

SENSITIVITY ANALYSIS FOR THE MULTIVARIATE EIGENVALUE PROBLEM*

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Abstract. This paper concerns with the sensitivity analysis for the multivariate eigenvalue problem (MEP). The concept of a simple multivariate eigenvalue of a matrix is generalized to the MEP and the first-order perturbation expansions of a simple multivariate eigenvalue and the corresponding multivariate eigenvector are presented. The explicit expressions of condition numbers, perturbation upper bounds and backward errors for multivariate eigenpairs are derived.

Key words. Multivariate eigenvalue problem, Simple eigenvalue, Condition number, Perturbation bound, Backward error.

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1. Introduction. Given a symmetric positive definite matrix $A \in \mathbb{R}^{n \times n}$, and a set of positive integers

$$\mathcal{P}_m = \{n_1, n_2, \dots, n_m\} \quad \text{with} \quad \sum_{i=1}^m n_i = n,$$

partition A into the block form

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix},$$

where $A_{ij} \in \mathbb{R}^{n_i \times n_j}$. Let $\lambda_1, \lambda_2, \dots, \lambda_m$ be m real scalars and

$$\Lambda := \text{diag}\{\lambda_1 I_{n_1}, \lambda_2 I_{n_2}, \dots, \lambda_m I_{n_m}\},$$

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where $I_{n_i} \in \mathbb{R}^{n_i \times n_i}$ is the identity matrix. Denote

$$\mathcal{S} := \{x = [x_1^T \ x_2^T \ \dots \ x_m^T]^T \in \mathbb{R}^n \mid x_j \in \mathbb{R}^{n_j}, \|x_j\|_2 = 1, j = 1, 2, \dots, m\}.$$

The *multivariate eigenvalue problem* (MEP) is to find a pair (Λ, x) such that

$$(1.1) \quad \begin{cases} Ax = \Lambda x, \\ x \in \mathcal{S}, \end{cases}$$

where Λ is called a *multivariate eigenvalue*, x a *multivariate eigenvector*, and (Λ, x) a *multivariate eigenpair* or a solution to MEP (1.1).

MEP arises as a necessary condition to the solution of the *maximal correlation problem* (MCP):

$$(1.2) \quad \begin{aligned} \max \quad & x^T Ax, \\ \text{s.t.} \quad & x \in \mathcal{S}, \end{aligned}$$

and finds applications in pattern analysis [10]. For more information about background materials, see [1, 6, 13, 21].

Several special cases of MEP have been well understood. It is a symmetric eigenvalue problem for a positive definite matrix when $m = 1$ (see e.g., [3]). When $m = n$, the MEP has exactly 2^n solutions which can be explicitly formulated.

Chu and Watterson [1] also discovered that there are precisely $\prod_{i=1}^m (2n_i)$ solutions for a generic matrix A whose n eigenvalues are distinct. Horst [8] proposed an iterative method (named the power method in [1] and the Horst method in [9, 13]) for computing a solution to the MEP. Detailed convergence analysis for the Horst iteration method are presented in [1] and [9]. Following the innate iterative structure, a SOR-style generalization (P-SOR) has been proposed by Sun [14] as a natural improvement upon the Horst method. The convergence of the P-SOR method has been established in [14]. Recently, several results on the global maximizer of the maximal correlation problem are presented by [5, 18, 19, 20, 21].

Although, as mentioned above, MEP is a symmetric eigenvalue problem for a positive definite matrix when $m = 1$, it must be pointed out that the generic case $1 < m < n$ has essential differences from the classical eigenvalue problem. For example, neither characteristic polynomial nor Schur-type decomposition can be readily employed.

It is known [3, 7] that both condition number and backward error play important roles in matrix computation. However, to our knowledge, little attention has been paid to the sensitivity analysis of the MEP. Golub and Zha [4] present a perturbation analysis of the canonical correlations of matrix pair.

This paper deals with the sensitivity analysis for MEP. The rest of this paper is organized as follows. Section 2 establishes the concept of simple multivariate eigenpair and presents first-order expansions. A sufficient and necessary condition for the simple multivariate eigenpairs is examined. In Section 3, following Rice's idea [12], we introduce the condition numbers for multivariate eigenvalues and eigenvectors and present explicit formulae for them. In Section 4, we derive perturbation bounds for multivariate eigenpair by applying the Brouwer fixed-point theorem. Section 5 deals with backward error of approximate multivariate eigenpair. In Section 6, some simple numerical examples are shown. Finally, we give our concluding remarks in Section 7.

2. First order expansions of simple multivariate eigenpair. The MEP (1.1) can be equivalently written as the following nonlinear system

$$(2.1) \quad \begin{cases} Ax = \Lambda x, \\ \frac{1}{2}x_i^T x_i = \frac{1}{2}, \quad x_i \in \mathbb{R}^{n_i}, \quad i = 1, 2, \dots, m. \end{cases}$$

Then the Jacobi matrix at a multivariate eigenpair (Λ, x) is the $(n+m) \times (n+m)$ matrix

$$J(\Lambda, x) := \begin{bmatrix} A - \Lambda & -B(x) \\ B(x)^T & 0 \end{bmatrix},$$

where

$$B(x) := \begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_m \end{bmatrix} \in \mathbb{R}^{n \times m}.$$

First we generalize the concept of simple root of a function (see, e.g., [11]) to the nonlinear system (2.1) as follows.

DEFINITION 2.1. A multivariate eigenpair (Λ, x) of the MEP (1.1) with nonsingular $J(\Lambda, x)$ is called a simple multivariate eigenpair.

REMARK 2.2. It is easy to see that if $m = 1$, then (λ, x) is a simple eigenpair of A if and only if λ is a simple eigenvalue of A .

After a careful look at the proof of [1, Lemma 3.6], we know that the following result holds.

THEOREM 2.3. Let $A \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix whose n eigenvalues are distinct. Then any multivariate eigenpair of the MEP (1.1) is simple.

To simplify the notation, without loss of generality, we assume that $n_j > 1$, $j = 1, 2, \dots, m$. For $x_j \in \mathbb{R}^{n_j}$, $\|x_j\|_2 = 1$, there exists an orthogonal matrix $Q_j = \begin{bmatrix} x_j & X_j \end{bmatrix}$ such that

$$Q_j^T x_j = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T, \quad j = 1, 2, \dots, m.$$

(If $n_j = 1$, we just take $Q_j = x_j$.)

Set

$$(2.2) \quad A_\lambda = G(x)^T A G(x), \quad G(x) = \text{diag}(X_1, \dots, X_m),$$

$$(2.3) \quad \bar{\Lambda} = \text{diag}(\lambda_1 I_{n_1-1}, \dots, \lambda_m I_{n_m-1}).$$

THEOREM 2.4. *Let (Λ, x) be a multivariate eigenpair of MEP (1.1). Then (Λ, x) is simple if and only if the matrix $A_\lambda - \bar{\Lambda}$ is nonsingular.*

Proof. Let $Q = \text{diag}(Q_1, \dots, Q_m)$, and let $P \in \mathbb{R}^{n \times n}$ be a permutation matrix such that

$$P \text{diag}(e_1^{(1)}, \dots, e_1^{(m)}) = \begin{bmatrix} I_m \\ 0 \end{bmatrix},$$

where

$$e_1^{(j)} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T \in \mathbb{R}^{n_j}, \quad j = 1, 2, \dots, m.$$

Then

$$(2.4) \quad \begin{bmatrix} PQ^T & 0 \\ 0 & I_m \end{bmatrix} J(\Lambda, x) \begin{bmatrix} QP^T & 0 \\ 0 & I_m \end{bmatrix} = J_{pq},$$

where

$$(2.5) \quad J_{pq} = \begin{bmatrix} M_{11} & M_{12} & -I_m \\ M_{12}^T & M_{22} & 0 \\ I_m & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} = PQ^T(A - \Lambda)QP^T, \quad M_{22} = A_\lambda - \bar{\Lambda}.$$

From (2.4) and (2.5), we see that

$$\det(J(\Lambda, x)) = \det(A_\lambda - \bar{\Lambda}),$$

and our conclusion holds. $\square$

In the rest of this paper, we assume that (Λ, x) is a simple multivariate eigenpair of the MEP (1.1).

Consider the perturbation of A

$$A(t) = A + tE,$$

where t is a real parameter and $E \in \mathbb{R}^{n \times n}$ is a symmetric matrix. Partition $A(t)$ as

$$A(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) & \cdots & A_{1m}(t) \\ A_{21}(t) & A_{22}(t) & \cdots & A_{2m}(t) \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}(t) & A_{m2}(t) & \cdots & A_{mm}(t) \end{bmatrix},$$

where $A_{ij}(t) \in \mathbb{R}^{n_i \times n_j}$ with $\sum_{j=1}^m n_j = n$.

We consider the associated MEP

$$A(t)x(t) = \Lambda(t)x(t),$$

where $\Lambda(t) = \text{diag}(\lambda_1(t)I_{n_1}, \lambda_2(t)I_{n_2}, \dots, \lambda_m(t)I_{n_m})$ and $x(t) \in \mathcal{S}$.

Set $\lambda(t) = [\lambda_1(t) \ \lambda_2(t) \ \dots \ \lambda_m(t)]^T$, and $\lambda = [\lambda_1 \ \lambda_2 \ \dots \ \lambda_m]^T$. First applying the implicit function theory, we prove the following result.

THEOREM 2.5. *Let E be a real symmetric matrix. Then there exist unique functions $x(t)$ and $\lambda(t)$ which are analytic in some neighborhood of $t_0 = 0$ such that*

$$(2.6) \quad \begin{cases} A(t)x(t) = \Lambda(t)x(t), \\ x(t) \in \mathcal{S}, \end{cases}$$

with $x(0) = x$ and $\Lambda(0) = \Lambda$.

Proof. Let

$$\begin{aligned} F(y, \mu, t) &:= A(t)y - \Omega y, \\ G(y, t) &:= \left[\frac{1}{2}(y_1^T y_1 - 1) \ \dots \ \frac{1}{2}(y_m^T y_m - 1) \right]^T. \end{aligned}$$

where $\mu = [\mu_1 \ \dots \ \mu_m]^T$, $\Omega = \text{diag}(\mu_1 I_{n_1}, \dots, \mu_m I_{n_m})$, and $y = [y_1^T \ \dots \ y_m^T]^T$, $y_j \in \mathbb{R}^{n_j}$. Then we have

1. $F(y, \mu, 0) = 0$, and $G(y, 0) = 0$.
2. $F(y, \mu, t)$ and $G(y, t)$ are real analytic functions of the real variables y , μ , and t .
3. $\det \begin{bmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial \mu} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial \mu} \end{bmatrix} \bigg|_{(y, \mu, t) = (x, \lambda, 0)} = \det(J(\Lambda, x)) \neq 0$.

Applying the implicit function theory establishes our conclusion. $\square$

Theorem 2.5 implies that $x(t)$ and $\lambda(t)$ in (2.6) have expansions of the form

$$(2.7) \quad x(t) = x + \dot{x}(0)t + O(t^2), \quad \lambda(t) = \lambda + \dot{\lambda}(0)t + O(t^2),$$

where $\dot{x}(0) = \left. \frac{dx(t)}{dt} \right|_{t=0}$ and $\dot{\lambda}(0) = \left. \frac{d\lambda(t)}{dt} \right|_{t=0}$.

The following theorem gives the expressions for the first order derivatives $\dot{x}(0)$ and $\dot{\lambda}(0)$.

THEOREM 2.6. *Assume that (Λ, x) is a simple multivariate eigenpair of MEP (1.1). Let P be an orthogonal matrix and Q a permutation matrix such that (2.4) and (2.5) hold. Then*

$$(2.8) \quad \dot{x}(0) = -Q^T P \begin{bmatrix} 0 & 0 \\ 0 & M_{22}^{-1} \end{bmatrix} P Q^T E x,$$

$$(2.9) \quad \dot{\lambda}(0) = [I_m \quad Z_{12}] P Q^T E x,$$

where $Z_{12} = -M_{12}M_{22}^{-1}$, and the matrices M_{12} and M_{22} are defined in (2.5).

Proof. Differentiating the first equation in (2.6) with respect to t at $t = 0$ gives

$$(2.10) \quad E x + A \dot{x}(0) = \begin{bmatrix} \dot{\lambda}_1(0)x_1 \\ \vdots \\ \dot{\lambda}_m(0)x_m \end{bmatrix} + \begin{bmatrix} \lambda_1 I_{n_1} & & \\ & \ddots & \\ & & \lambda_m I_{n_m} \end{bmatrix} \dot{x}(0).$$

Also, differentiating $x_i(t)^T x_i(t) = 1$ at $t = 0$ gives

$$(2.11) \quad x_i^T \dot{x}_i(0) = 0, \quad i = 1, 2, \dots, m.$$

Combining (2.10) with (2.11), we get

$$J(\Lambda, x) \begin{bmatrix} \dot{x}(0) \\ \dot{\lambda}(0) \end{bmatrix} = \begin{bmatrix} -E x \\ 0 \end{bmatrix}.$$

Since (Λ, x) is simple, then $J(\Lambda, x)$ is nonsingular. Thus, we get

$$(2.12) \quad \begin{bmatrix} \dot{x}(0) \\ \dot{\lambda}(0) \end{bmatrix} = J(\Lambda, x)^{-1} \begin{bmatrix} -E x \\ 0 \end{bmatrix}.$$

From (2.4), we get

$$J(\Lambda, x)^{-1} = \begin{bmatrix} Q^T P & 0 \\ 0 & I_m \end{bmatrix} J_{pq}^{-1} \begin{bmatrix} P Q^T & 0 \\ 0 & I_m \end{bmatrix},$$

where

$$(2.13) \quad J_{pq}^{-1} = \begin{bmatrix} 0 & 0 & I_m \\ 0 & M_{22}^{-1} & Z_{12}^T \\ -I_m & -Z_{12} & -Z_{11} \end{bmatrix},$$

$$Z_{12} = -M_{12}M_{22}^{-1}, \quad Z_{11} = M_{12}M_{22}^{-1}M_{12}^T - M_{11}.$$

Substituting this into (2.12) yields (2.8) and (2.9). $\square$

3. Condition numbers. Although there were earlier references dealing with conditioning, e.g. [17], the general framework of the conditioning theory was introduced by Rice [12]. Note that the multivariate eigenvalue problem (1.1) is equivalent to the nonlinear system (2.1). Naturally, the condition numbers of multivariate eigenvalue Λ and the corresponding eigenvector x of the MEP (1.1) are defined as follows.

DEFINITION 3.1. Let $(\lambda(t), x(t))$ satisfy (2.6). Then the condition number of the multivariate eigenvalue Λ is defined by

$$(3.1) \quad \kappa(\Lambda) := \lim_{t \rightarrow 0} \sup_{\|E\|_2 \leq \xi} \left\{ \frac{\|\lambda(t) - \lambda\|_2}{\alpha_1 |t|} \right\},$$

and the condition number of the corresponding eigenvector x is defined by

$$(3.2) \quad \kappa(x) := \lim_{t \rightarrow 0} \sup_{\|E\|_2 \leq \xi} \left\{ \frac{\|x(t) - x\|_2}{\alpha_2 |t|} \right\},$$

where α_1, α_2 and ξ are all positive parameters which allow us some flexibility.

For example:

- Taking $\alpha_1 = \alpha_2 = \xi = 1$, the condition numbers defined by (3.1) and (3.2) are the *absolute condition numbers*.
- Taking $\alpha_1 = \|\lambda\|_2$, $\alpha_2 = \|x\|_2$, $\xi = \|A\|_2$, the condition numbers defined by (3.1) and (3.2) are the *relative condition numbers*.

The following results give explicit expressions for $\kappa(\Lambda)$ and $\kappa(x)$.

THEOREM 3.2.

$$(3.3) \quad \kappa(\Lambda) = \frac{\sqrt{m}}{\alpha_1} \xi \| [I_m \quad Z_{12}] \|_2,$$

$$(3.4) \quad \kappa(x) = \frac{\sqrt{m}}{\alpha_2} \xi \| M_{22}^{-1} \|_2,$$

where the matrices Z_{12} and M_{22} are defined in (2.13) and (2.5), respectively.

Proof. From (2.7), (2.8) and (2.9), we see that

$$\begin{aligned}\kappa(\Lambda) &= \sup_{\|E\|_2 \leq \xi} \left\{ \frac{\|\dot{\lambda}(0)\|_2}{\alpha_1} \right\} \\ &= \sup_{\|E\|_2 \leq \xi} \left\{ \frac{1}{\alpha_1} \left\| \begin{bmatrix} I_m & Z_{12} \end{bmatrix} P Q^T E x \right\|_2 \right\} \\ &= \max_{y \in \mathbb{R}^n, \|y\|_2 \leq \sqrt{m}\xi} \left\{ \frac{1}{\alpha_1} \left\| \begin{bmatrix} I_m & Z_{12} \end{bmatrix} P Q^T y \right\|_2 \right\} \\ &= \frac{\sqrt{m}}{\alpha_1} \xi \left\| \begin{bmatrix} I_m & Z_{12} \end{bmatrix} \right\|_2.\end{aligned}$$

Similarly, we get

$$\begin{aligned}\kappa(x) &= \sup_{\|E\|_2 \leq \xi} \left\{ \frac{\|\dot{x}(0)\|_2}{\alpha_2} \right\} \\ &= \sup_{\|E\|_2 \leq \xi} \left\{ \frac{1}{\alpha_2} \left\| \begin{bmatrix} 0 & 0 \\ 0 & M_{22}^{-1} \end{bmatrix} P Q^T E x \right\|_2 \right\} \\ &= \frac{\sqrt{m}}{\alpha_2} \xi \|M_{22}^{-1}\|_2. \quad \square\end{aligned}$$

It is worthy pointing out that both $\kappa(\Lambda)$ and $\kappa(x)$ depend on M_{22}^{-1} .

The following corollary indicates a particular case when the eigenvalue Λ is well-conditioned.

COROLLARY 3.3.

$$(3.5) \quad \kappa(\Lambda) = \frac{\sqrt{m}}{\alpha_1} \xi \Leftrightarrow X_i^T A_{ij} x_j = 0, \quad i \neq j \text{ and } X_i^T (A_{ii} - \lambda_i I_{n_i}) x_i = 0.$$

Proof. From (3.3), we have

$$\begin{aligned}\kappa(\Lambda) = \frac{\sqrt{m}}{\alpha_1} \xi &\Leftrightarrow Z_{12} = 0 \\ &\Leftrightarrow M_{12} = 0 \quad (\text{see the second equality in (2.13)}) \\ &\Leftrightarrow X_i^T A_{ij} x_j = 0, \quad i \neq j \text{ and } X_i^T (A_{ii} - \lambda_i I_{n_i}) x_i = 0. \quad \square\end{aligned}$$

Although the matrix X_j is not unique, both $\kappa(\Lambda)$ and $\kappa(x)$ are independent of choices of X_j .

Several remarks on (3.3)–(3.5) are in order.

REMARK 3.4. Corollary 3.3 involves the condition $X_i^T (A_{ii} - \lambda_i I_{n_i}) x_i = 0$. This means that $(A_{ii} - \lambda_i I_{n_i}) x_i$ is orthogonal to all columns of X_i implying it must point in the direction of x_i . Therefore, $A_{ii} x_i = \mu_i x_i$ for some μ_i follows.

PROPOSITION 3.5. *If there exist orthogonal matrices $X_i \in \mathbb{R}^{n_i \times n_{i-1}}$, ($i = 1, \dots, m$) such that $Q_i^T A_{ij} Q_j$ are diagonal for $i, j = 1, \dots, m$, then for any eigenvalue Λ of MEP (1.1), $\kappa(\Lambda) = \frac{\sqrt{m}}{\alpha_1} \xi$.*

Proof. The hypotheses that $Q_i^T A_{ij} Q_j$ are diagonal imply $X_i^T A_{ij} x_j = 0$ for $i \neq j$ and $X_i^T (A_{ii} - \lambda_i I_{n_i}) x_i = 0$. Thus, the desired result follows from Corollary 3.3. $\square$

Below we outline several cases where Proposition 3.5 is applicable.

1. $m = 2$, $A_{11} = I_{n_1}$, $A_{22} = I_{n_2}$. For this case, we can take the matrices whose columns are comprised of the left and right singular vectors of A_{12} as Q_1 and Q_2 .
2. $m = n$. For this case, we can take $Q_i = (1)_{1 \times 1}$.
3. $A_{ij} = 0$, ($i \neq j$), i.e., $A = \text{diag}(A_{11}, \dots, A_{mm})$ is a block-diagonal matrix. For this case, we can take the eigenvector matrix of A_{ii} as Q_i .

REMARK 3.6. From (3.3)-(3.5), we see that both $\kappa(\Lambda)$ and $\kappa(x)$ depend on M_{22}^{-1} for general m and A . If $\|M_{22}^{-1}\|_2$ is large, then x is ill-conditioned, but λ may be well-conditioned (if, for example $M_{12} = 0$) or ill-conditioned. It is known [15, Theorem 3.6] that all eigenvalues are well-conditioned for the standard symmetric eigenvalue problem.

EXAMPLE 3.7. Consider the MEP (1.1) with $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}$, where $A_{11} = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$, $A_{12} = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \end{bmatrix}$, $A_{22} = \text{diag}(a_{33}, a_{44}, a_{55})$. Assume that A has n distinct eigenvalues. From [1], A has precisely 24 eigenpairs, one of them is

$$\Lambda = \text{diag}((a_{11} + \alpha_1)I_2, (a_{33} + \alpha_1)I_3),$$

$$x = [1 \quad 0 \quad 1 \quad 0 \quad 0]^T.$$

Accordingly,

$$A_\lambda - \bar{\Lambda} = \begin{bmatrix} a_{22} - a_{11} - \alpha_1 & \alpha_2 & 0 \\ \alpha_2 & a_{44} - a_{33} - \alpha_1 & 0 \\ 0 & 0 & a_{55} - a_{33} - \alpha_1 \end{bmatrix}.$$

If $a_{55} \approx a_{33} + \alpha_1$, then x is ill-conditioned.

On the other hand, from Proposition 3.5, all multivariate eigenvalues of A are

well-conditioned.

EXAMPLE 3.8. Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 4 & 0 & 0 \\ 1 & 0 & 3 & -1 \\ 1 & 0 & -1 & 4 + \varepsilon \end{bmatrix}, \quad n_1 = n_2 = 2, \quad 0 < \varepsilon \ll 1.$$

The matrix A is symmetric positive definite and has four distinct eigenvalues. Let

$$\Lambda = \text{diag}(2, 2, 4, 4), \quad x = [1 \quad 0 \quad 1 \quad 0]^T.$$

Then (Λ, x) is a multivariate eigenpair of A .

Note that

$$A - \Lambda = \begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & -1 & -1 \\ 1 & 0 & -1 & \varepsilon \end{bmatrix}.$$

From (2.4), we get

$$A_\lambda - \bar{\Lambda} = M_{22} = \begin{bmatrix} 2 & 0 \\ 0 & \varepsilon \end{bmatrix}, \quad M_{12}^T = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix},$$

$$Z_{12} = -M_{12}M_{22}^{-1} = -\frac{1}{\varepsilon} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad \|M_{22}^{-1}\|_2 = \frac{1}{\varepsilon}.$$

Thus, the absolute condition numbers are $\kappa(x) = \frac{\sqrt{2}}{\varepsilon}$ and $\kappa(\Lambda) = \sqrt{2 + \frac{4}{\varepsilon^2}}$. This means that Λ and x both are ill-conditioned.

Consider the perturbation of A

$$A(t) = A + tE.$$

Let $E \in \mathbb{R}^{4 \times 4}$ be a Householder transformation such that

$$Ex = QP^T \begin{bmatrix} 0 \\ y^{(1)} \end{bmatrix}, \quad y^{(1)} \in \mathbb{R}^2, \quad y^{(1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then from (2.8), we have

$$\dot{x}(0) = -Q^T P \begin{bmatrix} 0 & 0 \\ 0 & M_{22}^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ y^{(1)} \end{bmatrix} = -Q^T P h,$$

where $h = [0 \quad 0 \quad 0 \quad \frac{1}{\varepsilon}]^T$. Thus, $\|\dot{x}(0)\|_2 = \frac{1}{\varepsilon}$, and $x(t) = x + Q^T P h t + O(t^2)$.

From (2.9), we have

$$\begin{aligned}\dot{\lambda}(0) &= [I_m \quad Z_{12}] PQ^T QP^T \begin{bmatrix} 0 \\ y^{(1)} \end{bmatrix} \\ &= Z_{12}y^{(1)} = -\frac{1}{\varepsilon} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\varepsilon} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.\end{aligned}$$

Thus,

$$\lambda(t) = \lambda + \frac{1}{\varepsilon} \begin{bmatrix} -1 \\ 1 \end{bmatrix} t + O(t^2).$$

4. Perturbation bounds. Let (Λ, x) be a simple multivariate eigenpair of the MEP (1.1). Suppose that A is perturbed to $A + \Delta A$. Consider the following nonlinear system

$$(4.1) \quad \begin{cases} (A + \Delta A)(x + \Delta x) = (\Lambda + \Delta \Lambda)(x + \Delta x), \\ x + \Delta x \in \mathcal{S}, \end{cases}$$

where $\Delta \Lambda = \text{diag}(\Delta \lambda_1 I_{n_1}, \dots, \Delta \lambda_m I_{n_m})$ and $\Delta \lambda = [\Delta \lambda_1 \quad \dots \quad \Delta \lambda_m]^T$.

Set

$$(4.2) \quad \tau := \|J(\Lambda, x)^{-1}\|_2.$$

Assume that ΔA satisfies

$$(4.3) \quad \|\Delta A\|_2 < \min \left\{ \frac{1}{(4\sqrt{m}\tau + 2)\tau}, \lambda_{\min}(A) \right\},$$

where $\lambda_{\min}(A)$ denotes the smallest eigenvalue of A . Applying the technique of [14], we prove the following result.

THEOREM 4.1. *Assume ΔA satisfies (4.3). Then $A + \Delta A$ has a multivariate eigenvalue $\Lambda + \Delta \Lambda = \text{diag}(\mu_1 I_{n_1}, \dots, \mu_m I_{n_m})$ and a corresponding eigenvector $y \in \mathcal{S}$ such that*

$$(4.4) \quad \left\| \begin{bmatrix} y - x \\ \mu - \lambda \end{bmatrix} \right\|_2 \leq \frac{2\sqrt{m}\tau}{1 - \tau\|\Delta A\|_2} \|\Delta A\|_2,$$

where $\lambda = [\lambda_1 \quad \dots \quad \lambda_m]^T$ and $\mu = [\mu_1 \quad \dots \quad \mu_m]^T$.

Proof. Reformulating the nonlinear system (4.1) as

$$(4.5) \quad J(\Lambda, x) \begin{bmatrix} \Delta x \\ \Delta \lambda \end{bmatrix} = \begin{bmatrix} -\Delta A \cdot x \\ 0 \end{bmatrix} + \begin{bmatrix} -\Delta A \cdot \Delta x + \Delta \Lambda \cdot \Delta x \\ -\frac{1}{2} \|\Delta x_1\|_2^2 \\ \vdots \\ -\frac{1}{2} \|\Delta x_m\|_2^2 \end{bmatrix}.$$

By (4.5), we get

$$\begin{aligned} \left\| \begin{bmatrix} \Delta x \\ \Delta \lambda \end{bmatrix} \right\|_2 &\leq \tau \left\{ \sqrt{m} \|\Delta A\|_2 + \|\Delta A\|_2 \left\| \begin{bmatrix} \Delta x \\ \Delta \lambda \end{bmatrix} \right\|_2 \right. \\ &\quad \left. + \|\Delta \Lambda \cdot \Delta x\|_2 + \frac{1}{2} \left\| \begin{bmatrix} \|\Delta x_1\|_2^2 \\ \vdots \\ \|\Delta x_m\|_2^2 \end{bmatrix} \right\|_2 \right\} \\ &\leq \tau \left\{ \sqrt{m} \|\Delta A\|_2 + \|\Delta A\|_2 \left\| \begin{bmatrix} \Delta x \\ \Delta \lambda \end{bmatrix} \right\|_2 + \frac{1}{2} \left\| \begin{bmatrix} \Delta x \\ \Delta \lambda \end{bmatrix} \right\|_2^2 + \frac{1}{2} \|\Delta x\|_2^2 \right\}. \end{aligned}$$

Note that the last inequality follows from $\|\Delta x\|^2 + \|\Delta \lambda\|^2 \geq 2\|\Delta \lambda\|\|\Delta x\|$. Hence,

$$(4.6) \quad \left\| \begin{bmatrix} \Delta x \\ \Delta \lambda \end{bmatrix} \right\|_2 \leq \frac{\tau}{1 - \tau \|\Delta A\|_2} \left\{ \sqrt{m} \|\Delta A\|_2 + \left\| \begin{bmatrix} \Delta x \\ \Delta \lambda \end{bmatrix} \right\|_2^2 \right\}.$$

Consider the following quadratic equation about ξ

$$\xi = \frac{\tau}{1 - \tau \|\Delta A\|_2} (\sqrt{m} \|\Delta A\|_2 + \xi^2),$$

which has two positive solutions under the condition (4.3). The smaller one, denoted by ξ_* , satisfies

$$(4.7) \quad \xi_* = \frac{2\sqrt{m}\tau \|\Delta A\|_2}{1 - \tau \|\Delta A\|_2 + \sqrt{(1 - \tau \|\Delta A\|_2)^2 - 4\sqrt{m}\tau^2 \|\Delta A\|_2}} \leq \frac{2\sqrt{m}\tau \|\Delta A\|_2}{1 - \tau \|\Delta A\|_2}.$$

We now define a mapping $\mathcal{M}$ by

$$\mathcal{M} \begin{bmatrix} \Delta x \\ \Delta \lambda \end{bmatrix} = (J(\Lambda, x))^{-1} \left\{ \begin{bmatrix} -\Delta A \cdot x \\ 0 \end{bmatrix} + \begin{bmatrix} -\Delta A \cdot \Delta x + \Delta \Lambda \cdot \Delta x \\ -\frac{1}{2} \|\Delta x_1\|_2^2 \\ \vdots \\ -\frac{1}{2} \|\Delta x_m\|_2^2 \end{bmatrix} \right\},$$

and a set $\mathcal{D}$ by

$$(4.8) \quad \mathcal{D} := \left\{ \begin{bmatrix} \Delta x \\ \Delta \lambda \end{bmatrix} \in \mathbb{R}^{n+m} \left| \left\| \begin{bmatrix} \Delta x \\ \Delta \lambda \end{bmatrix} \right\|_2 \leq \xi_* \right. \right\}.$$

Obviously, $\mathcal{D}$ is a bounded closed convex subset of $\mathbb{R}^{n+m}$. From (4.6) and (4.7), we see that $\mathcal{M}$ is a continuous mapping which maps $\mathcal{D}$ into $\mathcal{D}$. By the Brouwer fixed-point theorem, the mapping $\mathcal{M}$ has a fixed-point $\begin{bmatrix} \widehat{\Delta x} \\ \widehat{\Delta \lambda} \end{bmatrix} \in \mathcal{D}$ such that $\begin{bmatrix} x \\ \lambda \end{bmatrix} + \begin{bmatrix} \widehat{\Delta x} \\ \widehat{\Delta \lambda} \end{bmatrix}$ is a solution of the nonlinear system (4.1). Note that $\|\Delta A\|_2 < \lambda_{\min}(A)$ implies the perturbed matrix $A + \Delta A$ is positive definite. From (4.5), $(\Lambda + \widehat{\Delta \lambda}, x + \widehat{\Delta x})$ is a multivariate eigenpair of $A + \Delta A$, and the desired conclusion is established. $\square$

The inequality (4.4) reveals the relationship between the sensitivity of the multivariate eigenpair (Λ, x) and τ .

Let $A_0 = \text{diag}(A_{11}, A_{22}, \dots, A_{mm})$ and the spectral decomposition of A_{jj} be

$$A_{jj} = U_j \Lambda_j U_j^T, \quad \Lambda_j = \text{diag}(\hat{\lambda}_1^{(j)}, \hat{\lambda}_2^{(j)}, \dots, \hat{\lambda}_{n_j}^{(j)}), \quad U_j = \begin{bmatrix} u_1^{(j)} & u_2^{(j)} & \dots & u_{n_j}^{(j)} \end{bmatrix}.$$

Suppose A_0 has n distinct eigenvalues. By Chu and Watterson [1], the MEP

$$\begin{aligned} A_0 y &= \Omega y, \quad y \in \mathcal{S}, \\ \Omega &= \text{diag}(\mu_1 I_{n_1}, \mu_2 I_{n_2}, \dots, \mu_m I_{n_m}), \end{aligned}$$

has exactly $\prod_{j=1}^m (2n_j)$ eigenpairs (Ω, y) :

$$\begin{aligned} \Omega &= \text{diag}(\hat{\lambda}_{k_1}^{(1)} I_{n_1}, \hat{\lambda}_{k_2}^{(2)} I_{n_2}, \dots, \hat{\lambda}_{k_m}^{(m)} I_{n_m}), \quad 1 \leq k_j \leq n_j, \quad j = 1, 2, \dots, m, \\ y &= \begin{bmatrix} \pm(u_{k_1}^{(1)})^T & \pm(u_{k_2}^{(2)})^T & \dots & \pm(u_{k_m}^{(m)})^T \end{bmatrix}^T. \end{aligned}$$

Let $\Lambda = \text{diag}(\Lambda_1, \Lambda_2, \dots, \Lambda_m)$, $e = \begin{bmatrix} e_1^{(n_1)T} & e_1^{(n_2)T} & \dots & e_1^{(n_m)T} \end{bmatrix}^T$, and $e_1^{(n_i)} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T \in \mathbb{R}^{n_i}$. Some calculation gives that

$$\tau^{(0)} = \|J(\Omega, y)^{-1}\|_2 = \left\| \begin{bmatrix} A_0 - \Omega & -B(y) \\ B(y)^T & 0 \end{bmatrix}^{-1} \right\|_2 = \left\| \begin{bmatrix} \Lambda - \Omega & -B(e) \\ B(e)^T & 0 \end{bmatrix}^{-1} \right\|_2 = \frac{1}{\alpha},$$

where $\alpha = \min_{1 \leq i \leq m} \min_{\substack{1 \leq j \leq n_i \\ j \neq k_i}} |\hat{\lambda}_{k_i}^{(i)} - \hat{\lambda}_j^{(i)}|$.

Let $E = A - A_0$. As a consequence of Theorem 4.1, we get the following result.

THEOREM 4.2. Suppose $A_0 = \text{diag}(A_{11}, A_{22}, \dots, A_{mm})$ has n distinct eigenvalues. Assume that $E = A - A_0$ satisfies

$$\|E\|_2 < \min \left\{ \frac{1}{(4\sqrt{m}\tau^{(0)} + 2)\tau^{(0)}}, \lambda_{\min}(A_0) \right\}.$$

Then the MEP (1.1) has exactly $\prod_{j=1}^m (2n_j)$ solutions. Furthermore, for any multivariate eigenpair (Ω, y) of A_0 , there exists a multivariate eigenpair (Λ, x) of A such that

$$\left\| \begin{bmatrix} y - x \\ \mu - \lambda \end{bmatrix} \right\|_2 \leq \frac{2\sqrt{m}\tau^{(0)}}{1 - \tau^{(0)}\|E\|_2} \|E\|_2,$$

where $\lambda = [\lambda_1 \ \lambda_2 \ \dots \ \lambda_m]^T$ and $\mu = [\mu_1 \ \mu_2 \ \dots \ \mu_m]^T$.

We can also prove the following inclusion results concerning multivariate eigenvalues.

THEOREM 4.3. *Let (Λ, x) be a multivariate eigenpair of (1.1). Then*

$$(4.9) \quad \sigma_{\min}(\lambda_i I - A_{ii}) \leq \min\{\nu_1^{(i)}, \nu_2^{(i)}\},$$

where

$$\begin{aligned} \nu_1^{(i)} &= \sum_{j \neq i} \|A_{ij}\|_2, \\ \nu_2^{(i)} &= \sqrt{m-1} \left\| \begin{bmatrix} A_{i1} & \dots & A_{i,i-1} & A_{i,i+1} & \dots & A_{im} \end{bmatrix} \right\|_2, \quad i = 1, 2, \dots, m. \end{aligned}$$

In particular, if x is a global maximizer of the MCP, then

$$(4.10) \quad 0 \leq \lambda_i - \lambda_{\max}(A_{ii}) \leq \min\{\nu_1^{(i)}, \nu_2^{(i)}\}, \quad i = 1, 2, \dots, m.$$

Proof. Since $Ax = \Lambda x$, we have that

$$(\lambda_i I - A_{ii})x_i = \sum_{j \neq i} A_{ij}x_j, \quad i = 1, 2, \dots, m.$$

The inequalities (4.9) are obtained straightforward.

If x is a global maximizer of the MCP (1.2), then the first inequality in (4.10) is already proved by Zhang and Chu [19]. Note that A_{ii} is symmetric, the second inequality in (4.10) follows from (4.9) due to $\lambda_i - \lambda_{\max}(A_{ii}) = \sigma_{\min}(\lambda_i I_{n_i} - A_{ii})$. $\square$

5. Backward errors. In this section, we deal with the normwise backward errors of approximate multivariate eigenpairs of the MEP (1.1).

Let $(\tilde{\Lambda}, \tilde{x})$ be an approximate multivariate eigenpair of (1.1), where

$$\tilde{\Lambda} = \text{diag}(\tilde{\lambda}_1 I_{n_1}, \tilde{\lambda}_2 I_{n_2}, \dots, \tilde{\lambda}_m I_{n_m}), \quad \tilde{x} = [\tilde{x}_1^T \ \tilde{x}_2^T \ \dots \ \tilde{x}_m^T]^T \in \mathcal{S}.$$

Consider the following matrix equation about E_0

$$(5.1) \quad (A + E_0)\tilde{x} = \tilde{\Lambda}\tilde{x},$$

where $E_0 \in \mathbb{R}^{n \times n}$ is a symmetric matrix.

The *normwise backward error* of $(\tilde{\Lambda}, \tilde{x})$ is defined by

$$\eta(\tilde{\Lambda}, \tilde{x}) := \min\{\|E_0\|_F \mid E_0 \text{ satisfies (5.1)}\}.$$

Note that (5.1) can be rewritten as

$$E_0 \tilde{x} = \tilde{\Lambda} \tilde{x} - A \tilde{x}.$$

Applying a result of Sun ([14, Lemma1.4]), we see that the following result holds.

THEOREM 5.1. *The normwise backward error $\eta(\tilde{\Lambda}, \tilde{x})$ is given by*

$$\eta(\tilde{\Lambda}, \tilde{x}) = \sqrt{\frac{2\|r\|_2^2}{m} - \frac{(\tilde{x}^T r)^2}{m^2}},$$

where $r = A \tilde{x} - \tilde{\Lambda} \tilde{x}$.

Theorem 5.1 may be used to analyze the accuracy of the computed results of iterative methods for MEP (1.1).

For example, we consider the Horst algorithm summarized below.

ALGORITHM 5.2. (Horst algorithm [1])

```

1  Given  $x^{(0)} \in \mathcal{S}$ ;
2    For  $k = 1, 2, \dots$ , until convergence;
3      For  $i = 1, 2, \dots, m$ ;
4        Compute  $y_i^{(k)} = \sum_{j=1}^m A_{ij} x_j^{(k)}$ ;
5        Compute  $\mu_i^{(k)} = \|y_i^{(k)}\|$ ;
6        Compute  $x_i^{(k+1)} = \frac{y_i^{(k)}}{\mu_i^{(k)}}$ ;
7      End
8    End
```

Let $x^{(k)} = \begin{bmatrix} (x_1^{(k)})^T & \dots & (x_m^{(k)})^T \end{bmatrix}^T$ and $\Omega^{(k)} = \text{diag}(\mu_1^{(k)} I_{n_1}, \dots, \mu_m^{(k)} I_{n_m}) \geq 0$. Then the algorithm can be formulated as

$$\Omega^{(k)} x^{(k+1)} = A x^{(k)}, \quad k = 1, 2, \dots$$

The next result shows that if $\|x^{(k+1)} - x^{(k)}\|_2 \leq \varepsilon$, then $(\Omega^{(k)}, x^{(k+1)})$ is an approximate multivariate eigenpair of A .

THEOREM 5.3. *There exists a symmetric matrix $E \in \mathbb{R}^{n \times n}$ such that*

$$(A + E)x^{(k+1)} = \Omega^{(k)} x^{(k+1)},$$

where $\|E\|_F \leq \sqrt{\frac{2}{m}} \|A(x^{(k+1)} - x^{(k)})\|_2$.

Proof. Let

$$(5.2) \quad r = Ax^{(k+1)} - \Omega^{(k)}x^{(k+1)}.$$

By Theorem 5.1, there exists a symmetric matrix $E \in \mathbb{R}^{n \times n}$ such that

$$Ex^{(k+1)} = -r,$$

where

$$\|E\|_F \leq \sqrt{\frac{2}{m}} \|r\|_2.$$

Recall that, by definition,

$$Ax^{(k)} = \Omega^{(k)}x^{(k+1)}.$$

Substituting it into (5.2), the desired conclusion is established. $\square$

From the definition of M_{22} in (2.5), we see that $\|M_{22}^{-1}\|_2 = 1/\sigma_{\min}(A_\lambda - \bar{\Lambda})$. Theorem 3.2 shows that the sensitivity of the eigenvector x depends on $\sigma_{\min}(A_\lambda - \bar{\Lambda})$. From (2.13) and (3.3), loosely speaking, the sensitivity of the eigenvalue Λ is also dependent on $\sigma_{\min}(A_\lambda - \bar{\Lambda})$.

Let

$$\mathcal{E}(n) := \{H \in \mathbb{R}^{n \times n} \mid H^T = H\}.$$

Suppose that (Λ, x) is a simple multivariate eigenpair of A and define

$$\mathcal{E}_1 := \{H \mid H \in \mathcal{E}(n), (\Lambda, x) \text{ is a multivariate eigenpair of } A + H, \text{ but not simple}\}.$$

Below we prove an interesting property of $\sigma_{\min}(A_\lambda - \bar{\Lambda})$ in connection with $\mathcal{E}_1$.

THEOREM 5.4. *If $\sigma_{\min}(A_\lambda - \bar{\Lambda}) < \lambda_{\min}(A)$, then*

$$\min_{H \in \mathcal{E}_1} \|H\|_F = \sigma_{\min}(A_\lambda - \bar{\Lambda}),$$

where A_λ and $\bar{\Lambda}$ are defined respectively by (2.2), so (2.3).

Proof. Eckhart-Young theorem [2] and Theorem 2.6 together imply that

$$\min_{H \in \mathcal{E}_1} \|H\|_F \geq \min_{H \in \mathcal{E}_1} \|G(x)^T H G(x)\|_F = \sigma_{\min}(A_\lambda - \bar{\Lambda}),$$

where $G(x)$ is defined in (2.2).

Let

$$(A_\lambda - \bar{\Lambda})v = \lambda_{\min}v, \quad v \in \mathbb{R}^{n-m}, \quad \|v\|_2 = 1,$$

where $\lambda_{\min}$ denotes the eigenvalue of $A_\lambda - \bar{\Lambda}$ satisfying $|\lambda_{\min}| = \sigma_{\min}(A_\lambda - \bar{\Lambda})$.

Now consider the matrix equation

$$(5.3) \quad G(x)^T N G(x) = -\lambda_{\min} v v^T$$

with the unknown $N \in \mathcal{E}(n)$. Let

$$u_j = X_j v_j, \quad u = [u_1^T \quad u_2^T \quad \dots \quad u_m^T]^T, \quad u_j \in \mathbb{R}^{n_j}.$$

Then, a simple calculation shows that

$$N = -\lambda_{\min} u u^T = -\lambda_{\min} G(x) v v^T G(x)^T$$

solves (5.3), and $\|N\|_F = \sigma_{\min}(A_\lambda - \bar{\Lambda})$. The assumption $\sigma_{\min}(A_\lambda - \bar{\Lambda}) < \lambda_{\min}(A)$ implies that $N \in \mathcal{E}_1$ and the desired result is obtained. $\square$

Finally, a relationship between $\tau_0 := 1/\sigma_{\min}(A_\lambda - \bar{\Lambda})$ and $\tau := \|J(\Lambda, x)^{-1}\|_2$ is given by the next theorem.

THEOREM 5.5.

$$\tau_0 \leq \tau \leq \max\{\tau_0(1 + \|M_{12}\|_2), 1 + \|M_{11}\|_2 + \tau_0(\|M_{12}\|_2^2 + \|M_{12}\|_2)\}.$$

Proof. From (2.4) and (4.2),

$$\tau = \|(J(\Lambda, x))^{-1}\|_2 = \|J_{pq}^{-1}\|_2.$$

Note that $M_{22}^{-1} = \begin{bmatrix} 0 & I_{n-m} & 0 \end{bmatrix} J_{pq}^{-1} \begin{bmatrix} 0 \\ I_{n-m} \\ 0 \end{bmatrix}$ and $M_{22} = A_\lambda - \bar{\Lambda}$, thus $\tau_0 \leq \tau$.

On the other hand, from (2.13),

$$\begin{aligned} \tau &= \|J_{pq}^{-1}\|_2 \\ &\leq \lambda_{\max} \left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & \tau_0 & \|Z_{12}\|_2 \\ 1 & \|Z_{12}\|_2 & \|Z_{11}\|_2 \end{bmatrix} \right) \leq \left\| \begin{bmatrix} 0 & 0 & 1 \\ 0 & \tau_0 & \|Z_{12}\|_2 \\ 1 & \|Z_{12}\|_2 & \|Z_{11}\|_2 \end{bmatrix} \right\|_1 \\ &= \max\{\tau_0 + \|Z_{12}\|_2, 1 + \|Z_{12}\|_2 + \|Z_{11}\|_2\} \\ &\leq \max\{\tau_0(1 + \|M_{12}\|_2), 1 + \|M_{11}\|_2 + \tau_0(\|M_{12}\|_2^2 + \|M_{12}\|_2)\}. \end{aligned}$$

The first inequality is obtained from [16, Lemma 1]. $\square$

In summary, Theorems 3.2, 4.1 and 5.5 show that the sensitivity of the multivariate eigenpair (Λ, x) of A is dependent on τ_0 .

6. Numerical examples. In this section, we present test results with the condition numbers and the backward errors of multivariate eigenvalue of A . All computations are carried out using MATLAB 7.4.0 with precision $\epsilon \approx 2.2 \times 10^{-16}$. In our experiments, the computed solutions are given by using Horst method.

EXAMPLE 6.1. ([1]) The matrix A is given by

$$A = \begin{bmatrix} 4.3299 & 2.3230 & -1.3711 & -0.0084 & -0.7414 \\ 2.3230 & 3.1181 & 1.0959 & 0.1285 & 0.0727 \\ -1.3711 & 1.0959 & 6.4920 & -1.9883 & -0.1878 \\ -0.0084 & 0.1285 & -1.9883 & 2.4591 & 1.8463 \\ -0.7414 & 0.0727 & -0.1878 & 1.8463 & 5.8875 \end{bmatrix}$$

with $m = 2$ and $\mathcal{P} = \{2, 3\}$. The computed multivariate eigenvalue and eigenvector of A are

$$\lambda = [6.5186 \quad 8.2116]^T, \quad x = [0.9357 \quad 0.3528 \quad -0.9341 \quad 0.3508 \quad 0.0667]^T.$$

EXAMPLE 6.2. ([21]) The matrix A is given by

$$A = \begin{bmatrix} 1 & 0 & 0.2617 & 0.6191 & 0.1068 \\ 0 & 1 & 0.2078 & 0.0118 & 0.0746 \\ 0.2617 & 0.2078 & 1 & 0 & 0 \\ 0.6191 & 0.0118 & 0 & 1 & 0 \\ 0.1068 & 0.0746 & 0 & 0 & 1 \end{bmatrix}$$

with $m = 2$ and $\mathcal{P} = \{2, 3\}$. The computed multivariate eigenvalue and eigenvector of A are

$$\lambda = [1.6889 \quad 1.6889]^T, \quad x = [0.9869 \quad 0.1615 \quad 0.4236 \quad 0.8897 \quad 0.1705]^T.$$

EXAMPLE 6.3. ([5]) The matrix A is given by

$$A = \begin{bmatrix} 45 & -20 & 5 & 6 & 16 & 3 \\ -20 & 77 & -20 & -25 & -8 & -21 \\ 5 & -20 & 74 & 47 & 18 & -32 \\ 6 & -25 & 47 & 54 & 7 & -11 \\ 16 & -8 & 18 & 7 & 21 & -7 \\ 3 & -21 & -32 & -11 & -7 & 70 \end{bmatrix}$$

with $m = 3$ and $\mathcal{P} = \{2, 2, 2\}$. The computed multivariate eigenvalue and eigenvector of A are

$$\lambda = [109.2864 \quad 179.7093 \quad 89.9667]^T,$$

$$x = [0.4921 \quad -0.8705 \quad 0.8004 \quad 0.5995 \quad 0.5684 \quad -0.8228]^T.$$

Table shows the condition numbers and the backward errors of multivariate eigenpairs of A .

Table 1. The condition numbers and the backward errors.

	$\kappa(\Lambda)$	$\kappa(x)$	$\eta(\tilde{\Lambda}, \tilde{x})$
Ex. 6.1	0.9403	11.0488	9.6422e-012
Ex. 6.2	0.6325	4.8225	9.0182e-012
Ex. 6.3	0.8658	4.2035	3.0518e-012

7. Conclusion and discussion. Sensitivity analysis for a multivariate eigenvalue problem has been performed. The first order perturbation expansions of a simple multivariate eigenvalue and the corresponding eigenvector are presented. The expressions are easy to compute. We also derive explicit expressions for the condition numbers, first-order perturbation upper bounds and backward errors for the multivariate eigenpairs.

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