ldots, x_n \in [m, M]$.

Proof. Clearly, the spectrum of $M + m - \sum_{i=1}^n x_i A_i$ is contained in $[m, M]$. Without loss of generality we may assume that $f(0) = 0$ (if $f(0) \neq 0$ we may consider the convex function $g(x) = f(x) - f(0)$ instead of f). There is a Hilbert space $\mathfrak{H}$ containing $\mathscr{H}$ and a family of mutually orthogonal projections $P_i \quad (i = 1, \ldots, n)$ on $\mathfrak{H}$ such that $\sum_{i=1}^n P_i = I_{\mathfrak{H}}$ and $A_i = PP_iP|_{\mathscr{H}}$ for each $i = 1, \ldots, n$, where P is the projection from $\mathfrak{H}$ onto $\mathscr{H}$ [17]. Therefore,

$$\begin{aligned}
f\left(M + m - \sum_{i=1}^n x_i A_i\right) &= f\left(M + m - \sum_{i=1}^n x_i PP_iP|_{\mathscr{H}}\right) \\
&= f\left(M + m - P\left(\sum_{i=1}^n x_i P_i\right)P|_{\mathscr{H}} - (I - P)0(I - P)|_{\mathscr{H}}\right) \\
&\leq f(M) + f(m) - Pf\left(\sum_{i=1}^n x_i P_i\right)P|_{\mathscr{H}} \\
&\quad -(I - P)f(0)(I - P)|_{\mathscr{H}} \\
&\qquad \text{(by Jensen–Mercer operator inequality (1.4) with} \\
&\qquad \Phi_1(A) = PAP \text{ and } \Phi_2(A) = (1 - P)A(1 - P)) \\
&\leq f(M) + f(m) - P\left(\sum_{i=1}^n f(x_i)P_i\right)P|_{\mathscr{H}} \\
&\qquad \text{(by } f(0) = 0 \text{ and the spectral theorem)} \\
&= f(M) + f(m) - \sum_{i=1}^n f(x_i)A_i. \quad \square
\end{aligned}$$

COROLLARY 3.2. *Let* f *and* $x_i \quad (i = 1, \ldots, n)$ *be as in Theorem 3.1 and* $f(0) = 0$. *If* $\sum_{i=1}^n A_i \leq I$, *then inequality* (3.1) *remains true.*

Proof. Put $B = I - \sum_{i=1}^n A_i$. Then $B + \sum_{i=1}^n A_i = I$. Hence,

$$f\left(M + m - \sum_{i=1}^n x_i A_i\right) = f\left(M + m - \sum_{i=1}^n x_i A_i - 0B\right)$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 26, pp. 742-753, October 2013

ELA

$$\leq f(M) + f(m) - \sum_{i=1}^{n} f(x_i)A_i - f(0)B \qquad \text{(by (3.1))}$$

$$\leq f(M) + f(m) - \sum_{i=1}^{n} f(x_i)A_i \qquad \text{(by } f(0) = 0\text{)}. \quad \square$$

The following particular case of (3.1) is of special interest.

Corollary 3.3. *Let A_i $(i = 1, \dots, n)$ be positive operators acting on a finite dimensional Hilbert space $\mathscr{H}$ with $\sum_{i=1}^{n} A_i = I$. If f is convex on the interval $[m, M]$ containing -1 and 1, then*

$$f\left(M + m + \sum_{i=1}^{k} A_i - \sum_{i=k+1}^{n} A_i\right) \leq f(M) + f(m) - f(-1)\sum_{i=1}^{k} A_i - f(1)\sum_{i=k+1}^{n} A_i.$$

Remark 3.4. It should be mentioned that (3.1) implies a weaker version of the Jensen–Mercer operator inequality. Assume that X_i $(i = 1, \dots, n)$ are self-adjoint operators on a finite dimensional Hilbert space $\mathscr{H}$ with spectra in $[m, M]$ and $w_i \in [0, 1]$ with $\sum_{i=1}^{n} w_i = 1$. Let $X_i = \sum_{j=1}^{k} \lambda_{ij} P_{ij}$ $(i = 1, \dots, n)$ be the spectral decomposition of X_i so that $\lambda_{ij} \in [m, M]$. It follows from $\sum_{i=1}^{n}\sum_{j=1}^{k} w_i P_{ij} = I$ that

$$\begin{aligned}
f\left(M + m - \sum_{i=1}^{n} w_i X_i\right) &= f\left(M + m - \sum_{i=1}^{n}\sum_{j=1}^{k} w_i \lambda_{ij} P_{ij}\right) \\
&\leq f(M) + f(m) - \sum_{i=1}^{n}\sum_{j=1}^{k} f(\lambda_{ij}) w_i P_{ij} \qquad \text{(by (3.1))} \\
&= f(M) + f(m) - \sum_{i=1}^{n} w_i f\left(\sum_{j=1}^{k} \lambda_{ij} P_{ij}\right) \\
&= f(M) + f(m) - \sum_{i=1}^{n} w_i f(X_i).
\end{aligned}$$

Let f be an operator convex function with $f(0) \leq 0$. It follows from the Jensen operator inequality (1.3) that

$$C^* f(A) C \leq f(C^* A C) \tag{3.2}$$

for any invertible expansive operator C and any self-adjoint operator A.

Theorem 3.5. *Let $m < M$ and let Φ be a positive linear map on $\mathbb{B}(\mathscr{H})$ with $0 < \Phi(I) \leq I$. Let*

$$m' = \min\{m\langle \Phi(I)x, x\rangle;\ \ \|x\| = 1\} \quad \textit{and} \quad M' = \max\{M\langle \Phi(I)x, x\rangle;\ \ \|x\| = 1\}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 26, pp. 742-753, October 2013

If $f : J \to \mathbb{R}$ is an operator convex function with $f(0) \le 0$ and $[m, M] \cup [m', M'] \subseteq J$, then

$$f((m + M)\Phi(I) - \Phi(A)) \le f(m) + f(M) - \Phi(f(A))$$

for any self-adjoint operator A with spectrum contained in $[m, M]$.

Proof. Define the positive linear map Ψ on $\mathbb{B}(\mathscr{H})$ by

$$\Psi(X) = \Phi(I)^{-\frac{1}{2}}\Phi(X)\Phi(I)^{-\frac{1}{2}}, \qquad (X \in \mathbb{B}(\mathscr{H})).$$

Then Ψ is unital, and it follows from (1.4) that

$$f(m + M - \Psi(A)) \le f(m) + f(M) - \Psi(f(A))$$

for each self-adjoint operator A with spectrum in $[m, M]$. Therefore,

$$f(m + M - \Phi(I)^{-\frac{1}{2}}\Phi(A)\Phi(I)^{-\frac{1}{2}}) \le f(m) + f(M) - \Phi(I)^{-\frac{1}{2}}\Phi(f(A))\Phi(I)^{-\frac{1}{2}}.$$

Hence,

$$\begin{aligned} &f(\Phi(I)^{-\frac{1}{2}}((m + M)\Phi(I) - \Phi(A))\Phi(I)^{-\frac{1}{2}}) \\ &\le \Phi(I)^{-\frac{1}{2}}((f(m) + f(M))\Phi(I) - \Phi(f(A)))\Phi(I)^{-\frac{1}{2}}. \end{aligned} \tag{3.3}$$

On the other hand, $\Phi(I)^{-\frac{1}{2}}$ is an expansive operator. Using (3.2) we obtain

$$\begin{aligned} &\Phi(I)^{-\frac{1}{2}}(f((m + M)\Phi(I) - \Phi(A))\Phi(I)^{-\frac{1}{2}} \\ &\le f(\Phi(I)^{-\frac{1}{2}}((m + M)\Phi(I) - \Phi(A))\Phi(I)^{-\frac{1}{2}}). \end{aligned} \tag{3.4}$$

Now, the result follows from inequalities (3.3) and (3.4). ☐

COROLLARY 3.6. *Let f and $\mathscr{H}$ be as in Theorem 3.5. Let Φ_i $(i = 1, \ldots, n)$ be positive linear maps on $\mathbb{B}(\mathscr{H})$ with $0 < \Phi(I) = \sum_{i=1}^n \Phi_i(I) \le I$. Then*

$$f\left((m + M)\Phi(I) - \sum_{i=1}^n \Phi_i(A_i)\right) \le f(m) + f(M) - \sum_{i=1}^n \Phi_i(f(A_i))$$

for all self-adjoint operators A_i with spectra in $[m, M]$.

Proof. Assume that $A_1, \ldots, A_n$ are self-adjoint operators on $\mathscr{H}$ with spectra in $[m, M]$ and $\Phi_1, \ldots, \Phi_n$ are positive linear maps on $\mathbb{B}(\mathscr{H})$ with $0 < \sum_{i=1}^n \Phi_i(I) \le I$. For $A, B \in \mathbb{B}(\mathscr{H})$ assume that $A \oplus B$ is the operator defined on $\mathbb{B}(\mathscr{H} \oplus \mathscr{H})$ by $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Now apply Theorem 3.5 to the self-adjoint operator A on the Hilbert space $\mathscr{H} \oplus \cdots \oplus \mathscr{H}$ defined by $A = A_1 \oplus \cdots \oplus A_n$ and the positive linear map Φ defined on $\mathbb{B}(\mathscr{H} \oplus \cdots \oplus \mathscr{H})$ by $\Phi(A) = \Phi_1(A_1) \oplus \cdots \oplus \Phi_n(A_n)$. ☐

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 26, pp. 742-753, October 2013

The next theorem yields a Jensen–Mercer operator type inequality for separately operator convex functions.

THEOREM 3.7. *Let $f:[m,M]\times[m',M']\longrightarrow\mathbb{R}$ be a separately operator convex function. Let $\Phi_i,\Psi_j,\ (1\le i\le r, 1\le j\le k)$ be positive linear maps on $\mathbb{B}(\mathscr{H})$ with $\sum_{i=1}^r\Phi_i(I)=I=\sum_{j=1}^k\Psi_j(I)$. Then*

$$\begin{aligned}&f\left(M+m-\sum_{i=1}^r\Phi_i(A_i),M'+m'-\sum_{j=1}^k\Psi_j(B_j)\right)\\&\le f(m,m')+f(m,M')+f(M,m')+f(M,M')-2f\left(\sum_{i=1}^r\Phi_i(A_i),\frac{M'+m'}{2}\right)\\&\quad-2f\left(\frac{m+M}{2},\sum_{j=1}^k\Psi_j(B_j)\right)+f\left(\sum_{i=1}^r\Phi_i(A_i),\sum_{j=1}^k\Psi_j(B_j)\right)\end{aligned}$$

for all self-adjoint operators A_i with spectra in $[m,M]$ and B_j with spectra in $[m',M']$.

Proof. Since f is separately convex, we have

$$f(m+M-t,s)\le f(m,s)+f(M,s)-f(t,s), \tag{3.5}$$

and

$$f(t,M'+m'-s)\le f(t,m')+f(t,M')-f(t,s) \tag{3.6}$$

for all $t\in[m,M]$ and $s\in[m',M']$. Adding inequalities (3.5) and (3.6) we obtain

$$\begin{aligned}f(t,s)\le\frac{1}{2}\,[&f(m,s)+f(M,s)+f(t,m')+f(t,M')\\&-f(t,M'+m'-s)-f(m+M-t,s)],\end{aligned} \tag{3.7}$$

for all $t\in[m,M]$ and $s\in[m',M']$. Using functional calculus for inequality (3.7) we get

$$\begin{aligned}&f\left(M+m-\sum_{i=1}^r\Phi_i(A_i),M'+m'-\sum_{j=1}^k\Psi_j(B_j)\right)\\&\le\frac{1}{2}\left[f\left(m,M'+m'-\sum_{j=1}^k\Psi_j(B_j)\right)+f\left(M,M'+m'-\sum_{j=1}^k\Psi_j(B_j)\right)\right.\\&\quad+f\left(M+m-\sum_{i=1}^r\Phi_i(A_i),m'\right)+f\left(M+m-\sum_{i=1}^r\Phi_i(A_i),M'\right)\\&\quad\left.-f\left(M+m-\sum_{i=1}^r\Phi_i(A_i),\sum_{j=1}^k\Psi_j(B_j)\right)-f\left(\sum_{i=1}^r\Phi_i(A_i),M'+m'-\sum_{j=1}^k\Psi_j(B_j)\right)\right].\end{aligned} \tag{3.8}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 26, pp. 742-753, October 2013

ELA

Since f is separately convex, the functions g_s and h_t defined by $g_s(t) = f(t,s) = h_t(s)$ are convex on $[m, M]$ and $[m', M']$ respectively. It follows from (1.4) that

$$f\left(m, M'+m'-\sum_{j=1}^{k}\Psi_j(B_j)\right) \le f(m,m')+f(m,M')-\sum_{j=1}^{k}\Psi_j(f(m,B_j)), \tag{3.9}$$

$$f\left(M, M'+m'-\sum_{j=1}^{k}\Psi_j(B_j)\right) \le f(M,m')+f(M,M')-\sum_{j=1}^{k}\Psi_j(f(M,B_j)), \tag{3.10}$$

$$f\left(M+m-\sum_{i=1}^{r}\Phi_i(A_i), m'\right) \le f(m,m')+f(M,m')-\sum_{i=1}^{r}\Phi_i(f(A_i,m')), \tag{3.11}$$

$$f\left(M+m-\sum_{i=1}^{r}\Phi_i(A_i), M'\right) \le f(m,M')+f(M,M')-\sum_{i=1}^{r}\Phi_i(f(A_i,M')). \tag{3.12}$$

Summing inequalities (3.9), (3.10), (3.11) and (3.12) we obtain

$$\begin{aligned}
& f\left(m, M'+m'-\sum_{j=1}^{k}\Psi_j(B_j)\right) + f\left(M, M'+m'-\sum_{j=1}^{k}\Psi_j(B_j)\right) \\
& + f\left(M+m-\sum_{i=1}^{r}\Phi_i(A_i), m'\right) + f\left(M+m-\sum_{i=1}^{r}\Phi_i(A_i), M'\right) \\
\le\ & 2\,[f(m,m')+f(m,M')+f(M,m')+f(M,M')] \\
& -\sum_{j=1}^{k}\Psi_j(f(m,B_j)+f(M,B_j)) - \sum_{i=1}^{r}\Phi_i(f(A_i,M')+f(A_i,m')) \\
\le\ & 2\,[f(m,m')+f(m,M')+f(M,m')+f(M,M')] \qquad \text{(by the convexity)} \\
& -2\left[\sum_{j=1}^{k}\Psi_j\left(f\left(\frac{m+M}{2}, B_j\right)\right) + \sum_{i=1}^{r}\Phi_i\left(f\left(A_i, \frac{M'+m'}{2}\right)\right)\right] \\
\le\ & 2\,[f(m,m')+f(m,M')+f(M,m')+f(M,M')] \qquad \text{(by the operator convexity)} \\
& -2\left[f\left(\frac{m+M}{2}, \sum_{j=1}^{k}\Psi_j(B_j)\right) + f\left(\sum_{i=1}^{r}\Phi_i(A_i), \frac{M'+m'}{2}\right)\right].
\end{aligned} \tag{3.13}$$

Also, since f is separately operator convex, we have

$$\frac{1}{2} f\left(M+m-\sum_{i=1}^{r}\Phi_i(A_i), \sum_{j=1}^{k}\Psi_j(B_j)\right) + \frac{1}{2} f\left(\sum_{i=1}^{r}\Phi_i(A_i), \sum_{j=1}^{k}\Psi_j(B_j)\right)$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 26, pp. 742-753, October 2013

$$\geq f\left(\frac{M+m}{2}, \sum_{j=1}^{k} \Psi_j(B_j)\right), \tag{3.14}$$

and

$$\frac{1}{2} f\left(\sum_{i=1}^{r} \Phi_i(A_i), M' + m' - \sum_{j=1}^{k} \Psi_j(B_j)\right) + \frac{1}{2} f\left(\sum_{i=1}^{r} \Phi_i(A_i), \sum_{j=1}^{k} \Psi_j(B_j)\right)$$

$$\geq f\left(\sum_{i=1}^{r} \Phi_i(A_i), \frac{M' + m'}{2}\right). \tag{3.15}$$

Combining inequalities (3.8), (3.13), (3.14) and (3.15), one can easily conclude the desired result. □

Acknowledgment. The authors would like to thank the referees for several useful comments improving the paper.

REFERENCES

[1] S. Abramovich, J. Barić, and J. Pečarić. A variant of Jessen's inequality of Mercer's type for superquadratic functions. *J. Inequal. Pure Appl. Math.*, 9(3):13, Article 62, 2008.

[2] J. Barić and A. Matković. Bounds for the normalized Jensen–Mercer functional. *J. Math. Inequal.*, 3(4):529–541, 2009.

[3] J. Barić, A. Matković, and J. Pečarić. A variant of the Jensen–Mercer operator inequality for superquadratic functions. *Math. Comput. Modelling*, 51:1230–1239, 2010.

[4] M.-D. Choi. A Schwarz inequality for positive linear maps on C^*-algebras. *Illinois J. Math.*, Volume 18:565–575, 1974.

[5] S.S. Dragomir. Hermite-Hadamards type inequalities for operator convex functions. *Appl. Math. Comput.*, 218(3):766–772, 2011.

[6] T. Furuta, H. Mićić, J. Pečarić, and Y. Seo. Mond–Pečarić Method in Operator Inequalities. *Zagreb Element*, 2005.

[7] S. Ivelić, A. Matković, and J.E. Pečarić. On a Jensen–Mercer operator inequality. *Banach J. Math. Anal.*, 5(1):19–28, 2011.

[8] M. Khosravi, J.S. Aujla, S.S. Dragomir, and M.S. Moslehian. Refinements of Choi–Davis–Jensen's inequality. *Bull. Math. Anal. Appl.*, 3(2):127–133, 2011.

[9] A. Matković and J. Pečarić. On a variant of the Jensen–Mercer inequality for operators. *J. Math. Inequal.*, 2(3):299–307, 2008.

[10] A. Matković, J. Pečarić, and I. Perić. A variant of Jensen's inequality of Mercer's type for operators with applications. *Linear Algebra Appl.*, 418:551–564, 2006.

[11] A. Matković, J. Pečarić, and I. Perić. Refinements of Jensen's inequality of Mercer's type for operator convex functions. *Math. Inequal. Appl.*, 11(1):113–126, 2008.

[12] A.McD. Mercer. A variant of Jensen's inequality. *J. Inequal. Pure Appl. Math.*, 4(4):2, Article 73, 2003.

[13] J. Mićić, Z. Pavić, and J. Pečarić. Jensen's inequality for operators without operator convexity. *Linear Algebra Appl.*, 434:1228–1237, 2011.

[14] B. Mond and J. Pečarić. Converses of Jensen inequality for several operators. *Rev. Anal. Numér. Théor. Approx.*, 23(2):179–183, 1994.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 26, pp. 742-753, October 2013

ELA

[15] M.S. Moslehian. Matrix Hermite-Hadamard type inequalities. *Houston J. Math.*, 39(1):177–189, 2013.

[16] M. Niezgoda. A generalization of Mercer's result on convex functions. *Nonlinear Analysis*, 71: 2771–2779, 2009.

[17] O.E. Tikhonov. A note on definition of matrix convex functions. *Linear Algebra Appl.*, 416:773–775, 2006.