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APERTURE ANGLE ANALYSIS FOR ELLIPSOIDS*

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Abstract. Let $\Omega \subseteq \mathbb{R}^n$ be a compact convex set and x be a point in the exterior of Ω. The aperture angle of x relative to Ω is defined as the maximal angle of the smallest closed convex cone that contains $\Omega - x$. This note provides an explicit formula, based on eigenvalues of symmetric matrices, for the aperture angle of a point relative to an ellipsoid.

Key words. Supporting cone, Ellipsoidal cone, Aperture angle, Incenter of a cone.

AMS subject classifications. 15A18, 15A63, 52A20, 90C26.

1. Introduction. Let $\mathbb{R}^n$ be equipped with its usual inner product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. Let Ω be a compact convex set in $\mathbb{R}^n$ and let $x \in \Omega^c$. The supporting cone to Ω at x is defined by

$$V(x) := \{t(u - x) : t \geq 0, u \in \Omega\},$$

i.e., as the smallest closed convex cone that contains $\Omega - x$. Recall that a convex cone is a nonempty set that is stable under addition and under multiplication by positive scalars. The aperture angle of x relative to Ω is defined as the number

$$\vartheta(x) := \theta_{\max}(V(x)), \tag{1.1}$$

where

$$\theta_{\max}(K) := \max_{p,q \in K \cap \mathbb{S}_n} \arccos \langle p, q \rangle$$

stands for the maximal angle of a closed convex cone $K \subseteq \mathbb{R}^n$. Here and in the sequel, $\mathbb{S}_n$ denotes the unit sphere of $\mathbb{R}^n$. The number (1.1) belongs to the interval $[0, \pi]$. From the very definition of $V(x)$ one sees that

$$\vartheta(x) = \max_{u,v \in \Omega} \arccos \left\langle \frac{u - x}{\|u - x\|}, \frac{v - x}{\|v - x\|} \right\rangle . \tag{1.2}$$

The concept of aperture angle plays a fundamental role in various approximation, illumination, and visibility problems, cf. [1, 2, 8]. It is an interesting concept also from

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a purely academic point of view. Aperture angle computation is a difficult numerical task, even if Ω has a relatively simple structure. Observe that (1.2) is a nonconvex optimization problem, because one needs to maximize a function that is not concave.

EXAMPLE 1.1. If $\Omega = \mathrm{co}\{a_1, \ldots, a_m\}$ is a polytope, then its supporting cone at $x \in \Omega^c$ is given by

$$V(x) = \left\{ \sum_{i=1}^{m} t_i \, \frac{a_i - x}{\|a_i - x\|} : t_1 \geq 0, \ldots, t_m \geq 0 \right\}. \tag{1.3}$$

As explained in [6, Theorem 3], computing the maximal angle of a polyhedral cone like (1.3) boils down to solve a collection of generalized eigenvalue problems. Unfortunately, the number of generalized eigenvalue problems to be solved increases exponentially with m.

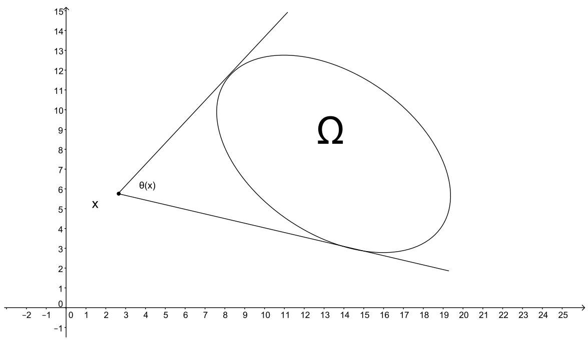

FIG. 1.1. *Aperture angle relative to an ellipsoid.*

The purpose of this note is to derive an explicit formula for evaluating $\vartheta(x)$ when Ω is a solid bounded ellipsoid. Such sort of set is the prototype example of a smooth convex body. The adjective "solid" applied to a set indicates that the set has nonempty interior.

REMARK 1.2. Figure 1.1 displays the aperture angle of x relative to an ellipsoid. For easy of visualization, instead of $V(x)$ we are drawing the translated set $x + V(x)$. We mention in passing that $x + V(x)$ is sometimes called the visual cone to Ω with vertex at x, cf. [3].

1.1. Preliminary material. Let $\mathbb{B}_n$ denote the closed unit ball of $\mathbb{R}^n$ and $\mathcal{O}_n$ be the group of orthogonal matrices of order n. The notation $\mathcal{S}_n$ refers to the space of symmetric matrices of order n and

$$\mathcal{P}_n := \{A \in \mathcal{S}_n : A \text{ is positive definite}\}.$$

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In the sequel, one assumes that Ω is a solid bounded ellipsoid in $\mathbb{R}^n$, i.e.,

$$\Omega = \{u \in \mathbb{R}^n : \langle u - z, A(u - z)\rangle \leq 1\} \tag{1.4}$$

with $z \in \mathbb{R}^n$ and $A \in \mathcal{P}_n$. The vector z corresponds to the "center" of the ellipsoid, whereas the matrix A determines its shape and orientation.

The next lemma shows that any supporting cone to (1.4) is an ellipsoidal cone. By the latter expression one means a set representable as

$$\mathcal{L}(Q, b) := \left\{w \in \mathbb{R}^n : \sqrt{\langle w, Qw\rangle} \leq \langle b, w\rangle\right\} \tag{1.5}$$

for some pair $(Q, b) \in \mathcal{P}_n \times \mathbb{R}^n$ satisfying the strict inequality

$$\langle b, Q^{-1}b\rangle \ > 1. \tag{1.6}$$

The condition (1.6) ensures that $\mathcal{L}(Q, b)$ is a proper cone. Recall that a closed convex cone in $\mathbb{R}^n$ is said to be proper if it is pointed and solid.

Remark 1.3. There are many equivalent ways of representing an ellipsoidal cone. For instance, (1.5) can be written as the image of the n-dimensional Lorentz cone

$$K_n := \left\{ w \in \mathbb{R}^n : \left(\sum_{i=1}^{n-1} w_i^2\right)^{1/2} \leq \ w_n \right\}$$

under a nonsingular matrix of order n.

Lemma 1.4. *Let Ω be an ellipsoid as in (1.4). Then, for all $x \in \Omega^c$, one has*

$$V(x) = \mathcal{L}(A, b_x) \tag{1.7}$$

with

$$b_x := [\langle x - z, A(x - z)\rangle - 1]^{-1/2}\, A(z - x). \tag{1.8}$$

Proof. The ellipsoid (1.4) can be written in the form

$$\Omega = \{z + A^{-1/2}\xi : \xi \in \mathbb{B}_n\}.$$

Hence, for all $x \in \Omega^c$, one has

$$\begin{aligned} V(x) &= \left\{t(z + A^{-1/2}\xi - x) : t \geq 0, \xi \in \mathbb{B}_n\right\} \\ &= A^{-1/2}\left[V_{\mathbb{B}_n}(A^{1/2}(x - z))\right], \end{aligned} \tag{1.9}$$

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where $V_{\mathbb{B}_n}(\xi)$ stands for the supporting cone to $\mathbb{B}_n$ at a point $\xi \in \mathbb{B}_n^c$. A supporting cone to a ball is known to be a revolution cone. Indeed, a matter of computation shows that

$$V_{\mathbb{B}_n}(\xi) = \Gamma\left(-\frac{\xi}{\|\xi\|}, \left(1 - \frac{1}{\|\xi\|^2}\right)^{1/2}\right), \tag{1.10}$$

where

$$\Gamma(y, s) := \{w \in \mathbb{R}^n : s\|w\| \leq \langle y, w\rangle\}$$

stands for the revolution cone with sharpness coefficient $s \in]0, 1[$ and central axis generated by $y \in \mathbb{S}_n$. The combination of (1.9) and (1.10) leads to

$$V(x) = A^{-1/2}\left[\Gamma\left(y_x, s_x\right)\right], \tag{1.11}$$

where

$$y_x := \frac{A^{1/2}(z - x)}{\|A^{1/2}(z - x)\|} \quad \text{and} \quad s_x := \left(1 - \frac{1}{\|A^{1/2}(z - x)\|^2}\right)^{1/2}.$$

It is not difficult to check that (1.11) can be written under the format (1.7) with $b_x = (1/s_x)A^{1/2}y_x$. Finally, note that $\langle b_x, A^{-1}b_x\rangle = 1/s_x^2$ is greater than 1. $\square$

The formula (1.7) has many useful consequences. However, such a characterization of $V(x)$ is not very helpful when it comes to evaluate the number (1.1). The proof of the next lemma explains how to convert $V(x)$ into a "standard" ellipsoidal cone by means of a suitable orthogonal transformation. By a standard ellipsoidal cone in $\mathbb{R}^n$ we understand a set of the form

$$\mathcal{E}(G) := \left\{ w \in \mathbb{R}^n : \left(\sum_{i,j=1}^{n-1} g_{i,j}\, w_i w_j\right)^{1/2} \leq\; w_n \right\} \tag{1.12}$$

with $G \in \mathcal{P}_{n-1}$. A crucial advantage of working with standard ellipsoidal cones is that the maximal angle of (1.12) is a well known function of the smallest eigenvalue of G. We shall come back to this point in Section 2.

Lemma 1.5. *Let $(Q, b) \in \mathcal{P}_n \times \mathbb{R}^n$ be a pair satisfying (1.6). Then there are matrices $G \in \mathcal{P}_{n-1}$ and $U \in \mathcal{O}_n$ such that*

$$\mathcal{L}(Q, b) = U[\mathcal{E}(G)]. \tag{1.13}$$

Proof. We explain how to construct the matrices G and U. Our first observation is that

$$\mathcal{L}(Q, b) = \{w \in \mathbb{R}^n : \langle w, Rw\rangle \leq 0,\ \langle b, w\rangle \geq 0\},$$

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where $R := Q - bb^T$ is a rank one perturbation of Q. The matrix R is clearly symmetric. Let

$$\lambda_1(R) \geq \cdots \geq \lambda_{n-1}(R) \geq \lambda_n(R)$$

be the eigenvalues of R arranged in nonincreasing order. Similarly, one arranges in nondecreasing order the eigenvalues of Q. The interlacing theorem for rank one perturbation of symmetric matrices yields the chain of inequalities

$$\lambda_1(Q) \geq \lambda_1(R) \geq \cdots \geq \lambda_{n-1}(Q) \geq \lambda_{n-1}(R) \geq \lambda_n(Q) \geq \lambda_n(R).$$

But $\lambda_n(R)$ is negative thanks to the assumption (1.6), and $\lambda_n(Q)$ is positive because $Q \in \mathcal{P}_n$. In short,

$$\lambda_{n-1}(R) > 0 > \lambda_n(R),$$

i.e., R has exactly one negative eigenvalue and $n-1$ positive eigenvalues (counting multiplicity). Our second observation is that

$$\text{“ } \langle b, w\rangle = 0 \;\text{ and }\; \langle w, Rw\rangle \leq 0 \text{ ” } \;\text{ implies }\; w = 0.$$

In view of the above property, one can apply [9, Proposition 2.1] and obtain

$$\mathcal{L}(Q,b) = \{w \in \mathbb{R}^n : \langle w, Rw\rangle \leq 0,\, \langle c, w\rangle \geq 0\}\,, \tag{1.14}$$

where c is the unique vector in $\mathbb{R}^n$ such that

$$Rc = \lambda_n(R)\, c, \quad \|c\| = 1, \quad \langle b, c\rangle > 0.$$

Note that c is a unit eigenvector of R associated with the eigenvalue $\lambda_n(R)$. The set on the right-hand side of (1.14) corresponds to an ellipsoidal cone in the Stern-Wolkowicz sense (cf. [9]). Consider now a matrix $U \in \mathcal{O}_n$ with columns formed with an orthonormal basis of eigenvectors of R. As last column of U we take the vector c. By working out the set on the right-hand side of (1.14), one arrives at the equality (1.13) with $U \in \mathcal{O}_n$ as just mentioned, and

$$G = \mathrm{Diag}\left(\frac{\lambda_1(R)}{-\lambda_n(R)}, \ldots, \frac{\lambda_{n-1}(R)}{-\lambda_n(R)}\right).$$

The above matrix G is a positive definite diagonal matrix of order $n-1$. This completes the proof of the lemma. ☐

2. The main result. As shown in [7, Theorem 1], the maximal angle of the standard ellipsoidal cone (1.12) admits the characterization

$$\theta_{\max}(\mathcal{E}(G)) = \arccos\left(\frac{\lambda_{\min}(G) - 1}{\lambda_{\min}(G) + 1}\right), \tag{2.1}$$

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where $\lambda_{\min}(G)$ is the smallest eigenvalue of G. With this information at hand, we now are ready to state the main result of this paper.

THEOREM 2.1. *Let Ω be an ellipsoid as in (1.4). Then, for all $x \in \Omega^c$, one has*

$$\vartheta(x) = \arccos\left(\frac{\lambda_{n-1}(x) + \lambda_n(x)}{\lambda_{n-1}(x) - \lambda_n(x)}\right), \tag{2.2}$$

where $\lambda_n(x)$ and $\lambda_{n-1}(x)$ denote, respectively, the smallest and the second smallest eigenvalue of the symmetric matrix

$$R_x := A - \frac{1}{\langle x - z, A(x-z)\rangle - 1}\, A(x-z)[A(x-z)]^T.$$

Proof. Note that $R_x = A - b_x b_x^T$ with b_x as in (1.8). Let $\lambda_1(x), \ldots, \lambda_n(x)$ be the whole collection of eigenvalues of R_x, which we arrange in nonincreasing order. The interlacing inequalities for R_x take the form

$$\lambda_1(A) \geq \lambda_1(x) \geq \cdots \geq \lambda_{n-1}(A) \geq \lambda_{n-1}(x) \geq \lambda_n(A) \geq \lambda_n(x).$$

As in Lemma 1.5, one deduces that

$$\lambda_{n-1}(x) > 0 > \lambda_n(x).$$

By combining Lemmas 1.4 and 1.5, one gets a representation formula for $V(x)$ of the type

$$V(x) = U[\mathcal{E}(G_x)] \tag{2.3}$$

with $U \in \mathcal{O}_n$ and

$$G_x := \mathrm{Diag}\left(\frac{\lambda_1(x)}{-\lambda_n(x)}, \ldots, \frac{\lambda_{n-1}(x)}{-\lambda_n(x)}\right).$$

Since the maximal angle of a proper cone is invariant under orthogonal transformations, the combination of (2.1) and (2.3) yields

$$\vartheta(x) \;=\; \theta_{\max}(\mathcal{E}(G_x)) \;=\; \arccos\left(\frac{\lambda_{\min}(G_x) - 1}{\lambda_{\min}(G_x) + 1}\right).$$

For completing the proof of (2.2), note that $\lambda_{\min}(G_x)$ is equal to the last diagonal entry of G_x. $\square$

The numerical computation of (2.2) offers no difficulty: one just needs to know the two smallest eigenvalues of R_x. Curiously enough, the remaining eigenvalues of R_x are irrelevant in connection with the evaluation of $\vartheta(x)$. As a matter of fact, $\vartheta(x)$ depends only on the ratio between $\lambda_{n-1}(x)$ and $-\lambda_n(x)$. We mention in passing that the smallest eigenvalue $\lambda_n(x)$ is always simple, but the second smallest eigenvalue $\lambda_{n-1}(x)$ could be multiple.

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3. Additional results and by-products. Many additional properties of $V(x)$ can be derived by exploiting the representation formula (2.3) and the fact that a standard ellipsoidal cone is a very well known mathematical object. The representation formula (2.3) holds for any matrix $U \in \mathcal{O}_n$ such that

$$\left\{\begin{array}{l}\text{the columns of } U \text{ are an orthonormal basis}\\ \text{of eigenvectors of } R_x, \text{ and } Ue_n = c_x.\end{array}\right. \tag{3.1}$$

Here, e_n is the n-th column of the identity matrix I_n and c_x stands for the unique solution to the system

$$R_x c = \lambda_n(x)\, c, \quad \|c\| = 1, \quad \langle b_x, c\rangle > 0.$$

Note that c_x is a unit vector in the interior of $V(x)$. As we shall see in the next proposition, c_x has a very interesting geometric interpretation.

3.1. Central axis of $V(x)$. Some comments on terminology are in order. The incenter of a proper cone $K \subseteq \mathbb{R}^n$ is defined as the unique solution to the maximization problem

$$\varrho(K) := \max_{w \in K \cap \mathbb{S}_n} \operatorname{dist}[w, \partial K],$$

where $\operatorname{dist}[w, \partial K]$ denotes the distance from w to the boundary of K. Geometrically speaking, the incenter of K is the "most interior" unit vectors of K. Hence, the ray generated by that vector can be seen as a sort of central axis of K. See [4, 5] for a long discussion on the theory of incenters for general proper cones.

PROPOSITION 3.1. *Let Ω be an ellipsoid as in (1.4) and let $x \in \Omega^c$. Then $V(x)$ is symmetric with respect to the line*

$$L_x := \operatorname{Ker}[R_x - \lambda_n(x) I_n].$$

In particular, the incenter of $V(x)$ is equal to c_x.

Proof. Symmetry relative to a line is to be understood in the classical sense, i.e., invariance with respect to reflections through that line. Take U as in (3.1), so that the representation formula (2.3) holds. Since the standard ellipsoidal cone $\mathcal{E}(G_x)$ is symmetric with respect to the line generated by e_n, the image of $\mathcal{E}(G_x)$ under U is symmetric with respect to the line generated by $Ue_n = c_x$. This takes care of the first part of the theorem. The second part is immediate, because if a proper cone is symmetric with respect to a line, then its incenter is a unit vector on that line. ◻

3.2. Antipodality in $V(x)$. The next theorem explains how to construct an antipodal pair of $V(x)$. By this expression one understands a pair $\{p, q\}$ of unit

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vectors in $V(x)$ that achieve the maximal angle of the cone $V(x)$.

THEOREM 3.2. *Let Ω be an ellipsoid as in (1.4) and let $x \in \Omega^c$. Take any unit vector w in the eigenspace*

$$E_x := \mathrm{Ker}[R_x - \lambda_{n-1}(x) I_n].$$

Then

$$p := \frac{1}{\sqrt{\lambda_{n-1}(x) - \lambda_n(x)}} \left(\sqrt{-\lambda_n(x)}\, w + \sqrt{\lambda_{n-1}(x)}\, c_x \right),$$
$$q := \frac{1}{\sqrt{\lambda_{n-1}(x) - \lambda_n(x)}} \left(-\sqrt{-\lambda_n(x)}\, w + \sqrt{\lambda_{n-1}(x)}\, c_x \right)$$

form an antipodal pair of $V(x)$.

Proof. Since $\lambda_{n-1}(x)$ and $\lambda_n(x)$ are distinct, the associated eigenvectors w and c_x are orthogonal. Let e_{n-1} denote the $(n-1)$-th column of I_n. Let U be a matrix as in (3.1) and with the additional property that $Ue_{n-1} = w$. In other words, the $(n-1)$-th column of U is equal to w. A direct application of [7, Theorem 1] shows that

$$\tilde{p} := \frac{1}{\sqrt{1 + \lambda_{\min}(G_x)}} \left[e_{n-1} + \sqrt{\lambda_{\min}(G_x)}\, e_n \right],$$
$$\tilde{q} := \frac{1}{\sqrt{1 + \lambda_{\min}(G_x)}} \left[-e_{n-1} + \sqrt{\lambda_{\min}(G_x)}\, e_n \right]$$

form an antipodal pair of $\mathcal{E}(G_x)$. It suffices now to use the representation formula (2.3) and observe that $(p, q) = (U\tilde{p}, U\tilde{q})$. ☐

One may see Theorem 3.2 as an extension of Theorem 2.1. Indeed, a quick computation shows that p and q are unit vectors in the boundary of $V(x)$, and that

$$\langle p, q \rangle = \frac{\lambda_{n-1}(x) + \lambda_n(x)}{\lambda_{n-1}(x) - \lambda_n(x)}.$$

This equality is consistent with the formula (2.2) for the maximal angle of $V(x)$.

3.3. A curious paradox. Our last result deals with the paradox of the ellipsoid with nondifferentiable aperture angle function. Although it is against geometric intuition, it is possible to construct a solid bounded ellipsoid in $\mathbb{R}^3$ whose aperture angle function is nondifferentiable. Such pathological ellipsoids can be constructed also in higher dimensional spaces, but not in $\mathbb{R}^2$. All this is explained next in a clear cut-manner.

THEOREM 3.3. *Let Ω be an ellipsoid as in (1.4) and let $x_* \in \Omega^c$. Let λ_* be the largest eigenvalue of the symmetric matrix*

$$S_* := A^{-1} - (x_* - z)(x_* - z)^T.$$

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Then the aperture angle function $\vartheta : \Omega^c \to \mathbb{R}$ is differentiable at x_* if and only if one of the following conditions holds:

(i) $E_* := \mathrm{Ker}\,(S_* - \lambda_* I_n)$ *has dimension* 1,
(ii) E_* *has dimension greater than* 1 *and it is contained in* $x_*^\perp$.

Proof. For all $x \in \Omega^c$, the smallest eigenvalue $\lambda_n(x)$ of R_x is simple. Hence, $\lambda_n : \Omega^c \to \mathbb{R}$ is a differentiable function. From (2.2) and the differentiability of λ_n, one deduces that ϑ is differentiable at x_* if and only if $\lambda_{n-1} : \Omega^c \to \mathbb{R}$ is differentiable at x_*. But the second smallest eigenvalue of R_x is related to the largest eigenvalue of the inverse matrix

$$R_x^{-1} = A^{-1} - (x-z)(x-z)^T.$$

In fact, for all $x \in \Omega^c$, one has

$$\lambda_{n-1}(x) = [\lambda_{\max}(R_x^{-1})]^{-1}.$$

So, everything boils down to study the differentiability at x_* of the real-valued function $x \mapsto f(x) := \lambda_{\max}(R_x^{-1})$. Note that $f(x)$ can be seen as the optimal-value

$$f(x) = \max_{\xi \in \mathbb{S}_n} \langle \xi, [A^{-1} - (x-z)(x-z)^T]\xi \rangle$$

of a parametric optimization problem. By applying Danskin's directional differentiability theorem, one sees that the directional derivative

$$f'(x_*; h) := \lim_{t \to 0^+} \frac{f(x_* + th) - f(x_*)}{t}$$

exists in all directions and it is given by

$$f'(x_*; h) \;=\; \max_{\xi \in E_* \cap \mathbb{S}_n} -2\langle x_* - z, \xi \rangle \langle \xi, h \rangle.$$

In other words,

$$f'(x_*; \cdot) \;=\; \max_{\eta \in N(x_*)} \langle \eta, \cdot \rangle$$

is the support function of the nonempty compact set

$$N(x_*) := \{-2\langle x_* - z, \xi \rangle\, \xi : \xi \in E_* \cap \mathbb{S}_n\}.$$

Hence, f is differentiable at x_* if and only if $N(x_*)$ is a singleton. This completes the proof of the theorem. $\square$

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Below we give the promised example of solid bounded ellipsoid in $\mathbb{R}^3$ with nondifferentiable aperture angle function. I thank my young colleague M. Torki (University of Avignon) who gave me a hand in building this example.

EXAMPLE 3.4. Consider the particular case

$$A = \begin{bmatrix} \frac{1}{1+\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad z = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad x_* = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

One can easily check that $x_* \in \Omega^c$. The largest eigenvalue of

$$S_* = \begin{bmatrix} 1+\sqrt{2} & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

is $\lambda_* = 1+\sqrt{2}$ and it has multiplicity equal to 2. Note that the associated eigenvector $\xi_* = (0, 1+\sqrt{2}\,, -1)^T$ is not orthogonal to x_*. Hence, ϑ is not differentiable at x_*.

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