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HARMONIC RECONSTRUCTION SYSTEMS*

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Abstract. This paper considers group reconstruction systems (GRS's), for finite dimensional real or complex Hilbert spaces $\mathcal{H}$, that are associated with unitary representations of finite abelian groups. The relation between these GRS's and the generalized Fourier matrix is established. A special type of Parseval GRS, called harmonic reconstruction system (HRS), is defined. It is shown that there exist HRS's that present maximal robustness to erasures given characterizations of certain families.

Key words. Reconstruction systems, Fusion frames, g-Frames, Maximal robustness to erasures, Group matrix, Generalized Fourier matrix.

AMS subject classifications. 42C15, 15B99, 20K15, 20C15, 15A03.

1. Introduction. Let $\mathcal{H}, \mathcal{K}$ be finite dimensional Hilbert spaces over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Let $L(\mathcal{H}, \mathcal{K})$ be the space of linear transformations from $\mathcal{H}$ to $\mathcal{K}$. Given $T \in L(\mathcal{H}, \mathcal{K})$, let T^* denote the adjoint of T. $GL(\mathcal{H})$ and $U(\mathcal{H})$ denote the group of invertible and unitary operators in $L(\mathcal{H}) = L(\mathcal{H}, \mathcal{H})$, respectively.

Let $m, n, d \in \mathbb{N}$ and $\mathbf{n} = (n_1, \dots, n_m) \in \mathbb{N}^m$. In the sequel, $\mathcal{H}$ will be a finite dimensional Hilbert space over $\mathbb{F}$ of dimension d.

Definition 1.1. A sequence $(T_i)_{i=1}^m$ where $T_i \in L(\mathbb{F}^{n_i}, \mathcal{H})$ is an $(m, \mathbf{n}, \mathcal{H})$-reconstruction system (RS) if

$$S = \sum_{i=1}^m T_i T_i^* \in GL(\mathcal{H}).$$

S is called the *RS operator* of $(T_i)_{i=1}^m$. If $S = \alpha I_{\mathcal{H}}$, $\alpha > 0$, we say that $(T_i)_{i=1}^m$ is a *tight RS*, and it is a *Parseval RS* if $S = I_{\mathcal{H}}$. If $n_1 = \cdots = n_m = n$, we write $(m, n, \mathcal{H})$-RS.

The set of $(m, \mathbf{n}, \mathcal{H})$-RS's will be denoted by $\mathcal{RS}(m, \mathbf{n}, \mathcal{H})$. The concept of $(m, n, \mathcal{H})$-RS (with $\mathbb{F}^n$ replaced by any Hilbert space $\mathcal{K}$ of dimension n) was intro-

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duced in [12] and $\big(m, \mathbf{n}, \mathbb{F}^d\big)$-RS's are considered in [13]. RS's are a generalization of frames [3, 6, 10] and fusion frames (or frames of subspaces) [4, 5]. Concretely, an $(m, 1, \mathcal{H})$-RS is a frame and projective RS's can be viewed as fusion frames (see Remark 2.3 and Remark 2.4 in [14]). In [16] RS's for not necessarily finite dimensional Hilbert spaces are called g-frames and are shown to be equivalent to stable space splittings of Hilbert spaces [15]. g-frames with $S = I_{\mathcal{H}}$ are considered in [2] under the name of coordinate operators.

In the sequel, we suppose that none of the T_i is a null operator. Suppose that $(T_i)_{i=1}^m \in \mathcal{RS}(m, n, \mathcal{H})$. In applications, an element $f \in \mathcal{H}$ (e.g., a signal) is converted into the data vectors $T_i^* f$, $i = 1, \ldots, m$. These vectors are transmitted and f is reconstructed by the receiver using

$$f = \sum_{i=1}^{m} S^{-1} T_i T_i^* f.$$

Sometimes some of the data vectors are lost, and it is necessary to reconstruct f with the partial information at hand. For this reason, it is important for $(T_i)_{i=1}^m$ to remain a RS after the erasure of certain operators T_i. If $(T_i)_{i=1}^m$ is an RS after the erasure of any $\lfloor \frac{nm-d}{m} \rfloor$ operators T_i, we say that $(T_i)_{i=1}^m$ is *maximal robust to erasures* (MRE). If we consider frames, i.e., $n = 1$, it is well known that some harmonic frames are MRE [9, 18].

In [14], two types of RS's associated with unitary representations of finite groups, called *group reconstruction systems* (GRS's) are studied. Here we consider the abelian case, i.e., GRS's associated with unitary representations of finite abelian groups. In Section 2, we study the relation between these GRS's and the generalized Fourier matrix. We present some results about diagonalization of group matrices (Theorem 2.7 and Theorem 2.9) with the aim to characterize the Gram matrix of a GRS in terms of the generalized Fourier matrix (Corollary 2.8 and Corollary 2.10). As a consequence, we obtain a characterization of these GRS's (Theorem 2.13 and Theorem 2.14).

Based on the characterization for GRS's obtained in Section 2, we define in Section 3 a special type of Parseval GRS, called *harmonic reconstruction system* (HRS). We characterize all MRE HRS's obtained by selecting d rows from a generalized Fourier matrix (Theorem 3.3). Then we characterize those with $\lceil \frac{d}{m} \rceil = 1$ (Corollary 3.4), $\lceil \frac{d}{m} \rceil = 2$ (Corollary 3.7) and n prime (Corollary 3.12). We present some examples that illustrate these characterizations.

We finish this Section introducing some notation. $\mathbb{F}^{d \times n}$ denotes the set of matrices of order $d \times n$ with entries in $\mathbb{F}$. If $M \in \mathbb{F}^{d \times n}$, then M^* denotes the conjugate transpose of M. The elements of $\mathbb{F}^n$ will be considered as column vectors, i.e., we identify $\mathbb{F}^n$ with $\mathbb{F}^{n \times 1}$, and if $f \in \mathbb{F}^n$ then $f(i)$ denotes the ith component of f. Given two elements $f, g \in \mathbb{F}^n$, we consider the product $\langle f, g \rangle_{\mathbb{F}^n} = g^* f$. The standard basis of $\mathbb{F}^n$

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will be denoted by $\left\{\delta_j^n\right\}_{j=1}^n$. Let $M \in \mathbb{F}^{d\times n}$. We denote the entry i,j, the ith row and the jth column of M with $M(i,j)$, $M(i,:)$ and $M(:,j)$, respectively. We also denote the submatrix of M consisting of the rows (columns) $M(i,:)$ with $i \in I$ ($M(:,j)$ with $j \in J$) with $M(I,:)$ ($M(:,J)$).

Given $A \in \mathbb{F}^{m\times n}, B \in \mathbb{F}^{p\times q}$ we consider the tensor product

$$A \otimes B = \begin{bmatrix} A(1,1)B & \cdots & A(1,n)B \\ \vdots & \ddots & \vdots \\ A(m,1)B & \cdots & A(m,n)B \end{bmatrix} \in \mathbb{F}^{mp\times nq}.$$

2. Group reconstruction systems and the generalized Fourier matrix. Let $\mathcal{G}$ be a group of order n. We recall that a (unitary) *representation* of $\mathcal{G}$ is a group homomorphism $\rho : \mathcal{G} \to U(\mathcal{H})$.

Definition 2.1. $(T_g)_{g\in\mathcal{G}} \in \mathcal{RS}(n,m,\mathcal{H})$ is a $(\mathcal{G},m,\mathcal{H})$-RS if there exists a representation $\rho : \mathcal{G} \to U(\mathcal{H})$ such that $\rho(g)T_h = T_{gh}$, for each $g,h \in \mathcal{G}$.

Definition 2.2. $(T_i)_{i=1}^m \in \mathcal{RS}(m,n,\mathcal{H})$ is an $(m,\mathcal{G},\mathcal{H})$-RS if there exists a representation $\rho : \mathcal{G} \to U(\mathcal{H})$ such that $\rho(g)T_i\delta_h^n = T_i\delta_{gh}^n$, for each $i = 1,\ldots,m$ and $g,h \in \mathcal{G}$.

Any of the RS's defined previously is called a *group reconstruction system* (GRS). The set of $(\mathcal{G},m,\mathcal{H})$-RS's will be denoted by $\mathcal{RS}(\mathcal{G},m,\mathcal{H})$ and the set of $(m,\mathcal{G},\mathcal{H})$-RS's will be denoted by $\mathcal{RS}(m,\mathcal{G},\mathcal{H})$.

For properties of RS's and GRS's and their relation with frames and fusion frames (or frames of subspaces), we refer the reader to [14].

Definition 2.3. $A \in \mathbb{F}^{n\times n}$ is a *$\mathcal{G}$-matrix* if there exists a function $\nu : \mathcal{G} \to \mathbb{F}$ such that $A(g,h) = \nu(g^{-1}h)$, $g,h \in \mathcal{G}$.

Definition 2.4. By a *block matrix of type* (n,m) is meant an $nm \times nm$ block matrix A with $n \times n$ blocks in $\mathbb{F}^{m\times m}$.

The sets of block matrices of type (n,m) will be denoted with $\mathcal{B}(\mathbb{F}^{nm\times nm})$ and if $A \in \mathcal{B}(\mathbb{F}^{nm\times nm})$ we denote the block (k,l) of A with $A_{k,l}$.

Definition 2.5. $A \in \mathcal{B}(\mathbb{F}^{nm\times nm})$ is a *block $\mathcal{G}$-matrix of type* (n,m) if there exists a function $\nu : \mathcal{G} \to \mathbb{F}^{m\times m}$ such that $A_{g,h} = \nu(g^{-1}h)$, $g,h \in \mathcal{G}$.

Definition 2.6. $A \in \mathcal{B}(\mathbb{F}^{mn\times mn})$ is a *block matrix of type* (m,n) *with $\mathcal{G}$-blocks* if there exist functions $\nu_{i,j} : \mathcal{G} \to \mathbb{F}$ such that $A_{i,j}(g,h) = \nu_{i,j}(g^{-1}h)$, $i,j = 1,\ldots,m$, $g,h \in \mathcal{G}$.

The set of block $\mathcal{G}$-matrices of type (n,m) will be denoted with $\mathcal{BG}(\mathbb{F}^{nm\times nm})$

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and the set of block matrices of type (m, n) with $\mathcal{G}$-blocks will be denoted with $\mathcal{GB}(\mathbb{F}^{mn\times mn})$. To express the relation between A and ν or $\{\nu_{i,j}\}_{i,j=1}^{m}$ described in the above definitions we write $A(\nu)$ or $A(\{\nu_{i,j}\}_{i,j=1}^{m})$, respectively. If $\mathcal{G}$ is a cyclic group then $A \in B\mathcal{G}(\mathbb{F}^{nm\times nm})$ is a block circulant matrix of type (n, m) and $A \in \mathcal{GB}(\mathbb{F}^{mn\times mn})$ is a block matrix of type (m, n) with circulant blocks (see [7]). The notation $\mathcal{BG}$ and $\mathcal{GB}$ used here is analogous to the notation $\mathcal{BC}$ and $\mathcal{CB}$ used in [7] in the circulant case.

In the sequel, we suppose that $\mathcal{G}$ is abelian.

It is well known that $\mathcal{G}$ is isomorphic to a direct product of cyclic groups $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$ where $\mathbb{Z}_{n_k}$ denotes as usual the cyclic group of integers modulo n_k and $n = \prod_{k=1}^{r} n_k$ (see, e.g., [1]). Given $g \in \mathcal{G}$, we denote its image under this isomorphism with $(g_1, \dots, g_r) \in \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$. In $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$ is defined the componentwise addition modulo n_k, $k = 1, \dots, r$. The generalized Fourier matrix of order n over $\mathcal{G}$ will be denoted with $\mathbf{F}_{\mathcal{G}}$ and it is given by $\mathbf{F}_{\mathcal{G}}(g, h) = \frac{1}{\sqrt{n}} \langle g, h\rangle$ where $\langle g, h\rangle = \prod_{k=1}^{r} e^{-2\pi i \frac{g_k h_k}{n_k}}$ with $g_k h_k$ taken modulo n_k. In particular, if $\mathcal{G}$ is cyclic, we have the usual Fourier matrix of order n denoted with $\mathbf{F}_n$ and given by $\mathbf{F}_n(k, l) = \frac{1}{\sqrt{n}} e^{-\frac{2\pi i}{n}(k-1)(l-1)}$. Clearly, $\mathbf{F}_{\mathcal{G}}$ is a unitary $\mathcal{G}$-matrix and $\mathbf{F}_{\mathcal{G}} = \mathbf{F}_{n_1} \otimes \cdots \otimes \mathbf{F}_{n_r}$. Recall that the characters of $\mathcal{G}$ are the group homomorphisms $\xi : \mathcal{G} \to \mathbb{C} \setminus \{0\}$ where $\mathbb{C} \setminus \{0\}$ is a group under multiplication. Denoting with $\{\xi_j\}_{j=1}^{n}$ the set of characters of $\mathcal{G}$, then $(\xi_j(g))_{j=1,\dots,n,g\in\mathcal{G}} = n\mathbf{F}_{\mathcal{G}}$ is called the *character table* of $\mathcal{G}$.

Given $g \in \mathcal{G}$, let

$$\Omega_{\mathcal{G},g}(f, h) = \begin{cases} \langle g, h\rangle, & f = h; \\ 0, & \text{otherwise,} \end{cases}$$

and $\nu_g : \mathcal{G} \to \mathbb{C}$ defined by

$$\nu_g(h) = \begin{cases} 1, & h = g; \\ 0, & \text{otherwise.} \end{cases}$$

Let $C(\nu_g)$ be the $\mathcal{G}$-matrix associated with ν_g and $A(\nu)$ any $\mathcal{G}$-matrix, then

$$A(\nu) = \sum_{g\in\mathcal{G}} \nu(g)\, C(\nu_g). \tag{2.1}$$

We have

$$C(\nu_g) = \mathbf{F}_{\mathcal{G}}^{*} \Omega_{\mathcal{G},g}^{*} \mathbf{F}_{\mathcal{G}}. \tag{2.2}$$

By (2.1) and (2.2),

$$A(\nu) = \mathbf{F}_{\mathcal{G}}^{*} \Omega_{\mathcal{G},\nu}^{*} \mathbf{F}_{\mathcal{G}}, \tag{2.3}$$

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where

$$\Omega_{\mathcal{G},\nu} = \sum_{g\in\mathcal{G}} \nu\left(g\right)\Omega_{\mathcal{G},g} = \begin{cases} \sum_{g\in\mathcal{G}} \nu\left(g\right)\left\langle g,h\right\rangle, & f=h; \\ 0, & \text{otherwise.}\end{cases}$$

Theorem 2.7 and Theorem 2.9 below give a generalization of (2.3) for the elements in $\mathcal{BG}\left(\mathbb{F}^{nm\times nm}\right)$ and $\mathcal{GB}\left(\mathbb{F}^{mn\times mn}\right)$ respectively.

THEOREM 2.7. *$A\left(\nu\right)\in\mathcal{BG}\left(\mathbb{F}^{nm\times nm}\right)$ if and only if*

$$A\left(\nu\right) = \left(\mathbf{F}_{\mathcal{G}}\otimes\mathbf{F}_m\right)^*\left(I_{\mathcal{G}}\otimes\mathbf{F}_m\right)\operatorname{diag}\left(A_g\right)_{g\in\mathcal{G}}\left(I_{\mathcal{G}}\otimes\mathbf{F}_m\right)^*\left(\mathbf{F}_{\mathcal{G}}\otimes\mathbf{F}_m\right) \tag{2.4}$$

where $A_g = \sum_{h\in\mathcal{G}}\left\langle -h,g\right\rangle A_{1,h}$.

Proof. If $A\left(\nu\right)\in\mathcal{BG}\left(\mathbb{F}^{nm\times nm}\right)$ then $A\left(\nu\right)=\sum_{g\in\mathcal{G}} C\left(\nu_g\right)\otimes A_{1,g}$. Therefore, from (2.2),

$$A\left(\nu\right) = \left(\mathbf{F}_{\mathcal{G}}^*\otimes\mathbf{F}_m^*\right)\left(\sum_{g\in\mathcal{G}}\Omega_{\mathcal{G},g}^*\otimes\left(\mathbf{F}_m A_{1,g}\mathbf{F}_m^*\right)\right)\left(\mathbf{F}_{\mathcal{G}}\otimes\mathbf{F}_m\right).$$

We have

$$\left(\sum_{g\in\mathcal{G}}\Omega_{\mathcal{G},g}^*\otimes\left(\mathbf{F}_m A_{1,g}\mathbf{F}_m^*\right)\right)_{f,h} = \begin{cases}\mathbf{F}_m\left(\sum_{g\in\mathcal{G}}\left\langle -g,h\right\rangle A_{1,g}\right)\mathbf{F}_m^*, & f=h; \\ 0, & \text{otherwise.}\end{cases}$$

Thus, (2.4) holds. Now suppose that (2.4) holds. So,

$$A = \left(\mathbf{F}_{\mathcal{G}}\otimes\mathbf{F}_m\right)^*\operatorname{diag}\left(B_g\right)_{g\in\mathcal{G}}\left(\mathbf{F}_{\mathcal{G}}\otimes\mathbf{F}_m\right)$$

for certain matrices $B_g\in\mathbb{F}^{m\times m}$. Thus,

$$A_{g,h} = \sum_{f\in\mathcal{G}}\mathbf{F}_{\mathcal{G}}^*\left(g,f\right)\mathbf{F}_m B_f \mathbf{F}_{\mathcal{G}}\left(f,h\right)\mathbf{F}_m.$$

Since $\mathbf{F}_{\mathcal{G}}^*\left(gh_1,f\right)\mathbf{F}_{\mathcal{G}}\left(f,gh_2\right)=\mathbf{F}_{\mathcal{G}}^*\left(h_1,f\right)\mathbf{F}_{\mathcal{G}}\left(f,h_2\right)$, then $A_{gh_1,gh_2}=A_{h_1,h_2}$, and by Lemma 4.4 in [14], $A\in\mathcal{BG}\left(\mathbb{F}^{nm\times nm}\right)$. ◻

Associated with $(T_i)_{i=1}^m$ where $T_i\in L(\mathbb{F}^{n_i},\mathcal{H})$, is the *Gramian operator*

$$G = T^*T : \bigoplus_{i=1}^m\mathbb{F}^{n_i}\to\bigoplus_{i=1}^m\mathbb{F}^{n_i},\; G\left(\left(x_i\right)_{i=1}^m\right) = \left(T_i^*\left(\sum_{j=1}^m T_j x_j\right)\right)_{i=1}^m.$$

In matrix form, $G\in\mathbb{F}^{\operatorname{tr}(\mathbf{n})\times\operatorname{tr}(\mathbf{n})}$ is a block matrix with blocks $T_i^*T_j\in\mathbb{F}^{n_i\times n_j}$.

COROLLARY 2.8. *$\left(T_g\right)_{g\in\mathcal{G}}\in\mathcal{RS}\left(\mathcal{G},m,\mathcal{H}\right)$ if and only if G can be written as in (2.4) and* $\operatorname{rank}\left(G\right)=d$.

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Proof. It follows from Theorem 4.5 in [14] and Theorem 2.7. ◻

THEOREM 2.9. *$A\left(\{\nu_{k,l}\}_{k,l=1}^{m}\right) \in \mathcal{GB}\left(\mathbb{F}^{mn\times mn}\right)$ if and only if*

$$A = \left(\mathbf{F}_m \otimes \mathbf{F}_{\mathcal{G}}\right)^* \left(\mathbf{F}_m \otimes I_{\mathcal{G}}\right) \left(\Omega^*_{\mathcal{G},\nu_{k,l}}\right)_{k,l=1}^{m} \left(\mathbf{F}_m \otimes I_{\mathcal{G}}\right)^* \left(\mathbf{F}_m \otimes \mathbf{F}_{\mathcal{G}}\right). \tag{2.5}$$

Proof. By (2.3), if $A\left(\{\nu_{k,l}\}_{k,l=1}^{m}\right) \in \mathcal{GB}\left(\mathbb{F}^{mn\times mn}\right)$ then $A_{k,l} = \mathbf{F}^*_{\mathcal{G}}\Omega^*_{\mathcal{G},\nu_{k,l}}\mathbf{F}_{\mathcal{G}}$. Thus, (2.5) holds. Now suppose that (2.5) holds. Then

$$A = \left(\mathbf{F}_m \otimes \mathbf{F}_{\mathcal{G}}\right)^* \left(\Lambda_{k,l}\right)_{k,l=1}^{m} \left(\mathbf{F}_m \otimes \mathbf{F}_{\mathcal{G}}\right)$$

where $\Lambda_{k,l}(f,g) = 0$ if $f \neq g$. Thus,

$$A_{k,l} = \sum_{r_1,r_2=1}^{m} \mathbf{F}^*_m(k,r_1)\,\mathbf{F}^*_{\mathcal{G}}\Lambda_{r_1,r_2}\mathbf{F}_m(r_2,l)\,\mathbf{F}_{\mathcal{G}}$$

$$\begin{aligned}
\left\langle \mathbf{F}^*_{\mathcal{G}}\Lambda_{r_1,r_2}\mathbf{F}_{\mathcal{G}}\delta_{gh_2}, \delta_{gh_1}\right\rangle_{\mathbb{F}^n} &= \left\langle \Lambda_{r_1,r_2}\mathbf{F}_{\mathcal{G}}\delta_{gh_2}, \mathbf{F}_{\mathcal{G}}\delta_{gh_1}\right\rangle_{\mathbb{F}^n} \\
&= \sum_{f\in\mathcal{G}} \Lambda_{r_1,r_2}(f,f)\,\mathbf{F}_{\mathcal{G}}(f,gh_2)\,\overline{\mathbf{F}_{\mathcal{G}}(f,gh_1)} \\
&= \sum_{f\in\mathcal{G}} \Lambda_{r_1,r_2}(f,f)\,\mathbf{F}_{\mathcal{G}}(f,h_2)\,\overline{\mathbf{F}_{\mathcal{G}}(f,h_1)} \\
&= \left\langle \mathbf{F}^*_{\mathcal{G}}\Lambda_{r_1,r_2}\mathbf{F}_{\mathcal{G}}\delta_{h_2}, \delta_{h_1}\right\rangle_{\mathbb{F}^n}.
\end{aligned}$$

Therefore, $A_{k,l}(gh_1,gh_2) = A_{k,l}(h_1,h_2)$, for each $g,h_1,h_2 \in \mathcal{G}$, and, by Lemma 4.9 in [14], $A \in \mathcal{GB}\left(\mathbb{F}^{mn\times mn}\right)$. ◻

COROLLARY 2.10. *$(T_k)_{k=1}^{m} \in \mathcal{RS}(m,\mathcal{G},\mathcal{H})$ if and only if G can be written as in (2.5) and* $\operatorname{rank}(G) = d$.

Proof. It follows from Theorem 4.10 in [14] and Theorem 2.9. ◻

REMARK 2.11. We can replace $\mathbf{F}_m$ in (2.4) and (2.5) by any other unitary matrix in $\mathbb{F}^{m\times m}$.

REMARK 2.12. Theorem 2.7 and Theorem 2.9 are a generalization of Theorem 5.6.4 and Theorem 5.7.3 in [7], respectively.

THEOREM 2.13. *Let $(T_g)_{g\in\mathcal{G}} \in \mathcal{RS}(n,m,\mathcal{H})$. $(T_g)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G},m,\mathcal{H})$ if and only if there exist $Q: \mathbb{F}^d \to \mathcal{H}$ unitary, $A \in \mathbb{F}^{d\times m}$ and unitary matrices $D_g \in \mathbb{F}^{d\times d}$ obtained from $\Omega_{\mathcal{G},g} \otimes I_m$, deleting $nm-d$ rows and columns, such that $T_g = QD_gA$ for each $g \in \mathcal{G}$.*

Proof. Let $(T_g)_{g\in\mathcal{G}} \in \mathcal{RS}(n,m,\mathcal{H})$.

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Suppose that $T_g = QD_gA$, for each $g \in \mathcal{G}$, where Q, D_g and A are as in the enunciation. Since

$$\begin{aligned} G_{gh_1,gh_2} &= T_{gh_1}^* T_{gh_2} = A^* D_{gh_1}^* Q^* Q D_{gh_2} A \\ &= A^* D_{gh_1}^* D_{gh_2} A = A^* D_{h_1}^* D_{h_2} A = T_{h_1}^* T_{h_2} = G_{h_1,h_2}, \end{aligned}$$

by Lemma 4.4 in [14], $G \in \mathcal{BG}(\mathbb{F}^{nm\times nm})$. Thus, by Theorem 4.5 in [14], $(T_g)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G}, m, \mathcal{H})$.

Suppose now that $(T_g)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G}, m, \mathcal{H})$. By Corollary 2.8, G can be written as in (2.4). Then

$$G = (\mathbf{F}_{\mathcal{G}} \otimes \mathbf{F}_m)^* \operatorname{diag}(B_g)_{g\in\mathcal{G}} (\mathbf{F}_{\mathcal{G}} \otimes \mathbf{F}_m), \tag{2.6}$$

for certain matrices $B_g \in \mathbb{F}^{m\times m}$. Since G is Hermitian and positive semidefinite, each B_g is Hermitian and positive semidefinite. Thus, we can write $B_g = U_g^* \Lambda_g^* \Lambda_g U_g$ where $U_g, \Lambda_g \in \mathbb{F}^{m\times m}$, U_g is unitary and Λ_g is diagonal. Thus,

$$G = (\mathbf{F}_{\mathcal{G}} \otimes \mathbf{F}_m)^* \operatorname{diag}\left(U_g^* \Lambda_g^*\right)_{g\in\mathcal{G}} \operatorname{diag}(\Lambda_g U_g)_{g\in\mathcal{G}} (\mathbf{F}_{\mathcal{G}} \otimes \mathbf{F}_m).$$

We have

$$\operatorname{diag}(\Lambda_g U_g)_{g\in\mathcal{G}} (\mathbf{F}_{\mathcal{G}} \otimes \mathbf{F}_m) = (\Omega_{\mathcal{G},h}(g,g)\mathbf{F}_{\mathcal{G}}(g,1)\Lambda_g U_g \mathbf{F}_m)_{g,h\in\mathcal{G}}.$$

Taking into account that $\operatorname{rank}(G) = d$, let $A \in \mathbb{F}^{d\times m}$ be the matrix obtained from

$$(\mathbf{F}_{\mathcal{G}}(g,1)\Lambda_g U_g \mathbf{F}_m)_{g\in\mathcal{G}} \in \mathbb{F}^{nm\times m}$$

deleting the $nm - d$ null rows corresponding to the null eigenvalues in $\operatorname{diag}(\Lambda_g)_{g\in\mathcal{G}}$ and let $D_g \in \mathbb{F}^{d\times d}$ be the diagonal matrix obtained from

$$\Omega_{\mathcal{G},g} \otimes I_m \in \mathbb{F}^{nm\times nm}$$

deleting accordingly $nm - d$ rows and columns. Since $\{D_g A\}_{g\in\mathcal{G}} \in \mathcal{RS}\left(n, m, \mathbb{F}^d\right)$ has Gram matrix G, by Lemma 2.5 in [14], there exists $Q : \mathbb{F}^d \to \mathcal{H}$ unitary such that $T_g = QD_gA$. $\square$

Theorem 2.14. *Let $(T_i)_{i=1}^m \in \mathcal{RS}(m, n, \mathcal{H})$. $(T_i)_{i=1}^m \in \mathcal{RS}(m, \mathcal{G}, \mathcal{H})$ if and only if there exist $Q : \mathbb{F}^d \to \mathcal{H}$ unitary, $f_i \in \mathbb{F}^d$, $i = 1, \ldots, m$, and unitary matrices $D_g \in \mathbb{F}^{d\times d}$ obtained from $\Omega_{\mathcal{G},g} \otimes I_m \in \mathbb{F}^{nm\times nm}$ deleting $nm - d$ rows and columns, such that $T_i\delta_g^n = QD_g f_i$ for each $i = 1, \ldots, m$ and each $g \in \mathcal{G}$.*

Proof. Let $(T_i)_{i=1}^m \in \mathcal{RS}(m, n, \mathcal{H})$.

Suppose that $T_i\delta_g^n = QD_g f_i$, for each $i = 1, \ldots, m$ and each $g \in \mathcal{G}$, where Q, D_g and f_i are as in the enunciation. Since

$$G_{i,j}(g,h) = \left\langle T_i^* T_j \delta_h^n, \delta_g^n \right\rangle_{\mathbb{F}^n} = \left\langle T_j \delta_h^n, T_i \delta_g^n \right\rangle_{\mathcal{H}} = \langle QD_h f_j, QD_g f_i \rangle_{\mathcal{H}}$$

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$$
\begin{aligned}
&= \langle D_h f_j, D_g f_i\rangle_{\mathbb{F}^d} = \left\langle D_g^* D_h f_j, f_i\right\rangle_{\mathbb{F}^d} = \left\langle D_{g^{-1}h} f_j, f_i\right\rangle_{\mathbb{F}^d} \\
&= \left\langle Q D_{g^{-1}h} f_j, Q f_i\right\rangle_{\mathcal{H}} = \left\langle T_j \delta^n_{g^{-1}h}, T_i \delta^n_1\right\rangle_{\mathcal{H}} = \left\langle T_i^* T_j \delta^n_{g^{-1}h}, \delta^n_1\right\rangle_{\mathbb{F}^n} \\
&= G_{i,j}\left(1, g^{-1}h\right),
\end{aligned}
$$

by Lemma 4.9 in [14], $G \in \mathcal{GB}(\mathbb{F}^{mn\times mn})$ and then, by Theorem 4.10 in [14], $(T_i)_{i=1}^m \in \mathcal{RS}(m, \mathcal{G}, \mathcal{H})$.

Suppose now that $(T_i)_{i=1}^m \in \mathcal{RS}(m, \mathcal{G}, \mathcal{H})$. By Remark 3.3 in [14], if

$$T_g \delta_i^m := T_i \delta_g^n$$

then $(T_g)_{g\in\mathcal{G}} \in \mathcal{RS}(\mathcal{G}, m, \mathcal{H})$. By Theorem 2.13, there exist $Q : \mathbb{F}^d \to \mathcal{H}$ unitary, $A \in \mathbb{F}^{d\times m}$ and unitary matrices $D_g \in \mathbb{F}^{d\times d}$ obtained from $\Omega_{\mathcal{G},g} \otimes I_m \in \mathbb{F}^{nm\times nm}$ deleting $nm - d$ rows and columns, such that

$$T_g = Q D_g A.$$

Therefore,

$$T_i \delta_g^n = T_g \delta_i^m = Q D_g A \delta_i^m. \quad \square$$

Remark 2.15. Let $A \in \mathbb{F}^{d\times m}$ and $\{D_g A\}_{g\in\mathcal{G}} \in \mathcal{RS}\left(\mathcal{G}, m, \mathbb{F}^d\right)$ where $\{D_g\}_{g\in\mathcal{G}}$ is obtained as in Theorem 2.13. $\rho : \mathcal{G} \to \mathcal{U}\left(\mathbb{F}^d\right)$ given by $\rho(g) = D_g$ is a reducible representation of $\mathcal{G}$ that can be decomposed as a sum of one-dimensional irreducible representations $\rho(g) f = \sum_{i=1}^d \rho_i(g) f_i$, where $\rho_i(g) = \langle g, h\rangle$ for some $h \in \mathcal{G}$ and $f = \oplus_{i=1}^d f_i$, $f_i \in \mathbb{F}$. By the one-dimensionality of the representations, the $\mathbb{F}\mathcal{G}$-modules corresponding to ρ_i and ρ_j are $\mathbb{F}\mathcal{G}$-isomorphic if and only if $\rho_i = \rho_j$. If $m = 1$, the ρ_i are all different, therefore, by Theorem 6.10 in [14], $\{D_g A\}_{g\in\mathcal{G}} \in \mathcal{RS}\left(\mathcal{G}, 1, \mathbb{F}^d\right)$ is Parseval if and only if $\|A(i, :)\| = \frac{1}{\sqrt{n}}$, $i = 1, \ldots, d$. If $m > 1$, by Theorem 6.10 in [14], $\{D_g A\}_{g\in\mathcal{G}} \in \mathcal{RS}\left(\mathcal{G}, m, \mathbb{F}^d\right)$ is Parseval if and only if $\|A(i, :)\| = \frac{1}{\sqrt{n}}$, $i = 1, \ldots, d$ and if $\rho_i = \rho_j$, $i \neq j$, then $\langle A(i, :), A(j, :)\rangle = 0$. If $m \leq d$, by Proposition 3.8(5) in [14], $\{D_g A\}_{g\in\mathcal{G}} \in \mathcal{RS}\left(\mathcal{G}, m, \mathbb{F}^d\right)$ is projective if A has orthogonal columns of equal norm.

3. Harmonic reconstruction systems.

The proof of Theorem 2.13 provides a method for constructing Parseval GRS's by considering in (2.6) diagonal matrices B_g with 1's as non-zero elements in the diagonal.

Definition 3.1. $(T_g)_{g\in\mathcal{G}}$, where $T_g \in L(\mathbb{F}^m, \mathcal{H})$, is a *harmonic* $(\mathcal{G}, m, \mathcal{H})$-RS ($(\mathcal{G}, m, \mathcal{H})$-HRS) if there exist $Q : \mathbb{F}^d \to \mathcal{H}$ unitary, $D_g \in \mathbb{F}^{d\times d}$ obtained from $\Omega_{\mathcal{G},g} \otimes I_m$ deleting $nm - d$ rows and columns, and A obtained from $e \otimes \frac{1}{\sqrt{n}} F_m$, $e = (1, \ldots, 1)^t \in \mathbb{F}^{n\times 1}$, deleting correspondingly $nm - d$ rows, such that $T_g = Q D_g A$ for each $g \in \mathcal{G}$. In particular, $(\mathcal{G}, m, \mathcal{H})$-HRS is a *cyclic* $(\mathcal{G}, m, \mathcal{H})$-RS ($(\mathcal{G}, m, \mathcal{H})$-CRS) if $\mathcal{G}$ is cyclic.

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The set of $(\mathcal{G}, m, \mathcal{H})$-HRS's will be denoted by $\mathcal{HRS}(\mathcal{G}, m, \mathcal{H})$, and the set of $(\mathcal{G}, m, \mathcal{H})$-CRS's will be denoted by $\mathcal{CRS}(\mathcal{G}, m, \mathcal{H})$.

We denote the set of permutations (i.e., bijections) on $\{1, \ldots, m\}$ with $\mathcal{S}_m$.

Definition 3.2. $(T_i)_{i=1}^m$, where $T_i \in L(\mathbb{F}^n, \mathcal{H})$, is a *harmonic* $(m, \mathcal{G}, \mathcal{H})$-RS ($(m, \mathcal{G}, \mathcal{H})$-HRS) if there exist $(QD_gA)_{g\in\mathcal{G}} \in \mathcal{HRS}(\mathcal{G}, m, \mathcal{H})$, an isomorphism $\sigma : \mathcal{G} \to \mathcal{G}$ and $\pi \in \mathcal{S}_m$ such that $T_i\delta_g^n = D_{\sigma(g)}A\delta_{\pi(i)}^m$. In particular, if $\mathcal{G}$ is cyclic, $(T_i)_{i=1}^m$ is a *cyclic* $(m, \mathcal{G}, \mathcal{H})$-RS ($(m, \mathcal{G}, \mathcal{H})$-CRS).

As a consequence of Remark 2.15, $(\mathcal{G}, m, \mathcal{H})$-HRS's are Parseval, and then, by Proposition 3.4 in [14], $(m, \mathcal{G}, \mathcal{H})$-HRS's are Parseval too.

The set of $(m, \mathcal{G}, \mathcal{H})$-HRS's will be denoted by $\mathcal{HRS}(m, \mathcal{G}, \mathcal{H})$, and the set of $(m, \mathcal{G}, \mathcal{H})$-CRS's will be denoted by $\mathcal{CRS}(m, \mathcal{G}, \mathcal{H})$.

We generically refer to any of the RS's defined previously as a *harmonic reconstruction system* (HRS).

Proposition 3.8(4) in [14] asserts that there exist one erasure robust tight GRS's. In case of HRS's, we can obtain maximal robustness to erasures.

From now on we restrict our attention to HRS's (for $\mathbb{F}^d$) of the form $\{D_gA\}_{g\in\mathcal{G}}$ where D_g and A are as in Definition 3.1. Note that each D_gA is formed by a set of m columns of a matrix obtained by selecting d rows from $\mathbf{F}_\mathcal{G} \otimes \mathbf{F}_m$. Therefore, if $m = 1$ they coincide, up to a factor of n, with harmonic frames as defined in [17]. By Lemma 2.5 in [14], if such a HRS for $\mathbb{F}^d$ is MRE then any other HRS for $\mathcal{H}$ of the form $\{QD_gA\}_{g\in\mathcal{G}}$, with $Q : \mathbb{F}^d \to \mathcal{H}$ unitary, is MRE too.

We are going to characterize HRS's that are MRE.

Let $\mathbf{F}_\mathcal{G} = \mathbf{F}_{n_1} \otimes \cdots \otimes \mathbf{F}_{n_r}$. If $k, l \in \{1, \ldots, n_1\} \times \cdots \times \{1, \ldots, n_r\}$ and $i, j \in \{1, \ldots, m\}$, then the element in the row (k, i) and column (l, j) of $\mathbf{F}_\mathcal{G} \otimes \mathbf{F}_m$ is $\mathbf{F}_\mathcal{G} \otimes \mathbf{F}_m((k, i), (l, j)) = \mathbf{F}_\mathcal{G}(k, l)\mathbf{F}_m(i, j)$. We consider the projections p_1 and p_2 given by $p_1(k_1, \ldots, k_r, i) = (k_1, \ldots, k_r)$ and $p_2(k_1, \ldots, k_r, i) = i$. If $I \subseteq \{1, \ldots, n_1\} \times \cdots \times \{1, \ldots, n_r\} \times \{1, \ldots, m\}$ we denote the partition of I associated to the surjection p_2 with I/p_2. Note that if $d > m(n-1)$ we can not erase blocks. In the sequel, we suppose that $2 \le d \le m(n-1)$ and $m \ge 2$. If the column blocks of $\mathbf{F}_\mathcal{G} \otimes \mathbf{F}_m(I, :) \in \mathbb{F}^{d\times nm}$ form a MRE HRS we say that $\mathbf{F}_\mathcal{G} \otimes \mathbf{F}_m(I, :)$ is MRE.

The next theorem gives a characterization of all HRS's that are MRE.

Theorem 3.3. *Let* $\mathbf{F}_\mathcal{G} = \mathbf{F}_{n_1} \otimes \cdots \otimes \mathbf{F}_{n_r}$. *Let* $I \subseteq \{1, \ldots, n_1\} \times \cdots \times \{1, \ldots, n_r\} \times \{1, \ldots, m\}$ *with* $|I| = d$. *Then* $\mathbf{F}_\mathcal{G} \otimes \mathbf{F}_m(I, :) \in \mathbb{F}^{d\times nm}$ *is MRE if and only if for each* $\widetilde{I} \in I/p_2$ *and* $L \subseteq \{1, \ldots, n\}$ *with* $|L| = \lceil \frac{d}{m} \rceil$, *we have:*

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1. $\left|\widetilde{I}\right| \leq \lceil \frac{d}{m} \rceil$.

2. $\left(\mathbf{F}_{\mathcal{G}}\left(k,l\right) \right)_{k\in p_1\left(\widetilde{I}\right),l\in L}$ *has rank* $\left|\widetilde{I}\right|$.

Proof. $\mathbf{F}_{\mathcal{G}} \otimes \mathbf{F}_m(I,:)$ is MRE if and only if each submatrix of the form

$$\left(\mathbf{F}_{\mathcal{G}}(k,l)\mathbf{F}_m(i,:) \right)_{(k,i)\in I, l\in J}$$

with $J \subseteq \{1,\ldots,n_1\} \times \ldots \times \{1,\ldots,n_r\}$, $|J| = \lceil \frac{d}{m} \rceil$, has rank equal d, i.e., its rows are linearly independent. Let $\lambda_{k,i} \in \mathbb{F}$, $(k,i) \in I$, be such that

$$\sum_{(k,i)\in I} \lambda_{k,i}\mathbf{F}_{\mathcal{G}}(k,l)\mathbf{F}_m(i,:) = 0,\ l \in L$$

or equivalently, by the linear independence of the rows of $\mathbf{F}_m$,

$$\sum_{(k,i)\in \widetilde{I}} \lambda_{k,i}\mathbf{F}_{\mathcal{G}}(k,l) = 0,\ \widetilde{I} \in I/p_2,\ l \in L.$$

From this equality the result follows. ☐

For certain families of HRS's we now give an explicit description of those $I \subseteq \{1,\ldots,n_1\} \times \cdots \times \{1,\ldots,n_r\} \times \{1,\ldots,m\}$ with $|I| = d$ such that $\mathbf{F}_{\mathcal{G}} \otimes \mathbf{F}_m(I,:)$ is MRE.

As an immediate consequence of Theorem 3.3, we obtain the following corollary.

COROLLARY 3.4. *Let* $\mathbf{F}_{\mathcal{G}} = \mathbf{F}_{n_1} \otimes \cdots \otimes \mathbf{F}_{n_r}$. *Let* $I \subseteq \{1,\ldots,n_1\} \times \cdots \times \{1,\ldots,n_r\} \times \{1,\ldots,m\}$ *with* $|I| = d$. *If* $\lceil \frac{d}{m} \rceil = 1$ *then* $\mathbf{F}_{\mathcal{G}} \otimes \mathbf{F}_m(I,:)$ *is MRE if and only if* I *has no two elements with the same second component.*

To prove Corollary 3.7 we need the following two lemmas.

LEMMA 3.5. *Let* n_1, n_2 *be natural numbers greater than* 1 *such that* $n_2|n_1$ *and let* k_1, k_2 *be natural numbers with* $1 \leq k_1 \leq n_1 - 1$ *and* $1 \leq k_2 \leq n_2 - 1$. *Then there exist integer numbers* l_1, l_2 *with* $0 < l_1 \leq n_1 - 1$, $0 \leq l_2 \leq n_2 - 1$ *such that* $\frac{k_1 l_1}{n_1} + \frac{k_2 l_2}{n_2} \in \mathbb{Z}$.

Proof. Take $l_1 = k_2 \frac{n_1}{n_2}$. We consider the following cases:

(a) $k_1 < n_2$. Take $l_2 = n_2 - k_1$.

(b) $k_1 \geq n_2$ and $k_1 = q_1 n_2 + p_1$ with $0 \leq p_1 < n_2$. If $p_1 \neq 0$ take $l_2 = n_2 - p_1$, if $p_1 = 0$ take $l_2 = 0$. ☐

LEMMA 3.6. *Let* n_i *and* k_i, $i = 1,\ldots,s$, *be natural numbers such that* $n_i > 1$, $1 \leq k_i \leq n_i - 1$, *and* $\gcd(n_i, n_j) = 1$ *for each* $i \neq j$. *Then for all integer numbers* l_i *with* $|l_i| \leq n_i - 1$, $i = 1,\ldots,s$, *and* $(l_1,\ldots,l_s) \neq (0,\ldots,0)$, *we have* $\sum_{i=1}^{s} \frac{k_i l_i}{n_i} \notin \mathbb{Z}$ *if and only if* $\gcd(k_i, n_i) = 1$ *for some* i.

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Proof. Suppose that there exist integer numbers l_i with $|l_i| \leq n_i - 1$, $i = 1, \ldots, s$, and $(l_1, \ldots, l_s) \neq (0, \ldots, 0)$, such that $\sum_{i=1}^{s} \frac{k_i l_i}{n_i} \in \mathbb{Z}$. Set $m_i = \prod_{j \neq i} n_j$ and $\sum_{i=1}^{s} \frac{k_i l_i}{n_i} = q$. Then

$$\sum_{i=1}^{s} k_i l_i m_i = q \prod_{i=1}^{s} n_i.$$

Let $c_i = \gcd(l_i, n_i)$, $n_i^{'} = \frac{n_i}{c_i}$, $l_i^{'} = \frac{l_i}{c_i}$ and $m_i^{'} = \prod_{j \neq i} n_j^{'}$. Note that since $l_i < n_i$ then $n_i^{'} > 1$. Dividing both sides of the previous equality by $\prod_{i=1}^{s} c_i$ we obtain,

$$\sum_{i=1}^{s} k_i l_i^{'} m_i^{'} = q \prod_{i=1}^{s} n_i^{'}.$$

Therefore, $n_i^{'} | k_i l_i^{'} m_i^{'}$. Since $\gcd(n_i^{'}, n_j^{'}) = 1$ for $i \neq j$, and $\gcd(n_i^{'}, l_i^{'}) = 1$, then $n_i^{'} | k_i$. So, k_i and n_i are not coprimes for each $i = 1, \ldots, s$.

Suppose now that k_i and n_i are not coprimes for each $i = 1, \ldots, s$. Thus, $k_i = n_i^{'} q_i$ with $n_i^{'}$ a proper divisor of n_i. Let l_i be such that $n_i = l_i n_i^{'}$. Then $1 < l_i < n_i$ and $\sum_{i=1}^{s} \frac{k_i l_i}{n_i} = \sum_{i=1}^{s} q_i \in \mathbb{Z}$. ▯

For $k, k' \in \{1, \ldots, n_1\} \times \cdots \times \{1, \ldots, n_r\}$ we consider the support of $k - k'$, $supp(k - k') = \{i \in \{1, \ldots, r\} : k_i \neq k_i'\}$.

Corollary 3.7. *Let $\lceil \frac{d}{m} \rceil = 2$ and $\mathbf{F}_{\mathcal{G}} = \mathbf{F}_{n_1} \otimes \cdots \otimes \mathbf{F}_{n_r}$. If $I \subseteq \{1, \ldots, n_1\} \times \cdots \times \{1, \ldots, n_r\} \times \{1, \ldots, m\}$ with $|I| = d$, then $\mathbf{F}_{\mathcal{G}} \otimes \mathbf{F}_m(I, :)$ is MRE if and only if for all $\widetilde{I} \in I/p_2$:*

1. $\left|\widetilde{I}\right| \leq 2$.
2. *for each $k = (k_1, \ldots, k_r), k' = (k_1', \ldots, k_r') \in p_1\left(\widetilde{I}\right)$ with $|supp(k - k')| \geq 1$, we have $gcd(n_i, n_j) = 1$ for each $i, j \in supp(k - k'), i \neq j$, and $\gcd(k_i' - k_i, n_i) = 1$ for some $i \in supp(k - k')$.*

Proof. By Theorem 3.3, $\mathbf{F}_{\mathcal{G}} \otimes \mathbf{F}_m(I, :)$ is MRE if and only if for each $\widetilde{I} \in I/p_2$, $\left|\widetilde{I}\right| \leq 2$ and $\left(\mathbf{F}_{\mathcal{G}}(k, l) \right)_{k \in p_1(\widetilde{I}), l \in L}$ has rank $\left|\widetilde{I}\right|$ for each $L \subseteq \{1, \ldots, n_1\} \times \cdots \times \{1, \ldots, n_r\}$ with $|L| = 2$.

Let $\widetilde{I} \in I/p_2$. If $\left|\widetilde{I}\right| = 1$, then clearly $\left(\mathbf{F}_{\mathcal{G}}(k, l) \right)_{k \in p_1(\widetilde{I}), l \in L}$ has rank $\left|\widetilde{I}\right|$ for each $L \subseteq \{1, \ldots, n_1\} \times \cdots \times \{1, \ldots, n_r\}$ with $|L| = 2$.

If $\left|\widetilde{I}\right| = 2$ set $p_1\left(\widetilde{I}\right) = \{(k_1, \ldots, k_r), (k_1', \ldots, k_r')\}$. Let $(l_1, \ldots, l_r), (l_1', \ldots, l_r') \in \{1, \ldots, n_1\} \times \ldots \times \{1, \ldots, n_r\}$ such that $(l_1, \ldots, l_r) \neq (l_1', \ldots, l_r')$. Then

$$\begin{bmatrix} e^{-2\pi i \frac{(k_1-1)(l_1-1)}{n_1}} \cdots e^{-2\pi i \frac{(k_r-1)(l_r-1)}{n_r}} & e^{-2\pi i \frac{(k_1-1)(l_1'-1)}{n_1}} \cdots e^{-2\pi i \frac{(k_r-1)(l_r'-1)}{n_r}} \\ e^{-2\pi i \frac{(k_1'-1)(l_1-1)}{n_1}} \cdots e^{-2\pi i \frac{(k_r'-1)(l_r-1)}{n_r}} & e^{-2\pi i \frac{(k_1'-1)(l_1'-1)}{n_1}} \cdots e^{-2\pi i \frac{(k_r'-1)(l_r'-1)}{n_r}} \end{bmatrix}$$

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has rank 1 if and only if $e^{-2\pi i\frac{(k_1'-k_1)(l_1'-l_1)}{n_1}}\cdots e^{-2\pi i\frac{(k_r'-k_r)(l_r'-l_r)}{n_r}}=1$, i.e.,

$$\frac{(k_1'-k_1)\,(l_1'-l_1)}{n_1}+\cdots+\frac{(k_r'-k_r)\,(l_r'-l_r)}{n_r}\in\mathbb{Z}.$$

Therefore, condition (2) follows from Lemma 3.5 and Lemma 3.6. ☐

The next three examples illustrate Corollary 3.4 and Corollary 3.7.

Example 3.8. Consider $\mathbf{F}_{\mathcal{G}}=\mathbf{F}_n$ and $m=2$. Then $\left(\ \mathbf{F}_n(k,:)\mathbf{F}_2(i,:)\ \right)_{(k,i)\in I}$, $|I|=d$, is MRE if and only if:

- $d=2$ and, by Corollary 3.4, $I=\{(k_1,i_1),(k_2,i_2)\}$ where $k_1,k_2\in\{1,\dots,n\}$, $i_1,i_2\in\{1,2\}$ and $i_1\neq i_2$.
- $d=3$ and, by Corollary 3.7, $I=\{(k_1,i_1),(k_2,i_2),(k_3,i_3)\}$ where $k_1,k_2,k_3\in\{1,\dots,n\}$, $i_1,i_2,i_3\in\{1,2\}$, $i_1=i_2\neq i_3$ and $\gcd(k_1-k_2,n)=1$.
- $d=4$ and, by Corollary 3.7, $I=\{(k_1,i_1),(k_2,i_2),(k_3,i_3),(k_4,i_4)\}$ where $k_1,k_2,k_3,k_4\in\{1,\dots,n\}$, $i_1,i_2,i_3,i_4\in\{1,2\}$, $i_1=i_2\neq i_3=i_4$ and $\gcd(k_1-k_2,n)=\gcd(k_3-k_4,n)=1$.

Example 3.9. Consider $\mathbf{F}_{\mathcal{G}}=\mathbf{F}_n$ and $m=3$. Then $\left(\ \mathbf{F}_n(k,:)\mathbf{F}_3(i,:)\ \right)_{(k,i)\in I}$, $|I|=d$, is MRE if and only if:

- $d=2$ and, by Corollary 3.4, $I=\{(k_1,i_1),(k_2,i_2)\}$ where $k_1,k_2\in\{1,\dots,n\}$, $i_1,i_2\in\{1,2,3\}$ and $i_1\neq i_2$.
- $d=3$ and, by Corollary 3.4, $I=\{(k_1,i_1),(k_2,i_2),(k_3,i_3)\}$ where $k_1,k_2,k_3\in\{1,\dots,n\}$, $i_1,i_2,i_3\in\{1,2,3\}$, $i_1\neq i_2$, $i_1\neq i_3$ and $i_2\neq i_3$.
- $d=4$ and, by Corollary 3.7, $I=\{(k_1,i_1),(k_2,i_2),(k_3,i_3),(k_4,i_4)\}$ where $k_1,k_2,k_3,k_4\in\{1,\dots,n\}$, $i_1,i_2,i_3,i_4\in\{1,2,3\}$, $i_1=i_2\neq i_3=i_4$ and $\gcd(k_1-k_2,n)=\gcd(k_3-k_4,n)=1$, or $i_3\neq i_1=i_2\neq i_4$, $i_3\neq i_4$ and $\gcd(k_1-k_2,n)=1$.
- $d=5$ and, by Corollary 3.7, $I=\{(k_1,i_1),(k_2,i_2),(k_3,i_3),(k_4,i_4),(k_5,i_5)\}$, where $k_1,k_2,k_3,k_4,k_5\in\{1,\dots,n\}$, $i_1,i_2,i_3,i_4,i_5\in\{1,2,3\}$, $i_5\neq i_1=i_2\neq i_3=i_4\neq i_5$ and $\gcd(k_1-k_2,n)=\gcd(k_3-k_4,n)=1$.
- $d=6$ and, by Corollary 3.7, $I=\{(k_1,i_1),\dots,(k_6,i_6)\}$, where $k_1,\dots,k_6\in\{1,\dots,n\}$, $i_1,\dots,i_6\in\{1,2,3\}$, $i_5=i_6\neq i_1=i_2\neq i_3=i_4\neq i_5=i_6$ and $\gcd(k_1-k_2,n)=\gcd(k_3-k_4,n)=\gcd(k_5-k_6,n)=1$.

Example 3.10. Consider $\mathbf{F}_{\mathcal{G}}=\mathbf{F}_{n_1}\otimes\mathbf{F}_{n_2}$ and $m=2$. If $|I|=d$, then $\left(\ (\mathbf{F}_{n_1}\otimes\mathbf{F}_{n_2})(k,l,:)\mathbf{F}_2(i,:)\ \right)_{(k,l,i)\in I}$ is MRE if and only if:

- $d=2$ and, by Corollary 3.4, $I=\{(k_1,l_1,i_1),(k_2,l_2,i_2)\}$ where $k_1,k_2\in\{1,\dots,n_1\}$, $l_1,l_2\in\{1,\dots,n_2\}$, $i_1,i_2\in\{1,2\}$ and $i_1\neq i_2$.
- $d=3$ and, by Corollary 3.7, $I=\{(k_1,l_1,i_1),(k_2,l_2,i_2),(k_3,l_3,i_3)\}$ where

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$k_1, k_2, k_3 \in \{1, \ldots, n_1\}$, $l_1, l_2, l_3 \in \{1, \ldots, n_2\}$, $i_1, i_2, i_3 \in \{1, 2\}$, $i_1 = i_2 \neq i_3$ and one of the following statements are true:
1. $k_1 \neq k_2$, $l_1 = l_2$ and $\gcd(k_1 - k_2, n_1) = 1$.
2. $k_1 = k_2$, $l_1 \neq l_2$ and $\gcd(l_1 - l_2, n_2) = 1$.
3. $k_1 \neq k_2$, $l_1 \neq l_2$, $\gcd(n_1, n_2) = 1$ and, $\gcd(k_1 - k_2, n_1) = 1$ or $\gcd(l_1 - l_2, n_2) = 1$.

- $d = 4$ and, by Corollary 3.7, $I = \{(k_1, l_1, i_1), \ldots, (k_4, l_4, i_4)\}$, $k_1, \ldots, k_4 \in \{1, \ldots, n_1\}$, $l_1, \ldots, l_4 \in \{1, \ldots, n_2\}$, $i_1, \ldots, i_4 \in \{1, 2\}$, $i_1 = i_2 \neq i_3 = i_4$, one of the following statements are true for k_1, k_2, l_1, l_2:
  1. $k_1 \neq k_2$, $l_1 = l_2$ and $\gcd(k_1 - k_2, n_1) = 1$.
  2. $k_1 = k_2$, $l_1 \neq l_2$ and $\gcd(l_1 - l_2, n_2) = 1$.
  3. $k_1 \neq k_2$, $l_1 \neq l_2$, $\gcd(n_1, n_2) = 1$ and, $\gcd(k_1 - k_2, n_1) = 1$ or $\gcd(l_1 - l_2, n_2) = 1$.

  and one of the following statements is true for k_3, k_4, l_3, l_4:
  1. $k_3 \neq k_4$, $l_3 = l_4$ and $\gcd(k_3 - k_4, n_3) = 1$.
  2. $k_3 = k_4$, $l_3 \neq l_4$ and $\gcd(l_3 - l_4, n_4) = 1$.
  3. $k_3 \neq k_4$, $l_3 \neq l_4$, $\gcd(n_3, n_4) = 1$ and, $\gcd(k_3 - k_4, n_3) = 1$ or $\gcd(l_3 - l_4, n_4) = 1$.

The proof of Corollary 3.12 requires the next lemma proved in, e.g., [8] (see also [11]).

LEMMA 3.11. *If n is prime then every minor of $\mathbf{F}_n$ is nonzero.*

COROLLARY 3.12. *Let $\mathbf{F}_{\mathcal{G}} = \mathbf{F}_n$ with n prime. Let $I \subseteq \{1, \ldots, n\} \times \{1, \ldots, m\}$ with $|I| = d$. Then $\mathbf{F}_{\mathcal{G}} \otimes \mathbf{F}_m(I, :)$ is MRE if and only if for each $\widetilde{I} \in I/p_2$ we have $\left|\widetilde{I}\right| \leq \lceil \frac{d}{m} \rceil$.*

Proof. If follows from Theorem 3.3 and Lemma 3.11. ∎

EXAMPLE 3.13. By Corollary 3.12, if in Example 3.8 and Example 3.9 n is prime and we only consider the restriction onto the component i, we obtain a description of sets $I \subseteq \{1, \ldots, n_1\} \times \cdots \times \{1, \ldots, n_r\} \times \{1, \ldots, m\}$ with $|I| = d$ such that $\mathbf{F}_n \otimes \mathbf{F}_m(I, :)$, $m = 2, 3$, is MRE.

We have computationally determined the existence of MRE HRS's for different abelian groups $\mathcal{G}$, and different values of m and d, apart from those in the classes described in the previous three corollaries. The study of non equivalent MRE HRS's, as was done for harmonic frames in [18], could be an interesting question for future investigations.

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