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THE COMBINATORIAL INVERSE EIGENVALUE PROBLEM: COMPLETE GRAPHS AND SMALL GRAPHS WITH STRICT INEQUALITY*

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Abstract. Let G be a simple undirected graph on n vertices and let $\mathcal{S}(G)$ be the class of real symmetric $n \times n$ matrices whose nonzero off-diagonal entries correspond exactly to the edges of G. Given $2n-1$ real numbers $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$, and a vertex v of G, the question is addressed of whether or not there exists $A \in \mathcal{S}(G)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $A(v)$ has eigenvalues $\mu_1, \ldots, \mu_{n-1}$, where $A(v)$ denotes the matrix with the vth row and column deleted. General results that apply to all connected graphs G are given first, followed by a complete answer to the question for K_n. Since the answer is constructive it can be implemented as an algorithm; a Mathematica code is provided to do so. Finally, for all connected graphs on 4 vertices it is shown that the answer is affirmative if all six inequalities are strict.

Key words. Graph, Interlacing inequalities, Inverse eigenvalue problem, Symmetric matrix.

AMS subject classifications. 05C50, 15A42, 15B57.

1. Introduction.

Various types of inverse eigenvalue problems have been of interest in a variety of subjects including control design, system identification, seismic tomography, principal component analysis, exploration and remote sensing, antenna array processing, geophysics, molecular spectroscopy, particle physics, and circuit theory, [3]. In this paper, we consider two variations of a structured inverse eigenvalue problem. Let $G = (V, E)$ be a simple undirected graph with vertex set $V = \{1, 2, \ldots, n\}$. Let $\mathcal{S}(G)$ be the set of all real symmetric $n \times n$ matrices $A = [a_{ij}]$ such that for $i \neq j, a_{ij} \neq 0$ if and only if $ij \in E$. There is no condition on the diagonal entries of A. The inverse eigenvalue problem for graphs asks: Given a graph G on n vertices and n real numbers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, is there a matrix M in $\mathcal{S}(G)$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$? We refer to this problem as the λ problem. A modified inverse eigenvalue problem for graphs asks: Given a graph G on n vertices, a vertex v of G, and $2n-1$ real numbers satisfying the interlacing inequalities $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$, is there a matrix M in $\mathcal{S}(G)$ with

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eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $M(v)$ has eigenvalues $\mu_1, \ldots, \mu_n$? We refer to this problem as the λ, μ problem.

The path, P_n, is the connected graph on n vertices with all vertices of degree 2 except for exactly 2 pendant vertices (vertices of degree 1). The star, S_n, is the connected graph on n vertices with one vertex of degree $n-1$ and $n-1$ pendant vertices. The λ, μ problem has been partially answered for P_n and S_n. In the 1970's both Hald [5] and Hochstadt [6] showed that given any $2n-1$ distinct numbers $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n$, there is a tridiagonal matrix M with eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $M(1)$ has eigenvalues $\mu_1, \ldots, \mu_{n-1}$. In terms of graphs, this states that there is a matrix $M \in \mathcal{S}(P_n)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $M(v)$ has eigenvalues $\mu_1, \ldots, \mu_{n-1}$ where v is a pendant vertex of P_n. A result of Mirsky in 1958 [7, Lemma 2] implies that given $2n-1$ real numbers $\lambda_1 > \mu_1 > \lambda_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n$, there exists a matrix M in $\mathcal{S}(S_n)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $M(1)$ has eigenvalues $\mu_1, \ldots, \mu_{n-1}$ where 1 corresponds to the dominating vertex in S_n. In 1989, Duarte [4] proved that given any tree T on n vertices, any vertex v in T, and real numbers $\lambda_1 > \mu_1 > \lambda_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n$, there exists a matrix M in $\mathcal{S}(T)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $M(v)$ has eigenvaluse $\mu_1, \ldots, \mu_{n-1}$.

The results in this paper are categorized into three areas. First, we prove some general results on the λ, μ problem. Second, we completely solve the λ, μ problem for the graph K_n, which is the graph on n vertices where every vertex is adjacent to every other vertex. Third, we prove that given any graph G on n vertices, $n \leq 4$, any vertex v in G, and real numbers $\lambda_1 > \mu_1 > \lambda_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n$, there exists a matrix M in $\mathcal{S}(G)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $M(v)$ has eigenvalues $\mu_1, \ldots, \mu_{n-1}$. The results of this paper are used in a companion paper [1] to completely solve the λ, μ problem for graphs on 4 or fewer vertices.

2. General results. In this section, we prove some results that apply to all graphs which we will later use to extend our knowledge of the inverse eigenvalue problem for stars and paths to complete graphs and graphs on four vertices. The first result is due to Mirsky, but we give it in a somewhat different form as found in Boley-Golub [2]. We note that this result solves the λ, μ problem for stars in the case when all λ_i's and μ_i's are distinct.

LEMMA 2.1 (Boley-Golub). *Given $2n-1$ real numbers $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$ such that $\mu_i \neq \mu_j$ for all $i \neq j$, there exists an $n \times n$ bordered matrix $A = \begin{bmatrix} a & b^T \\ b & M \end{bmatrix}$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ where $M = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_{n-1})$ and*

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$$b_i^2 = -\frac{\displaystyle\prod_{j=1}^{n}(\mu_i - \lambda_j)}{\displaystyle\prod_{\substack{j=1\\ j\neq i}}^{n-1}(\mu_i - \mu_j)}, \quad 1 \le i \le n-1. \tag{2.1}$$

In the following two lemmas, we prove that solutions to the λ, μ problem for a given graph G can be extended to a graph obtained from G by adding/deleting edges in two various manners. Before stating the lemmas we provide the following definition.

DEFINITION 2.2. Let v be a vertex of a graph G. The *neighborhood* of v is the set of vertices adjacent to v in G and is denoted by $N(v)$. The *closed neighborhood* of v is the set $v \cup N(v)$ and is denoted by $N[v]$.

LEMMA 2.3. *Let G be a connected graph on $n > 2$ vertices, let u, v be adjacent vertices of G, and let w be any other vertex of G. Let H be the graph obtained from G by inserting an edge between u and every vertex in $N(v) \setminus N[u]$ and between v and every vertex in $N(u) \setminus N[v]$. Let $\lambda_1 \ge \mu_1 \ge \lambda_2 \ge \mu_2 \ge \cdots \ge \lambda_{n-1} \ge \mu_{n-1} \ge \lambda_n$. Then if there exists $A \in \mathcal{S}(G)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $A(w)$ has eigenvalues $\mu_1, \ldots, \mu_{n-1}$, there exists $B \in \mathcal{S}(H)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $B(w)$ has eigenvalues $\mu_1, \ldots, \mu_{n-1}$.*

Proof. Label G so that u and v are the first two vertices of G, the i vertices in $N(u) \setminus (\{w\} \cup N[v])$ come next, the j vertices in $N(v) \setminus (\{w\} \cup N[u])$ come next, the k vertices in $[N(v) \cap N(u)] \setminus \{u, v, w\}$ come next, the ℓ vertices in $(N(u) \cup N(v) \cup \{w\})^c$ come next, and w is last. There exist $a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_i \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_j \end{bmatrix}$, $e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_k \end{bmatrix}$, and $d = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_k \end{bmatrix}$ where $a_1, \ldots, a_i$, $b_1, \ldots, b_j$, $e_1, \ldots, e_k$, and $d_1, \ldots, d_k$ are nonzero, such that we may write A in the form

$$A = \left[\begin{array}{cc|cccc|c} d_{uu} & h & a^T & 0_j^T & e^T & 0_\ell^T & f \\ h & d_{vv} & 0_i^T & b^T & d^T & 0_\ell^T & g \\ \hline a & 0_i & & & & & \\ 0_j & b & & & & & y \\ e & d & & & \mathbf{D} & & \\ 0_\ell & 0_\ell & & & & & \\ \hline f & g & & & y^T & & d_{ww} \end{array}\right]$$

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where h is nonzero. We allow the possibility that i, j, k or ℓ is 0. Let $Q = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ be a 2×2 orthogonal matrix such that $cs \neq 0$, and $\begin{bmatrix} p & r \\ r & q \end{bmatrix} = Q^T \begin{bmatrix} d_{uu} & h \\ h & d_{vv} \end{bmatrix} Q$ is not diagonal, $ce^T + sd^T$ and $-se^T + cd^T$ have all nonzero entries, and if either f or g is nonzero that $cf + sg$ and $-sf + cg$ are nonzero. Note that if $f = g = 0$, then w is not adjacent to u and not adjacent to v in both graphs G and H, and $cf + sg = -sf + cg = 0$ as desired. Let $Q_n = Q \oplus I_{n-2}$ and $Q_{n-1} = Q \oplus I_{n-3}$. Let $B = Q_n^T A Q_n$. Then

$$B = \left[\begin{array}{cc|cccc|c} p & r & ca^T & sb^T & ce^T + sd^T & 0_\ell^T & cf + sg \\ r & q & -sa^T & cb^T & -se^T + cd^T & 0_\ell^T & -sf + cg \\ \hline ca & -sa & & & & & \\ sb & cb & & & & & \\ ce + sd & -se + cd & & & \mathbf{D} & & y \\ 0_\ell & 0_\ell & & & & & \\ \hline cf + sg & -sf + cg & & & y^T & & d_{ww} \end{array}\right]$$

so $B \in \mathcal{S}(H)$ and has eigenvalues $\lambda_1, \ldots, \lambda_n$. We also have $B(w) = Q_{n-1}^T A(w) Q_{n-1}$ so that $B(w)$ has eigenvalues $\mu_1, \ldots, \mu_{n-1}$. $\square$

The following analogue of the previous lemma will be used in Section 5.

Lemma 2.4. *Let G be a connected graph on $n > 2$ vertices, let u, v be adjacent vertices of G such that $N(u) \cap N(v) = \emptyset$, and let w be any other vertex of G. Let H be the graph obtained from G by inserting an edge between u and every vertex in $N(v) \setminus \{u\}$ and between v and every vertex in $N(u) \setminus \{v\}$ and deleting the edge uv. Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$. Then if there exists $A \in \mathcal{S}(G)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $A(w)$ has eigenvalues $\mu_1, \ldots, \mu_{n-1}$, there exists $B \in \mathcal{S}(H)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $B(w)$ has eigenvalues $\mu_1, \ldots, \mu_{n-1}$.*

Proof. Label G so that u and v are the first two vertices of G, the i vertices in $N(u) \setminus \{v, w\}$ come next, the j vertices in $N(v) \setminus \{u, w\}$ come next, the ℓ vertices in $(N(u) \cup N(v) \cup \{w\})^c$ come next, and w is last. There exist $a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_i \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_j \end{bmatrix}$ where $a_1, \ldots, a_i$, $b_1, \ldots, b_j$, are nonzero, such that we may write A in the form

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$$A = \left[\begin{array}{cc|ccc|c} d_{uu} & h & a^T & 0_j^T & 0_\ell^T & f \\ h & d_{vv} & 0_i^T & b^T & 0_\ell^T & g \\ \hline a & 0_i & & & & \\ 0_j & b & & \mathbf{D} & & y \\ 0_\ell & 0_\ell & & & & \\ \hline f & g & & y^T & & d_{ww} \end{array}\right]$$

where h is nonzero. We allow the possibility that i, j or ℓ is 0. Let $Q = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ be a 2×2 orthogonal matrix with $cs \neq 0$ that diagonalizes $\begin{bmatrix} d_{uu} & h \\ h & d_{vv} \end{bmatrix}$. Thus, $Q^T \begin{bmatrix} d_{uu} & h \\ h & d_{vv} \end{bmatrix} Q = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$ for some p, q. Furthermore, if exactly one of f and g is nonzero, then $cf+sg$ and $-sf+cg$ are nonzero. If both f and g are zero, then $cf+sg$ and $-sf+cg$ are zero. Both f and g cannot be nonzero since the neighborhoods of u and v are disjoint. Let $Q_n = Q \oplus I_{n-2}$ and $Q_{n-1} = Q \oplus I_{n-3}$. Let $B = Q_n^T A Q_n$. Then

$$B = \left[\begin{array}{cc|ccc|c} p & 0 & ca^T & sb^T & 0_\ell^T & cf+sg \\ 0 & q & -sa^T & cb^T & 0_\ell^T & -sf+cg \\ \hline ca & -sa & & & & \\ sb & cb & & \mathbf{D} & & y \\ 0_\ell & 0_\ell & & & & \\ \hline cf+sg & -sf+cg & & y^T & & d_{ww} \end{array}\right]$$

so $B \in \mathcal{S}(H)$ and has eigenvalues $\lambda_1, \ldots, \lambda_n$. We also have $B(w) = Q_{n-1}^T A(w) Q_{n-1}$ so that $B(w)$ has eigenvalues $\mu_1, \ldots, \mu_{n-1}$. □

LEMMA 2.5. *Let G be a connected graph on n vertices. Assume there exists*

- *a vertex w of G such that $G - w = K_{n-1}$,*
- *$A \in \mathcal{S}(G)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $A(w)$ has eigenvalues $\mu_1, \ldots, \mu_{n-1}$.*

Then, given $j \in \{1, 2, \ldots, n\}$, there exists $B \in \mathcal{S}(K_n)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ such that the eigenvalues of $B(j)$ are $\mu_1, \ldots, \mu_{n-1}$.

Proof. If necessary relabel the vertices of G so that w is the jth vertex. Suppose w is adjacent to k vertices in G ($k \geq 1$ since G is connected). Label the vertices of $G - w$ as $u_1, u_2, \ldots, u_{n-1}$ such that u_i is adjacent to w if and only if $i \leq k$. Since $G - w = K_{n-1}$, u_1 is adjacent to u_{k+1}. Applying Lemma 2.3 with $u = u_1$ and $v = u_{k+1}$ we produce a new graph G' by inserting an edge between u_{k+1} and every vertex in

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$N(u_1) \setminus N[u_{k+1}]$, which, in this case, is just w. Then w is adjacent to $k+1$ vertices in G'. By the lemma there exists $B' \in \mathcal{S}(G')$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $B'(w)$ has eigenvalues $\mu_1, \ldots, \mu_{n-1}$. We can repeat the same process on G' and keep the same λ's and μ's. Eventually w will be adjacent to u_i for all $i \in \{1, \ldots, n-1\}$, in which case the resulting graph becomes K_n and we have a matrix $B \in \mathcal{S}(K_n)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ such that the eigenvalues of $B(j)$ are $\mu_1, \ldots, \mu_{n-1}$. □

The following two results are an immediate corollary of Rayleigh's Theorem and an extension of the corollary to connected graphs.

Corollary 2.6. *Let A be a real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then for all $i = 1, 2, \ldots, n$, $\lambda_1 \geq a_{ii} \geq \lambda_n$*

Lemma 2.7. *Let G be a connected graph on at least 2 vertices. Let $A \in \mathcal{S}(G)$ and let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of A. Then for $i = 1, \ldots, n$, $\lambda_1 > a_{ii} > \lambda_n$.*

Proof. By Corollary 2.6, $\lambda_1 \geq a_{ii} \geq \lambda_n$ for $i = 1, 2, \ldots, n$. Suppose $\lambda_1 = a_{ii}$ for some i. Let $j \in \{1, 2, \ldots, n\}, j \neq i$. Let $C = \begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}$ and let $\alpha, \beta, \alpha \geq \beta$ be the eigenvalues of C. By repeated application of the interlacing inequalities, $\alpha \leq \lambda_1 = a_{ii}$. By Corollary 2.6, $\alpha \geq a_{ii}$. Thus, $\alpha = a_{ii}$. Therefore, $\beta = \operatorname{tr} C - \alpha = a_{jj}$. Furthermore, the determinant of C is equal to the product of its eigenvalues, and thus,

$$a_{ii}a_{jj} - a_{ij}^2 = \alpha\beta = a_{ii}a_{jj} \Rightarrow a_{ij} = 0.$$

Then i is an isolated vertex, contradicting that G is connected. Thus, $\lambda_1 > a_{ii}$ for $i = 1, 2, \ldots, n$. Similarly, $\lambda_n < a_{ii}$ for $i = 1, 2, \ldots, n$. □

The following results give in the next section a necessary condition for a solution to the λ, μ problem for complete graphs.

Theorem 2.8 ([9], Lemma 1.2). *Let A be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and suppose that $B = A(1)$ has eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1}$. If the multiset $\{\mu_1, \ldots, \mu_{n-1}\} \subseteq \{\lambda_1, \ldots, \lambda_n\}$, then $A = a_{11} \oplus B$.*

Corollary 2.9. *Let G be a graph with vertex v. Assume the matrix $A \in \mathcal{S}(G)$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ and $A(v)$ has eigenvalues $\mu_1, \ldots, \mu_{n-1}$. If the multiset $\{\mu_1, \ldots, \mu_{n-1}\} \subseteq \{\lambda_1, \ldots, \lambda_n\}$, then v is an isolated vertex of G.*

3. Complete graphs. We begin with a solution to the λ problem for complete graphs. We make use of the concept of the minimum rank of a graph defined by

$$\operatorname{mr}(G) = \min\{\operatorname{rank} A : A \in \mathcal{S}(G)\}.$$

Theorem 3.1. *Let $n \geq 2$ and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then there exists an $A \in \mathcal{S}(K_n)$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ if and only if $\lambda_1 > \lambda_n$.*

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Proof. We begin by proving the forward implication. Let $A \in \mathcal{S}(K_n)$ with $n \geq 2$ have eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. If $\lambda_1 = \lambda_n$, then $\lambda_1 = \lambda_2 = \cdots = \lambda_n$, and thus, $A - \lambda_1 I \in \mathcal{S}(K_n)$ has eigenvalues $0, 0, \ldots, 0$. Thus, $\operatorname{rank}(A - \lambda_1 I) = 0 < \operatorname{mr}(K_n) = 1$, a contradiction.

We now prove the reverse implication by proceeding with induction on n. If $n = 2$ let $A = \frac{1}{2}\begin{bmatrix} \lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 \end{bmatrix}$. Note that this matrix is in $\mathcal{S}(K_2)$ since $\lambda_1 \neq \lambda_2$ and has eigenvalues λ_1 and λ_2.

Assume the reverse implication holds for $n-1$. Let $n > 2$ and let $\lambda_1 \geq \cdots \geq \lambda_n$ be given with $\lambda_1 > \lambda_n$.

Case 1. Suppose $\lambda_2 > \lambda_n$. By the inductive hypothesis, there exists $B \in \mathcal{S}(K_{n-1})$ with eigenvalues $\lambda_2, \ldots, \lambda_n$. Let b_{ij} denote the i,j entry of B. Since K_{n-1} is connected, by Lemma 2.7, $\lambda_2 > b_{ii}$ for $i = 1, \ldots, n-1$.

Let $A = \begin{bmatrix} \lambda_1 & 0^T \\ 0 & B \end{bmatrix}$ which has $\lambda_1, \lambda_2, \ldots, \lambda_n$ as eigenvalues. Let $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ be any orthogonal matrix with $ab \neq 0$. Let $Q = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \oplus I_{n-2}$. Then,

$$Q^T A Q = \begin{bmatrix} a^2\lambda_1 + b^2 b_{11} & ab(b_{11} - \lambda_1) & bb_{12} & \cdots & bb_{1,n-1} \\ ab(b_{11} - \lambda_1) & b^2\lambda_1 + a^2 b_{11} & ab_{12} & \cdots & ab_{1,n-1} \\ bb_{12} & ab_{12} & b_{22} & \cdots & b_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ bb_{1,n-1} & ab_{1,n-1} & b_{2,n-1} & \cdots & b_{n-1,n-1} \end{bmatrix}.$$

Since $ab \neq 0$ and $\lambda_1 > b_{11}$, $Q^T A Q \in \mathcal{S}(K_n)$. Note also that $Q^T A Q$ has the same eigenvalues as A, i.e., $\lambda_1, \ldots, \lambda_n$.

Case 2. Suppose that $\lambda_2 = \lambda_n$. Then $\lambda_1 > \lambda_2$. Let $A = \frac{\lambda_1 - \lambda_2}{n} J + \lambda_2 I$ where J is the $n \times n$ matrix with all entries equal to 1. Then $A \in \mathcal{S}(K_n)$ and has eigenvalues $\frac{\lambda_1 - \lambda_2}{n} \cdot n + \lambda_2 = \lambda_1$ with multiplicity 1 and λ_2 with multiplicity $n-1$. $\square$

Corollary 3.2. *Let $n \geq 2$. Let D be an $n \times n$ diagonal matrix whose diagonal entries are not all equal. Then there exists an $n \times n$ orthogonal matrix Q such that $E = Q^T D Q \in \mathcal{S}(K_n)$.*

Proof. Let $\lambda_1, \ldots, \lambda_n$ be the diagonal entries of D. By Theorem 3.1 there is an $E \in \mathcal{S}(K_n)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$. Since E is a real symmetric matrix, it is orthogonally diagonalizable. Thus, there exists an orthogonal matrix Q such that $QEQ^T = D$. Thus, $E = Q^T D Q$. $\square$

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We now state and prove our main result, a complete solution to the λ, μ problem for K_n.

THEOREM 3.3. *Given $2n-1$ real numbers $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$, there exists $A \in \mathcal{S}(K_n)$ such that λ_i's are the eigenvalues of A and μ_i's are the eigenvalues of $A(v)$, where v is a vertex of K_n, if and only if $\mu_1 > \mu_{n-1}$ and the multiset $\{\mu_1, \ldots, \mu_{n-1}\} \nsubseteq \{\lambda_1, \ldots, \lambda_n\}$. The condition of $\mu_1 > \mu_{n-1}$ is excluded when $n = 2$.*

Proof. We begin with the forward implication. Let $A \in \mathcal{S}(K_n)$ with the interlaced λ_i's and μ_i's. Since K_n does not have an isolated vertex, by the contrapositive of Corollary 2.9, the multiset $\{\mu_1, \ldots, \mu_{n-1}\} \nsubseteq \{\lambda_1, \ldots, \lambda_n\}$. Without loss of generality let the μ_i's be the eigenvalues of $A(1)$. Since $A(1)$ is symmetric, it is diagonalizable. Suppose that $\mu_1 = \mu_{n-1}$. Then $A(1)$ is simliar to $\mu_1 I_{n-1}$. Thus, $A(1) = \mu_1 I_{n-1}$. This contradicts that $A(1) \in \mathcal{S}(K_{n-1})$. Thus, $\mu_1 > \mu_{n-1}$.

We now prove the reverse implication. Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$ be given such that $\mu_1 > \mu_{n-1}$ and $\{\mu_1, \ldots, \mu_{n-1}\} \nsubseteq \{\lambda_1, \ldots, \lambda_n\}$. Let β_i, $1 \leq i \leq k$, be the distinct values of μ_i such that $\beta_i > \beta_{i+1}$ for $i = 1, \ldots, k-1$. Let the multiplicity of each β_i be m_i, so that $\sum m_i = n-1$. We can partition the λ_i's into 2 groups: the first group contains α_i, $1 \leq i \leq k+1$ where $\alpha_i \geq \beta_i \geq \alpha_{i+1}$ for $i = 1, \ldots, k$; the second group consists of β_i's, each with multiplicity $m_i - 1$.

Note that $\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \cdots \geq \alpha_k \geq \beta_k \geq \alpha_{k+1}$ and $\beta_i \neq \beta_j$ for $i \neq j$. Thus, by Lemma 2.1, there exists a $k+1 \times k+1$ bordered matrix $A = \begin{bmatrix} a & b'^T \\ b' & M \end{bmatrix}$ where $M = \operatorname{diag}(\beta_1, \beta_2, \ldots, \beta_k)$ and the eigenvalues of A are $\alpha_1, \ldots, \alpha_{k+1}$. Since $\{\mu_1, \ldots, \mu_{n-1}\} \nsubseteq \{\lambda_1, \ldots, \lambda_n\}$, there exists i such that $\beta_i \neq \alpha_j$ for all j. Thus, from Equation (2.1), $b'_i \neq 0$. Therefore, $b' \neq 0$.

Let $B = \begin{bmatrix} a & b'^T & 0^T \\ b' & M & 0 \\ 0 & 0 & D' \end{bmatrix}$, where D' is the diagonal matrix with entries β_i, each with multiplicity $m_i - 1$. Note that the eigenvalues of B are the λ_i's and the eigenvalues of $B(1)$ are the μ_i's. For convenience, we let $b = \begin{bmatrix} b' \\ 0 \end{bmatrix}$ and $D = M \oplus D'$. Then $B = \begin{bmatrix} a & b^T \\ b & D \end{bmatrix}$. Since $\mu_1 > \mu_{n-1}$, not all the diagonal entries of D are the same. By Corollary 3.2, there exists an $n-1 \times n-1$ orthogonal matrix Q such that $E = Q^T D Q \in \mathcal{S}(K_{n-1})$. Thus,

$$C = \begin{bmatrix} 1 & 0^T \\ 0 & Q^T \end{bmatrix} \begin{bmatrix} a & b^T \\ b & D \end{bmatrix} \begin{bmatrix} 1 & 0^T \\ 0 & Q \end{bmatrix} = \begin{bmatrix} a & b^T Q \\ Q^T b & E \end{bmatrix}$$

is similar to B while E is similar to D. So, C also has eigenvalues $\lambda_1, \ldots, \lambda_n$, and

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$C(1)$ has eigenvalues $\mu_1, \ldots, \mu_{n-1}$.

Note that since $b' \neq 0$, $Q^T b \neq 0$. Since $E \in \mathcal{S}(K_{n-1})$ and $Q^T b \neq 0$, $C \in \mathcal{S}(G)$ where G is a connected graph on n vertices with K_{n-1} as a subgraph. Now the result follows from Lemma 2.5. ∎

4. Mathematica code. The following is annotated code written in Wolfram Mathematica 8.0 that implements the construction given in Theorem 3.3. There is one slight variation from the construction outlined in the theorem; here the Boley-Golub matrix is inserted as a non-contiguous sub-matrix. The following is what shows on the screen when the code has been run with $n = 4$, $\lambda_1 = 45$, $\lambda_2 = 39$, $\lambda_3 = 6$, $\lambda_4 = 1$, $\mu_1 = 39$, $\mu_2 = 8$, and $\mu_3 = 3$.

Let λ be the desired eigenvalues of M and μ the desired eigenvalues of M(1).

```
λ = {1, 6, 39, 45};
μ = {3, 8, 39};
```

We first use the desired eigenvalues to construct a bordered matrix (B) using the process described in Lemma 2.1.

```
λ = Sort[λ];
μ = Sort[μ];
redundant = insertion = {};


For[i = 2, i < Length[μ]+1, i++,
    If[μ[[i]] == μ[[i − 1]], insertion = redundant = Append[redundant, {i}]]];

α = Delete[λ, redundant];
β = Delete[μ, redundant];


B = DiagonalMatrix[Join[{Apply[Plus, λ] − Apply[Plus, μ]}, μ]];
b = Sqrt[−(Apply[Times, #]) &[Table[(j − i), {i, α}, {j, β}]]/(Apply[Times, #])
    &[Table[If[j ≠ i, (i − j), 1], {j, β}, {i, β}]]];


For[i = 2, i ≤ Length[redundant], i++, insertion[[i, 1]] = (redundant[[i, 1]]−i + 1)];
b = Insert[b, 0, insertion];
B[[1]] = B[[1]] + Join[{0}, b];
B[[All, 1]] = B[[All, 1]] + Join[{0}, b];
```

We now multiply the bordered matrix by orthogonal matrices (Q), maintaining all eigenvalues while creating new edges. We first try a convenient orthogonal matrix

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Q such that Q(1) has non-zero rational entries. If that multiplication fails to form all of the required edges, we choose random orthogonal matrices until one succeeds.

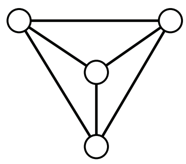
```
Q = ArrayPad[IdentityMatrix[Length[b]] −
    Table[2/Length[b], {Length[b]}, {Length[b]}], {1, 0}];
Q[[1, 1]] = 1;
M = Map[Chop, Transpose[Q].B.Q, {2}];


While[Apply[Times, Flatten[ReplacePart[M, {i_, i_} → 1]]] === 0, Q =
    ArrayPad[Orthogonalize[Table[RandomReal[], {Length[b]}, {Length[b]}]], {1,0}];
    Q[[1, 1]] = 1;
    M = Map[Chop, Transpose[Q].M.Q, {2}]]
```

We now have a matrix corresponding to a complete graph stored in the variable M. The remaining code displays M and its corresponding graph as well as displaying the eigenvalues of M and M(1).

```
MatrixForm[M]
Chop[Sort[Eigenvalues[M // N]]]
Chop[Sort[Eigenvalues[M[[2 ;; -1, 2 ;; -1]] // N]]]
GraphPlot[M]
```

$$\begin{bmatrix} 41 & 2\sqrt{\frac{7}{5}}-\frac{2\sqrt{\frac{518}{5}}}{3} & -4\sqrt{\frac{7}{5}}+\frac{2\sqrt{\frac{518}{5}}}{3} & -4\sqrt{\frac{7}{5}}-\frac{2\sqrt{\frac{518}{5}}}{3} \\ 2\sqrt{\frac{7}{5}}-\frac{2\sqrt{\frac{518}{5}}}{3} & \frac{191}{9} & \frac{134}{9} & -\frac{52}{9} \\ -4\sqrt{\frac{7}{5}}+\frac{2\sqrt{\frac{518}{5}}}{3} & \frac{134}{9} & \frac{176}{9} & -\frac{82}{9} \\ -4\sqrt{\frac{7}{5}}-\frac{2\sqrt{\frac{518}{5}}}{3} & -\frac{52}{9} & -\frac{82}{9} & \frac{83}{9} \end{bmatrix}$$

$\{1., 6., 39., 45.\}$
$\{3., 8., 39.\}$



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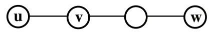
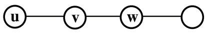
5. Connected graphs on 4 or fewer vertices. Duarte's work solves the λ, μ problem for distinct eigenvalues for trees, which accounts for many of the graphs on 4 or fewer vertices. The following is a special case of the theorem in his paper [4], rephrased in terms of the λ, μ problem.

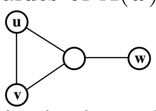
THEOREM 5.1. *Let T be a tree on n vertices and let w be a vertex of T. Given $2n-1$ distinct real numbers $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n$, there exists a matrix $A \in \mathcal{S}(T)$ such that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A and $\mu_1, \mu_2, \ldots, \mu_{n-1}$ are the eigenvalues of $A(w)$.*

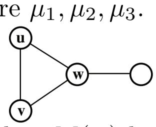
Theorem 5.1 solves the λ, μ problem with strict inequalities for (except K_3) all connected graphs on 3 or fewer vertices, P_4, and S_4. The λ, μ problems for K_3 and K_4 were solved in Section 3. The remaining graphs to be considered are C_4; the graph obtained from K_4 by deleting two incident edges, which we refer to as the paw; and the graph obtained from K_4 by deleting one edge, which we refer to as the diamond. It is also necessary to consider each of the three types of vertices of the paw and the two types of vertices of the diamond. Cycles of course have only one type of vertex.

THEOREM 5.2. *Let w be either the pendant vertex of the paw or the degree 3 vertex of the paw. Given $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$, there exists a matrix $M \in \mathcal{S}(\text{paw})$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that the eigenvalues of $M(w)$ are μ_1, μ_2, μ_3.*

Proof. We apply Lemma 2.3 to P_4 with vertices labeled as either 

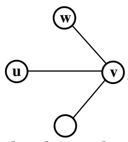
or 

. Except for labeling, the proof works the same for either choice of w. By Theorem 5.1, there exists $A \in \mathcal{S}(P_4)$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that the eigenvalues of $A(w)$ are μ_1, μ_2, μ_3. Letting $G = P_4$ in Lemma 2.3, then the paw labeled as 

or 

is H and there exists $M \in \mathcal{S}(\text{paw})$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that $M(w)$ has eigenvalues μ_1, μ_2, μ_3. $\square$

THEOREM 5.3. *Let w be a degree 2 vertex of the diamond. Given $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$, there exists a matrix $M \in \mathcal{S}(\text{diamond})$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that the eigenvalues of $M(w)$ are μ_1, μ_2, μ_3.*

Proof. We apply Lemma 2.3 to S_4, labeled as 

. By Theorem 5.1, there exists $A \in \mathcal{S}(S_4)$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that the eigenvalues of $A(w)$ are

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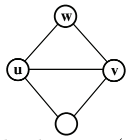
μ_1, μ_2, μ_3. Letting $G = S_4$ in Lemma 2.3, then the diamond labeled as  is

H and there exists $M \in \mathcal{S}(\text{diamond})$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that $M(w)$ has eigenvalues μ_1, μ_2, μ_3. ☐

It is not possible to apply Lemma 2.3 to obtain a solution to the λ, μ problem with distinct eigenvalues for the paw with w a degree 2 vertex, the diamond with w a degree 3 vertex, or to C_4. The proofs for these cases will include more involved arguments.

LEMMA 5.4. *If* $A = \begin{bmatrix} a & w & x \\ w & b & y \\ x & y & c \end{bmatrix}$, $wxy \neq 0$ *and*

$$wy^2 + (a-b)xy - wx^2 = (b-c)wx + (x^2 - w^2)y = 0,$$

then $b - wy/x$ *is an eigenvalue of* A *with multiplicity* 2.

Proof. Solving $wy^2 + (a-b)xy - wx^2 = 0$ and $(b-c)wx + (x^2 - w^2)y = 0$ for a and c respectively, we have $a = w(x^2 - y^2)/(xy) + b$ and $c = y(x^2 - w^2)/(wx) + b$. Let I be the 3×3 identity matrix. Then

$$A - (b - \frac{wy}{x})I = \begin{bmatrix} wx/y & w & x \\ w & wy/x & y \\ x & y & xy/w \end{bmatrix}$$

and has rank 1 (this can been seen by verifying that every 2×2 principal minor is 0). Thus, $b - wy/x$ is an eigenvalue of A with multiplicity 2. ☐

THEOREM 5.5. *Let* v *be a degree* 2 *vertex of the paw. Given* $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$, *there exists a matrix* $M \in \mathcal{S}(\text{paw})$ *with eigenvalues* $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ *such that the eigenvalues of* $M(v)$ *are* μ_1, μ_2, μ_3.

Proof. By Theorem 5.2, there exists a matrix $A = \begin{bmatrix} a & w & x & 0 \\ w & b & y & z \\ x & y & c & 0 \\ 0 & z & 0 & d \end{bmatrix}$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that $A(4)$ has eigenvalues μ_1, μ_2, μ_3. Let

$$Q = \frac{1}{\sqrt{x^2 + y^2}} \begin{bmatrix} y & x & 0 & 0 \\ -x & y & 0 & 0 \\ 0 & 0 & \sqrt{x^2 + y^2} & 0 \\ 0 & 0 & 0 & \sqrt{x^2 + y^2} \end{bmatrix}$$

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and let $B = Q^T AQ =$

$$\frac{1}{x^2+y^2}\begin{bmatrix} ay^2-2xwy+bx^2 & wy^2+(a-b)xy-wx^2 & 0 & -zx\sqrt{x^2+y^2} \\ wy^2+(a-b)xy-wx^2 & ax^2+2xwy+by^2 & (x^2+y^2)^{\frac{3}{2}} & zy\sqrt{x^2+y^2} \\ 0 & (x^2+y^2)^{\frac{3}{2}} & c(x^2+y^2) & 0 \\ -zx\sqrt{x^2+y^2} & zy\sqrt{x^2+y^2} & 0 & d(x^2+y^2) \end{bmatrix}.$$

Case 1. Suppose $wy^2 + (a-b)xy - wx^2 \neq 0$. Then B is in $\mathcal{S}(\text{paw})$ where row and column 4 correspond to a vertex of degree 2. Since $A \sim B$ and $A(4) \sim B(4)$, B has eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $B(4)$ has eigenvalues μ_1, μ_2, μ_3.

Case 2. Suppose $wy^2 + (a-b)xy - wx^2 = 0$. Let

$$P = \frac{1}{\sqrt{x^2+w^2}}\begin{bmatrix} \sqrt{x^2+w^2} & 0 & 0 & 0 \\ 0 & x & w & 0 \\ 0 & -w & x & 0 \\ 0 & 0 & 0 & \sqrt{x^2+w^2} \end{bmatrix}.$$

Let $C = P^T AP =$

$$\frac{1}{x^2+w^2}\begin{bmatrix} a(x^2+w^2) & 0 & (x^2+w^2)^{\frac{3}{2}} & 0 \\ 0 & bx^2-2xwy+cw^2 & (b-c)wx+(x^2-w^2)y & xz\sqrt{x^2+w^2} \\ (x^2+w^2)^{\frac{3}{2}} & (b-c)wx+(x^2-w^2)y & bw^2+2xwy+cx^2 & wz\sqrt{x^2+w^2} \\ 0 & xz\sqrt{x^2+w^2} & wz\sqrt{x^2+w^2} & d(x^2+w^2) \end{bmatrix}.$$

Since $\mu_1 > \mu_2 > \mu_3$, by the contrapositive of Lemma 5.4, $(b-c)wx+(x^2-w^2)y \neq 0$. Thus, C is in $\mathcal{S}(\text{paw})$ where row and column 4 correspond to a vertex of degree 2. Since $A \sim C$ and $A(4) \sim C(4)$, C has eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $C(4)$ has eigenvalues μ_1, μ_2, μ_3. $\square$

THEOREM 5.6. *Let v be a vertex of C_4. Given $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$, there exists a matrix $M \in \mathcal{S}(\mathrm{C}_4)$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that the eigenvalues of $M(v)$ are μ_1, μ_2, μ_3.*

Proof. By Theorem 5.2, there exists a matrix $A = \begin{bmatrix} a & w & x & 0 \\ w & b & y & z \\ x & y & c & 0 \\ 0 & z & 0 & d \end{bmatrix}$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that $A(4)$ has eigenvalues μ_1, μ_2, μ_3.

Case 1. Suppose $\begin{bmatrix} x \\ y \end{bmatrix}$ is not an eigenvector of $\begin{bmatrix} a & w \\ w & b \end{bmatrix}$. Let

$$Q = \begin{bmatrix} \mathfrak{c} & \mathfrak{s} & 0 & 0 \\ -\mathfrak{s} & \mathfrak{c} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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where $\begin{bmatrix} \mathfrak{c} & \mathfrak{s} \\ -\mathfrak{s} & \mathfrak{c} \end{bmatrix}$ is an orthogonal matrix that diagonalizes $\begin{bmatrix} a & w \\ w & b \end{bmatrix}$. Let

$$B = Q^T A Q = \begin{bmatrix} a\mathfrak{c}^2 - 2\mathfrak{c}\mathfrak{s}w + b\mathfrak{s}^2 & 0 & \mathfrak{c}x - \mathfrak{s}y & -\mathfrak{s}z \\ 0 & a\mathfrak{s}^2 + 2\mathfrak{c}\mathfrak{s}w + b\mathfrak{c}^2 & \mathfrak{s}x + \mathfrak{c}y & \mathfrak{c}z \\ \mathfrak{c}x - \mathfrak{s}y & \mathfrak{s}x + \mathfrak{c}y & c & 0 \\ -\mathfrak{s}z & \mathfrak{c}z & 0 & d \end{bmatrix}.$$

Since $\begin{bmatrix} x \\ y \end{bmatrix}$ is not an eigenvector of $\begin{bmatrix} a & w \\ w & b \end{bmatrix}$, $\begin{bmatrix} \mathfrak{c} \\ -\mathfrak{s} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \neq 0$ and $\begin{bmatrix} \mathfrak{s} \\ \mathfrak{c} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \neq 0$. Therefore, B is in $\mathcal{S}(\mathrm{C}_4)$. Since $A \sim B$ and $A(4) \sim B(4)$, B has eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $B(4)$ has eigenvalues μ_1, μ_2, μ_3.

Case 2. Suppose $\begin{bmatrix} x \\ y \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} a & w \\ w & b \end{bmatrix}$. Let

$$Q = \frac{1}{\sqrt{x^2+y^2}} \begin{bmatrix} y & x & 0 & 0 \\ -x & y & 0 & 0 \\ 0 & 0 & \sqrt{x^2+y^2} & 0 \\ 0 & 0 & 0 & \sqrt{x^2+y^2} \end{bmatrix}.$$

Since $\begin{bmatrix} x \\ y \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} a & w \\ w & b \end{bmatrix}$, $\begin{bmatrix} y & x \\ -x & y \end{bmatrix}$ diagonalizes $\begin{bmatrix} a & w \\ w & b \end{bmatrix}$. Thus,

$$B = \frac{1}{x^2+y^2} \begin{bmatrix} ay^2 - 2xwy + bx^2 & 0 & 0 & -zx\sqrt{x^2+y^2} \\ 0 & ax^2 + 2xwy + by^2 & (x^2+y^2)^{\frac{3}{2}} & zy\sqrt{x^2+y^2} \\ 0 & (x^2+y^2)^{\frac{3}{2}} & c(x^2+y^2) & 0 \\ -zx\sqrt{x^2+y^2} & zy\sqrt{x^2+y^2} & 0 & d(x^2+y^2) \end{bmatrix}.$$

As seen in the proof of Theorem 5.5, the entry in the first row and second column of B is $wy^2 + (a-b)xy - wx^2$. Therefore, $wy^2 + (a-b)xy - wx^2 = 0$.

Subcase 1. Suppose $\begin{bmatrix} w \\ x \end{bmatrix}$ is not an eigenvector of $\begin{bmatrix} b & y \\ y & c \end{bmatrix}$. Let

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathfrak{c} & \mathfrak{s} & 0 \\ 0 & -\mathfrak{s} & \mathfrak{c} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\begin{bmatrix} \mathfrak{c} & \mathfrak{s} \\ -\mathfrak{s} & \mathfrak{c} \end{bmatrix}$ is an orthogonal matrix that diagonalizes $\begin{bmatrix} b & y \\ y & c \end{bmatrix}$. Let

$$C = P^T A P = \begin{bmatrix} a & \mathfrak{c}w - \mathfrak{s}x & \mathfrak{s}w + \mathfrak{c}x & 0 \\ \mathfrak{c}w - \mathfrak{s}x & b\mathfrak{c}^2 - 2\mathfrak{c}\mathfrak{s}y + c\mathfrak{s}^2 & 0 & \mathfrak{c}z \\ \mathfrak{s}w + \mathfrak{c}x & 0 & b\mathfrak{s}^2 + 2\mathfrak{c}\mathfrak{s}y + c\mathfrak{c}^2 & \mathfrak{s}z \\ 0 & \mathfrak{c}z & \mathfrak{s}z & d \end{bmatrix}.$$

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Since $\begin{bmatrix} w \\ x \end{bmatrix}$ is not an eigenvector of $\begin{bmatrix} b & y \\ y & c \end{bmatrix}$, $\begin{bmatrix} \mathfrak{c} \\ -\mathfrak{s} \end{bmatrix} \cdot \begin{bmatrix} w \\ x \end{bmatrix} \neq 0$ and $\begin{bmatrix} \mathfrak{s} \\ \mathfrak{c} \end{bmatrix} \cdot \begin{bmatrix} w \\ x \end{bmatrix} \neq 0$. Therefore, C is in $\mathcal{S}(\mathrm{C}_4)$. Since $A \sim C$ and $A(4) \sim C(4)$, B has eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $B(4)$ has eigenvalues μ_1, μ_2, μ_3.

Subcase 2. Suppose $\begin{bmatrix} w \\ x \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} b & y \\ y & c \end{bmatrix}$. Let

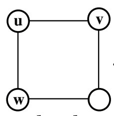
$$P = \frac{1}{\sqrt{x^2+w^2}} \begin{bmatrix} \sqrt{x^2+w^2} & 0 & 0 & 0 \\ 0 & x & w & 0 \\ 0 & -w & x & 0 \\ 0 & 0 & 0 & \sqrt{x^2+w^2} \end{bmatrix}.$$

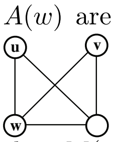
Since $\begin{bmatrix} w \\ x \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} b & y \\ y & c \end{bmatrix}$, $\begin{bmatrix} x & w \\ -w & x \end{bmatrix}$ diagonalizes $\begin{bmatrix} b & y \\ y & c \end{bmatrix}$. Thus,

$$C = \frac{1}{x^2+w^2} \begin{bmatrix} a(x^2+w^2) & 0 & (x^2+w^2)^{\frac{3}{2}} & 0 \\ 0 & bx^2 - 2xwy + cw^2 & 0 & xz\sqrt{x^2+w^2} \\ (x^2+w^2)^{\frac{3}{2}} & 0 & bw^2 + 2xwy + cx^2 & wz\sqrt{x^2+w^2} \\ 0 & xz\sqrt{x^2+w^2} & wz\sqrt{x^2+w^2} & d(x^2+w^2) \end{bmatrix}.$$

As seen in the proof of Theorem 5.5, the entry in the second row and third column of C is $(b-c)wx + (x^2-w^2)y$, and thus, $(b-c)wx + (x^2-w^2)y = 0$. However, since $\mu_1 > \mu_2 > \mu_3$ by the contrapositive of Lemma 5.4, $(b-c)wx + (x^2-w^2)y \neq 0$, a contradiction. $\square$

THEOREM 5.7. *Let w be a degree 3 vertex of the diamond. Given $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$, there exists a matrix $M \in \mathcal{S}$(diamond) with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that the eigenvalues of $M(w)$ are μ_1, μ_2, μ_3.*

Proof. We will apply Lemma 2.4 to C_4, labeled as 

. By Theorem 5.6, there exists $A \in \mathcal{S}(C_4)$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that the eigenvalues of $A(w)$ are μ_1, μ_2, μ_3. Letting $G = C_4$ in Lemma 2.4, then the diamond labeled as 

is H and there exists $M \in \mathcal{S}$(diamond) with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that $M(w)$ has eigenvalues μ_1, μ_2, μ_3. $\square$

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Summarizing, we have a solution to the λ, μ problem for small connected graphs and for distinct λ_i's and μ_i's.

THEOREM 5.8. *Let G be a connected graph on $2 \leq n \leq 4$ vertices, let w be a vertex of G, and let $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n$. Then there exists $A \in \mathcal{S}(G)$ such that the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$ and the eigenvalues of $A(w)$ are $\mu_1, \ldots, \mu_{n-1}$.*

We also have a solution to the λ problem for small graphs and distinct λ_i's.

COROLLARY 5.9. *Let G be a connected graph on $1 \leq n \leq 4$ vertices, and let $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. Then there exists $A \in \mathcal{S}(G)$ such that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of G.*

It is obvious that the corollary also holds for disconnected graphs.

6. Conclusion. Our last theorem naturally leads to the following:

Question. Let G be any connected graph on n vertices and let w be any vertex of G. Given $2n-1$ distinct real numbers $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n$ is there a matrix $A \in \mathcal{S}(G)$ such that

- $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, and
- $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of $A(w)$?

Shader and Monfared [8] now have a proof by means of the Nilpotent Jacobian method that the answer to this question is yes. Since their proof is not constructive, our method above is of independent interest. Of course, our construction depends on the specific graph and deleted vertex.

We solved the λ, μ problem completely for K_n. Can the problem be solved completely for the graph obtained by removing a single edge from K_n?

In [1], we present a solution to the λ, μ problem for all graphs on 4 or fewer vertices and for all cases of $(>, \geq)$ among the λ_i's and μ_i's. With one exception, a solution is possible in cases other than the case in which the λ_i's and μ_i's are all distinct.

We also give in [1] a complete solution to the λ, μ problem for S_n if the deleted vertex is the dominant vertex, and for P_n if the deleted vertex is a pendant vertex. In the latter case we will see that the case in which the λ_i's and μ_i's are all distinct, as in [5, 6], is in fact the only case in which a solution is possible.

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