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A NOTE ON THE CONVEXITY OF THE REALIZABLE SET OF EIGENVALUES FOR NONNEGATIVE SYMMETRIC MATRICES*

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Abstract. Geometric properties of the set $\mathcal{R}_n$ of n–tuples of realizable spectra of nonnegative symmetric matrices, and the Soules set $\mathcal{S}_n$ introduced by McDonald and Neumann, are examined. It is established that $\mathcal{S}_5$ is properly contained in $\mathcal{R}_5$. Two interesting examples are presented which show that neither $\mathcal{R}_n$ nor $\mathcal{S}_n$ need be convex. It is proved that $\mathcal{R}_n$ and $\mathcal{S}_n$ are star convex and centered at $(1, 1, \ldots, 1)$.

Key words. Symmetric matrices, inverse eigenvalue problem, realizable set, Soules set.

AMS subject classifications. 15A18,15A29,15A57

1. Introduction, Definitions.

The inverse eigenvalue problem for $n \times n$ symmetric nonnegative matrices can be stated as follows:
Find necessary and sufficient conditions for a set of real numbers $\lambda_1, \ldots, \lambda_n$ to be the eigenvalues of an $n \times n$ symmetric nonnegative matrix.
If there is a symmetric nonnegative matrix A with eigenvalues $\lambda_1, \ldots, \lambda_n$, then we shall say that the set of numbers $\lambda_1, \ldots, \lambda_n$ is *realizable.*

Many necessary conditions for an n–tuple of numbers to be realizable are known, and there are also several known sufficient conditions. Nevertheless, for $n > 4$, the realizable n–tuples have not yet been fully characterized. Sources of information on the inverse eigenvalue problem for symmetric nonnegative matrices include [7], [4], [2], [1, Ch. 4 and Supplement Sec. 2], and [5].

Without loss of generality we can assume throughout that $1 = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq -1$. As in [6], we discuss three important sets in connection with the inverse eigenvalue problem for symmetric matrices. The set of all points which satisfy $\lambda_2 + \ldots + \lambda_n \geq -1$ is referred to as the *trace nonnegative polytope* and denoted by $\mathcal{T}_n$. The set of all points in $\mathcal{T}_n$ which are realizable is referred to as the *realizable set* and denoted by $\mathcal{R}_n$. In [8], Soules provides an algorithm for constructing orthogonal matrices. This method is generalized by Elsner, Nabben, and Neumann [3] and they call the resulting matrices Soules matrices. In [6], the topological closure of the Soules set of matrices is used to construct nonnegative symmetric matrices. We explain how to construct a Soules matrix below. If R is an orthogonal matrix, we say that $(1, \lambda_2, \ldots, \lambda_n)$ is in the *feasible region* for R if $R\Lambda R^T \geq 0$, where Λ is the matrix whose diagonal entries are $1, \lambda_2, \ldots, \lambda_n$. The set of all n–tuples which are in the feasible region for some Soules matrix is referred to as the *Soules set* and denoted by $\mathcal{S}_n$.

In [6] it is shown that for $n \leq 4$, $\mathcal{S}_n = \mathcal{R}_n = \mathcal{T}_n$, and for $n \geq 5$, that $\mathcal{R}_n$ is a proper subset of $\mathcal{T}_n$. The question as to whether or not $\mathcal{S}_5$ is equal to $\mathcal{R}_5$ is posed.

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Our examples in the present note show that $\mathcal{S}_5$ is properly contained in $\mathcal{R}_5$. This proves that for $n = 5$ there are nonnegative symmetric matrices which are not similar to nonnegative matrices constructed from Soules matrices.

The set $\mathcal{T}_n$ is always convex. For $n \leq 4$, the set $\mathcal{R}_n$ is also convex, but as we show, $\mathcal{R}_5$ and $\mathcal{R}_6$ are not convex. It is shown in [6] that for $n \leq 5$, the set $\mathcal{S}_n$ is convex. We show here that $\mathcal{S}_6$ is not convex.

A set is *star convex centered at* p if the line from any point in the set to p is also contained in the set. We conclude our paper by showing that $\mathcal{S}_n$ and $\mathcal{R}_n$ are star convex centered at $(1, 1, \ldots, 1)$.

2. Examples, Results, and Convexity Arguments. Following Elsner, Nabben, and Neumann [3], we can describe the construction of a Soules matrix as follows. Start with a positive vector $n \times 1$ unit vector w. Partition w into $\begin{bmatrix} u \\ v \end{bmatrix}$. Then it is straightforward to see that the vector

$$\tilde{w} \;=\; \begin{bmatrix} (\|v\|_2 \,/\, \|u\|_2)u \\ -(\|u\|_2 \,/\, \|v\|_2)v \end{bmatrix}$$

satisfies that $\|\tilde{w}\|_2 = 1$, and $\tilde{w}^T w = 0$. We can then create additional orthogonal vectors by further re–partitioning the vectors $(\|v\|_2 \,/\, \|u\|_2)u$ and $(\|u\|_2 \,/\, \|v\|_2)v$ to obtain a set $w_1, w_2, \ldots, w_k$, $k \leq n$, of $n \times 1$ vectors which are mutually orthogonal. After $n-1$ steps we can construct an orthogonal matrix R with columns $w_1, \ldots, w_n$. The matrices which can be constructed in this way are referred to as *Soules matrices.* By letting a subvector of w approach zero and taking the limit of the matrices constructed, we generate additional matrices in the topological closure of the original Soules set which we also consider to be Soules matrices.

It is observed in [6] that $\mathcal{T}_5$ is the convex hull of the following points:

$$\begin{aligned}
a &= (1,1,1,1,1),\\
b &= (1,1,1,1,-1),\\
c &= (1,1,1,-1,-1),\\
d &= (1,1,0,-1,-1),\\
e &= (1,0,0,0,-1),\\
f &= (1,1,-\tfrac{1}{2},-\tfrac{1}{2},-1),\\
g &= (1,-\tfrac{1}{4},-\tfrac{1}{4},-\tfrac{1}{4},-\tfrac{1}{4}),\\
h &= (1,1,-\tfrac{2}{3},-\tfrac{2}{3},-\tfrac{2}{3}),\\
i &= (1,\tfrac{1}{2},\tfrac{1}{2},-1,-1),
\end{aligned}$$

and that the Soules set $\mathcal{S}_5$ is the convex hull of $a, b, c, d, e, f, g, j, k, l$ where

$$j \;=\; (1,1,-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}),$$

$$\begin{aligned} k &= (1, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}), \\ l &= (1, 0, 0, -\frac{1}{2}, -\frac{1}{2}). \end{aligned}$$

We introduce one additional point:

$$m = (1, \frac{-1+\sqrt{5}}{4}, \frac{-1+\sqrt{5}}{4}, \frac{-1-\sqrt{5}}{4}, \frac{-1-\sqrt{5}}{4}).$$

This point is interesting for a variety of reasons. Foremost, we will show that it is a realizable point on the line from l to i. In [6] it was shown that l is the only point on this line which is contained in $\mathcal{S}_5$. Secondly, the level curves $1+\lambda_2+\lambda_3+\lambda_4+\lambda_5 = 0$ and $1+\lambda_2^3+\lambda_3^3+\lambda_4^3+\lambda_5^3 = 0$ intersect at m. In fact, m is the last point on the line from l to i which can be realizable since the remaining points on the line fail the necessary condition that $1+\lambda_2^q+\lambda_3^q+\lambda_4^q+\lambda_5^q \geq 0$ for all integers $q \geq 1$. Quantitatively, m consists of 1, a multiple of the golden mean, and a multiple of its reciprocal.

Observation 2.1. *The line from l to m is contained in $\mathcal{R}_5$, and this is the only portion of the line from l to i which is realizable.*

Proof. Consider the matrix

$$R = \begin{bmatrix} 1/\sqrt{5} & (-3+\sqrt{5})/2\alpha & (-1-\sqrt{5})/2\beta & (-3-\sqrt{5})/2\gamma & (-1+\sqrt{5})/2\delta \\ 1/\sqrt{5} & (1-\sqrt{5})/\alpha & 0 & (1+\sqrt{5})/\gamma & 0 \\ 1/\sqrt{5} & (-3+\sqrt{5})/2\alpha & (1+\sqrt{5})/2\beta & (-3-\sqrt{5})/2\gamma & (+1-\sqrt{5})/2\delta \\ 1/\sqrt{5} & 1/\alpha & 1/\beta & 1/\gamma & 1/\delta \\ 1/\sqrt{5} & 1/\alpha & -1/\beta & 1/\gamma & -1/\delta \end{bmatrix},$$

where $\alpha = \sqrt{5(3-\sqrt{5})}$, $\beta = \sqrt{5+\sqrt{5}}$, $\gamma = \sqrt{5(3+\sqrt{5})}$ and $\delta = \sqrt{5-\sqrt{5}}$. Let $S = \operatorname{diag}(l)$ and $Q = \operatorname{diag}(m)$, the 5×5 diagonal matrices formed from the entries of l and m respectively. Then

$$RSR^T =$$
$$\begin{bmatrix} 0 & (5+2\sqrt{5})/\gamma^2 & \beta^2/2\gamma^2 & \beta^2/2\gamma^2 & (5+2\sqrt{5})/\gamma^2 \\ (5+2\sqrt{5})/\gamma^2 & 0 & (5+2\sqrt{5})/\gamma^2 & \beta^2/2\gamma^2 & \beta^2/2\gamma^2 \\ \beta^2/2\gamma^2 & (5+2\sqrt{5})/\gamma^2 & 0 & (5+2\sqrt{5})/\gamma^2 & \beta^2/2\gamma^2 \\ \beta^2/2\gamma^2 & \beta^2/2\gamma^2 & (5+2\sqrt{5})/\gamma^2 & 0 & (5+2\sqrt{5})/\gamma^2 \\ 5+2\sqrt{5}\gamma^2 & \frac{\beta^2}{2\gamma^2} & \beta^2/2\gamma^2 & (5+2\sqrt{5})/\gamma^2 & 0 \end{bmatrix}$$

and

$$RQR^T = \begin{bmatrix} 0 & 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}.$$

Let $p = tm + (1-t)l$, for $0 \leq t \leq 1$, and $P = \operatorname{diag}(p)$. Then the set of inequalities formed by $RPR^T \geq 0$ are linear in t and satisfied at $t = 0$ and $t = 1$, thus they are satisfied for any $0 \leq t \leq 1$. Hence the line from l to m is realizable. It is easy to check that the remaining points on the line from l to i fail the necessary condition that $1 + \lambda_2^q + \lambda_3^q + \lambda_4^q + \lambda_5^q \geq 0$ for all integers $q \geq 1$. ☐

Corollary 2.2. *The set $\mathcal{S}_5$ is properly contained in $\mathcal{R}_5$.*

Proof. In Observation 2.1 we see that $m \in \mathcal{R}_5$. By [6, Lemma 4.3], $m \notin \mathcal{S}_5$. ☐

Observation 2.3. *The set $\mathcal{R}_5$ is not convex.*

Proof. The point

$$p = (1, \frac{1+3\sqrt{5}}{16}, \frac{1+3\sqrt{5}}{16}, \frac{-7-3\sqrt{5}}{16}, \frac{-7-3\sqrt{5}}{16})$$

is on the line from m to the realizable point c, but

$$1 + 2(\frac{1+3\sqrt{5}}{16})^3 + 2(\frac{-7-3\sqrt{5}}{16})^3 < 0,$$

thus p fails a necessary condition for realizability and hence is not in $\mathcal{R}_5$. ☐

We now turn our attention to $\mathcal{S}_6$ and $\mathcal{R}_6$.

Observation 2.4. *The sets $\mathcal{S}_6$ and $\mathcal{R}_6$ are not convex.*

Proof. Let

$$R1 = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{12} & 1/2 & 1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{6} & 1/\sqrt{12} & 1/2 & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{6} & 1/\sqrt{12} & -1/2 & 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{12} & -1/2 & 0 & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & -1/\sqrt{3} & 0 & 0 & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & -1/\sqrt{3} & 0 & 0 & 0 & -1/\sqrt{2} \end{bmatrix}$$

and

$$R2 = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} & 0 & 0 & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{6} & -1/\sqrt{6} & 0 & 0 & 2/\sqrt{6} & 0 \\ 1/\sqrt{6} & -1/\sqrt{6} & 0 & 0 & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{6} & -1/\sqrt{6} & 0 & 0 & -1/\sqrt{6} & -1/\sqrt{2} \end{bmatrix}.$$

Then $R1$ and $R2$ are Soules matrices. The point

$$r = (1, 1, 1, -1, -1, -1)$$

is feasible for $R1$ and the point

$$s = (1, 1, -1/2, -1/2, -1/2, -1/2)$$

is feasible for the matrix $R2$. The points $tr + (1-t)s$ are not realizable for any $0 < t < 1$, since by the Perron-Frobenius Theorem the realizing matrix would have to be reducible, however there is no way to split these points into two or more sets such that the sum of the numbers in each set is nonnegative. Thus $\mathcal{S}_6$ and $\mathcal{R}_6$ cannot be convex. $\square$

We conclude with the following result.

THEOREM 2.5. *The sets $\mathcal{S}_n$ and $\mathcal{R}_n$ are star convex centered at the point* $(1, 1, \ldots, 1)$.

Proof. Let p be a point in $\mathcal{S}_n$ or $\mathcal{R}_n$. Let $P = \operatorname{diag}(p)$, the $n \times n$ diagonal matrix formed from the elements of p. If we are working with the set $\mathcal{S}_n$, then there is a Soules matrix R such that $RPR^T \geq 0$. If we are working with $\mathcal{R}_n$, then we can choose a nonnegative symmetric matrix A whose eigenvalues are listed in p. Since A can be diagonalized using an orthogonal similarity, there is an orthogonal matrix R such that $A = RPR^T \geq 0$. Let $\Lambda = \operatorname{diag}(1, \lambda_2, \ldots, \lambda_n)$. Then $R\Lambda R^T \geq 0$ is a set of linear inequalities in $\lambda_2, \lambda_3, \ldots, \lambda_n$, which combined with the inequalities $1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq -1$, generate a polytope. Since $RIR^T = I \geq 0$ we know that $(1, 1, \ldots, 1)$ is in the polytope, as is the line from p to $(1, 1, \ldots, 1)$. $\square$

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