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REFINED INERTIAS OF TREE SIGN PATTERNS*

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Abstract. The refined inertia $(n_+, n_-, n_z, 2n_p)$ of a real matrix is the ordered 4-tuple that subdivides the number n_0 of eigenvalues with zero real part in the inertia (n_+, n_-, n_0) into those that are exactly zero (n_z) and those that are nonzero $(2n_p)$. For $n \geq 2$, the set of refined inertias $\mathbb{H}_n = \{(0, n, 0, 0), (0, n-2, 0, 2), (2, n-2, 0, 0)\}$ is important for the onset of Hopf bifurcation in dynamical systems. Tree sign patterns of order n that require or allow the refined inertias $\mathbb{H}_n$ are considered. For $n = 4$, necessary and sufficient conditions are proved for a tree sign pattern (necessarily a path or a star) to require $\mathbb{H}_4$. For $n \geq 3$, a family of $n \times n$ star sign patterns that allows $\mathbb{H}_n$ is given, and it is proved that if a star sign pattern requires $\mathbb{H}_n$, then it must have exactly one zero diagonal entry associated with a leaf in its digraph.

Key words. Eigenvalues, Tree sign pattern, Refined inertia, Hopf bifurcation.

AMS subject classifications. 15B35, 15A18, 05C50.

1. Introduction. An $n \times n$ *sign pattern* is an $n \times n$ matrix with entries from $\{+, -, 0\}$. The sign, $\text{sgn}(a)$, of a real number a is defined by $\text{sgn}(a) = +, -,$ or 0 when $a > 0, a < 0$, or $a = 0$, respectively. The sign pattern of a real matrix $A = [a_{ij}]$ is the sign pattern $\mathcal{A} = \text{sgn}(A) = [\text{sgn}(a_{ij})]$; matrix A is called a *realization* of $\mathcal{A}$. The *sign pattern class* $Q(\mathcal{A})$ of the sign pattern $\mathcal{A}$ is the set $Q(\mathcal{A}) = \{A | \text{sgn}(A) = \mathcal{A}\}$. The *digraph* $D(\mathcal{A})$ *of a sign pattern* $\mathcal{A} = [\alpha_{ij}]$ has n vertices, an arc from i to j if $\alpha_{ij} \neq 0$ and a loop at vertex i if $\alpha_{ii} \neq 0$. The *signed digraph of sign pattern* $\mathcal{A}$ is the digraph of $\mathcal{A}$ with α_{ij} on the arc from i to j if $\alpha_{ij} \neq 0$ and α_{ii} on the loop at vertex i if $\alpha_{ii} \neq 0$.

As defined in [7], the *refined inertia* $ri(A)$ of a real $n \times n$ matrix A is the ordered 4-tuple $(n_+, n_-, n_z, 2n_p)$ such that n_+ (resp., n_-) is the number of eigenvalues (including multiplicities) of A with positive (resp., negative) real part, and n_z (resp., $2n_p$) is the number of zero eigenvalues (resp., nonzero pure imaginary eigenvalues) of A. Here $n_+ + n_- + n_z + 2n_p = n$. The *inertia* of A is $(n_+, n_-, n_z + 2n_p)$, thus the refined inertia subdivides those eigenvalues with zero real part and distinguishes between those that are exactly zero and those that are nonzero.

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A sign pattern $\mathcal{A}$ is *sign nonsingular* if $n_z = 0$ (i.e., $\det(A) \neq 0$) for all $A \in Q(\mathcal{A})$; see [3]. An $n \times n$ sign pattern $\mathcal{A}$ is *potentially stable* if there is a matrix $A \in Q(\mathcal{A})$ such that $n_- = n$. An $n \times n$ sign pattern $\mathcal{A}$ is *sign stable* if $n_- = n$ for all $A \in Q(\mathcal{A})$. Each of these properties is invariant under sign pattern equivalence (i.e., transposition, permutation similarity and signature similarity). Multiplying a matrix A by a positive scalar, one nonzero diagonal entry can be set to have magnitude 1 when refined inertia is considered. Furthermore, for an $n \times n$ irreducible matrix $A \in Q(\mathcal{A})$, without loss of generality $n-1$ nonzero off-diagonal entries corresponding to a spanning tree of $D(\mathcal{A})$ can be set to have magnitude 1 by a positive diagonal similarity (see, e.g., [2, Lemma 2.3]).

The following observation, which is Lemma 3.4 *(iii)* in [4], is used to prove some results in Section 2.

Observation 1.1. [4] Suppose $\mathcal{A}$ is a sign pattern that has a realization A with $ri(A) = (n_+, n_-, n_z, 2n_p)$ that allows a full rank Jacobian matrix. If $n_p \geq 1$, then there exist $A_1, A_2 \in Q(\mathcal{A})$ such that the refined inertias of A_1 and A_2 are $(2 + n_+, n_-, n_z, 2(n_p - 1))$ and $(n_+, 2 + n_-, n_z, 2(n_p - 1))$, respectively.

Hopf bifurcation is of interest in the study of dynamical systems. To connect Hopf bifurcation in a dynamical system to refined inertia, for $n \geq 2$ let $\mathbb{H}_n = \{(0, n, 0, 0), (0, n-2, 0, 2), (2, n-2, 0, 0)\}$, as defined in [1]. A sign pattern $\mathcal{A}$ *requires refined inertia* $\mathbb{H}_n$ if $\mathbb{H}_n = \{ri(A) | A \in Q(\mathcal{A})\}$. A sign pattern $\mathcal{A}$ *allows refined inertia* $\mathbb{H}_n$ if $\mathbb{H}_n \subseteq \{ri(A) | A \in Q(\mathcal{A})\}$. Consider an n-dimensional dynamical system linearized about an equilibrium with Jacobian matrix having sign pattern $\mathcal{A}$. Let a be a parameter of the system and $A(a_i)$ be the Jacobian matrix with $a = a_i$. If $a_1 < a_2 < a_3$ or $a_1 > a_2 > a_3$ and $ri(A(a_1)) = (0, n, 0, 0)$, $ri(A(a_2)) = (0, n-2, 0, 2)$, and $ri(A(a_3)) = (2, n-2, 0, 0)$, then $\mathcal{A}$ allows $\mathbb{H}_n$ and Hopf bifurcation may occur giving rise to periodic solutions. The same idea applies for dynamical systems with magnitude restrictions on some entries of $\mathcal{A}$; for examples from different applications, see [1].

Clearly, if $\mathcal{A}$ requires $\mathbb{H}_n$ then $\mathcal{A}$ is potentially stable and sign nonsingular with $\text{sgn}(\det(A)) = \text{sgn}((-1)^n)$ for all $A \in Q(\mathcal{A})$. Furthermore, if $\mathcal{A}$ requires $\mathbb{H}_n$ then $\mathcal{A}$ is not sign stable and $-\mathcal{A}$ is not potentially stable.

Some results for the requires $\mathbb{H}_n$ problem can be found in [1]. It is shown in [1, Theorem 2.1] that a 3×3 sign nonsingular sign pattern allows $\mathbb{H}_3$ if and only if it requires $\mathbb{H}_3$. Theorem 2.3 in [1] states that if a 4×4 sign nonsingular sign pattern requires a negative trace and allows $\mathbb{H}_4$, then it requires $\mathbb{H}_4$. It is conjectured in [1] that no $n \times n$ sign pattern requires $\mathbb{H}_n$ for $n \geq 8$ and an example of a 7×7 sign pattern that requires $\mathbb{H}_7$ is given.

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In this paper, we focus on (irreducible) tree sign patterns, i.e., sign patterns $\mathcal{A}$ for which $D(\mathcal{A})$ is a doubly directed tree. In Section 2, we characterize the 4×4 tree sign patterns that require $\mathbb{H}_4$. Using one of these sign patterns, in Section 3 we describe a set of $n \times n$ star sign patterns that allow $\mathbb{H}_n$ for $n \geq 3$. With results from [5], we prove that an $n \times n$ star sign pattern that requires $\mathbb{H}_n$ must have exactly one zero diagonal entry associated with a leaf in its digraph. In Section 4, we extend a result from [1] on reducible sign patterns that require $\mathbb{H}_n$ and give a new example of a surprising reducible pattern that allows $\mathbb{H}_9$. Some concluding remarks are given in Section 5.

2. Tree sign patterns. If $D(\mathcal{A})$ is a doubly directed path, then $\mathcal{A}$ is called a *path sign pattern.* If $D(\mathcal{A})$ is a doubly directed star, then $\mathcal{A}$ is called a *star sign pattern.* In any realization $A = [a_{ij}]$ of a path sign pattern, without loss of generality assume that adjacent vertices on the path are numbered $1, 2, \ldots, n$, and entries $a_{i,i+1} = 1$ for $i = 1, \ldots, n-1$. In any realization $A = [a_{ij}]$ of a star sign pattern $\mathcal{A}$, without loss of generality take vertex 1 in $D(\mathcal{A})$ as the center vertex and the $n-1$ entries $a_{1,i} = 1$ for $i = 2, \ldots, n$.

Results from [1] can be used to show the following characterization.

THEOREM 2.1. [1] *A 3×3 tree sign pattern requires $\mathbb{H}_3$ if and only if it is potentially stable and sign nonsingular, but not sign stable.*

If the digraph of a 4×4 sign pattern $D(\mathcal{A})$ is a doubly directed tree, then $\mathcal{A}$ is either a path sign pattern or a star sign pattern.

OBSERVATION 2.2. If $\mathcal{A}$ is a 4×4 sign pattern that requires a positive determinant, then $A \in Q(\mathcal{A})$ can have one of only six possible refined inertias, namely the three refined inertias in $\mathbb{H}_4$, $(4, 0, 0, 0)$, $(2, 0, 0, 2)$ or $(0, 0, 0, 4)$.

The potentially stable 4×4 path and star sign patterns are listed in [6] and [8]. Beginning with these sign patterns, we show that up to equivalence there are exactly 5 path sign patterns and 5 star sign patterns that require $\mathbb{H}_4$.

2.1. Path sign patterns of order 4. Up to equivalence, the following are the only 4×4 path sign patterns that are potentially stable, sign nonsingular, not sign stable, and for which its negative is not potentially stable (see [6] and [8]):

$$\mathcal{P}_1 = \begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & - & - \end{bmatrix}, \ \mathcal{P}_2 = \begin{bmatrix} 0 & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & + & - \end{bmatrix}, \ \mathcal{P}_3 = \begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & - \end{bmatrix},$$

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$$\mathcal{P}_4 = \begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & 0 \end{bmatrix}, \ \mathcal{P}_5 = \begin{bmatrix} - & + & 0 & 0 \\ + & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & 0 \end{bmatrix}.$$

Since $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$, and $\mathcal{P}_4$ have only nonpositive entries on the diagonal, Theorem 2.3 in [1] applies; i.e., if any one of them allows $\mathbb{H}_4$, then it also requires $\mathbb{H}_4$. The following result is immediate from this theorem and the table of realizations below.

THEOREM 2.3. *The path sign patterns $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, and $\mathcal{P}_4$ require $\mathbb{H}_4$.*

Realization of Sign Pattern	ri $(0,4,0,0)$	ri $(0,2,0,2)$	ri $(2,2,0,0)$
$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & a & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \in Q(\mathcal{P}_1)$	$a = 0.76$	some $a \in (0.76, 0.77)$	$a = 0.77$
$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & -a & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \in Q(\mathcal{P}_2)$	$a = 3.24$	some $a \in (3.23, 3.24)$	$a = 3.23$
$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & a & -1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \in Q(\mathcal{P}_3)$	$a = 1.65$	some $a \in (1.65, 1.66)$	$a = 1.66$
$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & a & -1 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \in Q(\mathcal{P}_4)$	$a = 0.5$	$a = 1$	$a = 2$

The next result shows that the above is also true for $\mathcal{P}_5$, which is a superpattern of a sign pattern equivalent to $\mathcal{P}_2$.

THEOREM 2.4. *The path sign pattern $\mathcal{P}_5$ requires $\mathbb{H}_4$.*

Proof. To consider refined inertia, any realization of $\mathcal{P}_5$ can be normalized to

$$M = \begin{bmatrix} -a & 1 & 0 & 0 \\ d & b & 1 & 0 \\ 0 & -e & -c & 1 \\ 0 & 0 & f & 0 \end{bmatrix} \in Q(\mathcal{P}_5),$$

where $a, b, c, d, e, f \in \mathbb{R}^+$. The characteristic polynomial of M is $x^4 + p_1 x^3 + p_2 x^2 + p_3 x + p_4$ with

$$\begin{aligned} p_1 &= a + c - b \\ p_2 &= ac + e - ab - bc - d - f \\ p_3 &= ae + bf - abc - cd - af \\ p_4 &= abf + fd. \end{aligned}$$

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Define the map $\chi : \mathbb{R}^6 \to \mathbb{R}^4$ by $\chi(a,b,c,d,e,f) = (p_1,p_2,p_3,p_4)$. The Jacobian matrix of the map χ, i.e., $[\frac{\partial(p_1,\ldots,p_4)}{\partial(a,\ldots,f)}]$ is

$$Jac_\chi = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ c-b & -a-c & a-b & -1 & 1 & -1 \\ e-f-bc & f-ac & -ab-d & -c & a & b-a \\ bf & af & 0 & f & 0 & d+ab \end{bmatrix}.$$

The 4×4 submatrix consisting of columns 1, 2, 4 and 5 has determinant fe, and hence, Jac_χ has rank 4. Since $-\mathcal{P}_5$ is not potentially stable [6, 8], $\mathcal{P}_5$ does not allow refined inertia $(4,0,0,0)$, and thus by Observation 1.1 it also does not allow $(2,0,0,2)$ or $(0,0,0,4)$. Fix $a = 1$, $b = 0.5$, $c = 2$, $d = 0.05$, and $f = 0.1$. If $e = 1.24$, then $ri(M) = (0,4,0,0)$; if $e = 1.23$, then $ri(M) = (2,2,0,0)$. Therefore, by continuity, there is a value of e such that $1.23 < e < 1.24$ with $ri(M) = (0,2,0,2)$. Hence, $\mathcal{P}_5$ allows $\mathbb{H}_4$ and since it does not allow $(4,0,0,0)$, $(2,0,0,2)$, or $(0,0,0,4)$, by Observation 2.2 it requires $\mathbb{H}_4$. ☐

2.2. Star sign patterns of order 4. Up to equivalence, the following are the only 4×4 star sign patterns that are potentially stable, sign nonsingular, not sign stable, and for which its negative is not potentially stable (see [6]):

$$\mathcal{S}_1 = \begin{bmatrix} - & + & + & + \\ - & - & 0 & 0 \\ + & 0 & - & 0 \\ - & 0 & 0 & 0 \end{bmatrix},\ \mathcal{S}_2 = \begin{bmatrix} - & + & + & + \\ + & - & 0 & 0 \\ + & 0 & - & 0 \\ - & 0 & 0 & 0 \end{bmatrix},\ \mathcal{S}_3 = \begin{bmatrix} 0 & + & + & + \\ - & 0 & 0 & 0 \\ + & 0 & - & 0 \\ - & 0 & 0 & - \end{bmatrix},$$

$$\mathcal{S}_4 = \begin{bmatrix} - & + & + & + \\ + & - & 0 & 0 \\ - & 0 & + & 0 \\ + & 0 & 0 & 0 \end{bmatrix},\ \mathcal{S}_5 = \begin{bmatrix} + & + & + & + \\ - & 0 & 0 & 0 \\ - & 0 & - & 0 \\ - & 0 & 0 & - \end{bmatrix}.$$

Since each of the patterns $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ requires negative trace, the following result is immediate from [1, Theorem 2.3] and the table of realizations below.

THEOREM 2.5. *The star sign patterns $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ require $\mathbb{H}_4$.*

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Realization of Sign Pattern	ri $(0,4,0,0)$	ri $(0,2,0,2)$	ri $(2,2,0,0)$
$\begin{bmatrix} -0.01 & 1 & 1 & 1 \\ -1 & -a & 0 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \in Q(\mathcal{S}_1)$	$a = 0.9$	some $a \in (0.8, 0.9)$	$a = 0.8$
$\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -a & 0 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \in Q(\mathcal{S}_2)$	$a = 2.6$	some $a \in (2.5, 2.6)$	$a = 2.5$
$\begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ -a & 0 & 0 & -1 \end{bmatrix} \in Q(\mathcal{S}_3)$	$a = 2$	$a = 1$	$a = 0.5$

By eliminating the other three refined inertias in Observation 2.2 and finding a realization for each refined inertia in $\mathbb{H}_4$, we now show that sign patterns $\mathcal{S}_4$ and $\mathcal{S}_5$ require $\mathbb{H}_4$.

THEOREM 2.6. *The star sign pattern $\mathcal{S}_4$ requires $\mathbb{H}_4$.*

Proof. To consider refined inertia, any realization of $\mathcal{S}_4$ can be normalized to

$$M = \begin{bmatrix} -1 & 1 & 1 & 1 \\ c & -a & 0 & 0 \\ -d & 0 & b & 0 \\ e & 0 & 0 & 0 \end{bmatrix},$$

where $a, b, c, d, e \in \mathbb{R}^+$. The characteristic polynomial of M is $c_M(x) = x^4 + p_1x^3 + p_2x^2 + p_3x + p_4$, where

$$\begin{aligned} p_1 &= a - b + 1 \\ p_2 &= a - ab - b - c + d - e \\ p_3 &= ad - ab - ae + bc + be \\ p_4 &= abe. \end{aligned}$$

The normalized form M satisfies the conditions of Observation 1.1 by defining the map $\chi : \mathbb{R}^5 \to \mathbb{R}^4$ as $\chi(a, b, c, d, e) = (p_1, p_2, p_3, p_4)$. The Jacobian matrix of the map χ is

$$Jac_\chi = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 1-b & -1-a & -1 & 1 & -1 \\ d-b-e & c-a+e & b & a & b-a \\ be & ae & 0 & 0 & ab \end{bmatrix}.$$

Taking the 4×4 submatrix formed by the first, third, fourth and fifth columns of Jac_χ, the determinant is $-(a^2b + ab^2)$, which is nonzero since $a, b > 0$. Therefore,

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Jac_χ has rank 4. Since $-\mathcal{S}_4$ does not appear in [6], it is not potentially stable, and thus $\mathcal{S}_4$ does not allow refined inertia $(4,0,0,0)$. Consequently $\mathcal{S}_4$ does not allow refined inertia $(2,0,0,2)$ or $(0,0,0,4)$ by Observation 1.1. Fix $a = c = e = 1$ and $b = 0.1$. If $d = 2$, then $ri(M) = (0,4,0,0)$; if $d = 1.9$, then $ri(M) = (2,2,0,0)$. By continuity and the sign nonsingularity of $\mathcal{S}_4$, there exists a value of d such that $1.9 < d < 2$ with $ri(M) = (0,2,0,2)$. Therefore, by Observation 2.2, the star sign pattern $\mathcal{S}_4$ requires $\mathbb{H}_4$. □

Lemma 2.7. *The star sign pattern $\mathcal{S}_5$ allows $\mathbb{H}_4$.*

Proof. Consider the matrix

$$M = \begin{bmatrix} f & 1 & 1 & 1 \\ -c & 0 & 0 & 0 \\ -d & 0 & -a & 0 \\ -e & 0 & 0 & -b \end{bmatrix} \in Q(\mathcal{S}_5),$$

where $a,b,c,d,e,f \in \mathbb{R}^+$. Fix $a = b = c = d = e = 1$. If $f = 0.5$, then $ri(M) = (0,4,0,0)$; if $f = 0.6$, then $ri(M) = (2,2,0,0)$. Therefore, by continuity, negativity of the trace and the sign nonsingularity of $\mathcal{S}_5$, there exists a value of f such that $0.5 < f < 0.6$ with $ri(M) = (0,2,0,2)$. Thus $\mathcal{S}_5$ allows $\mathbb{H}_4$. □

Theorem 2.8. *The star sign pattern $\mathcal{S}_5$ requires $\mathbb{H}_4$.*

Proof. To consider refined inertia, any realization of $\mathcal{S}_5$ can be normalized to

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -c & 0 & 0 & 0 \\ -d & 0 & -a & 0 \\ -e & 0 & 0 & -b \end{bmatrix}$$

where $a,b,c,d,e \in \mathbb{R}^+$. The characteristic polynomial of M is $c_M(x) = x^4 + p_1x^3 + p_2x^2 + p_3x + p_4$, where

$$\begin{aligned} p_1 &= a + b - 1 \\ p_2 &= ab - a - b + c + d + e \\ p_3 &= ac - ab + ae + bc + bd \\ p_4 &= abc. \end{aligned}$$

If $\mathcal{S}_5$ allows refined inertia $(0,0,0,4)$, then there exists an M with characteristic polynomial $x^4 + p_2x^2 + p_4$, where $p_2, p_4 > 0$. If $p_1 = a + b - 1 = 0$ then $a = 1 - b$. Thus since $a > 0$ it follows that $b < 1$. Now substituting $a = 1 - b$ into $p_3 = 0$ gives

$$\begin{aligned} c &= e(b-1) + b(1-b-d) \\ &= (e-b)(b-1) - bd. \end{aligned}$$

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Since $c > 0$ and $b < 1$, the first equality gives $d < 1 - b$ and the second equality gives $e < b$. Now substituting a and c into p_2 gives

$$\begin{aligned} p_2 &= 2b - 2b^2 + d - 1 + be - bd \\ &= (1-b)(d-1+b) + b(e-b). \end{aligned}$$

Then $1 - b > 0$, $d - 1 + b < 0$ and $e - b < 0$ imply that $p_2 < 0$. Thus $\mathcal{S}_5$ does not allow refined inertia $(0, 0, 0, 4)$. Since $-\mathcal{S}_5$ is not potentially stable [6], it follows that $\mathcal{S}_5$ does not allow refined inertia $(4, 0, 0, 0)$.

Finally, suppose that $\mathcal{S}_5$ allows refined inertia $(2, 0, 0, 2)$. Then the coefficients of the characteristic polynomial satisfy

$$p_1 < 0,\ p_2 > 0,\ p_3 < 0, \text{ and } p_4 > 0.$$

Thus $p_1 = a + b - 1 < 0$. Now consider the following three cases.

Case 1 $a = b$. Coefficients p_1, p_2, and p_3 become

$$\begin{aligned} p_1 &= 2b - 1 \\ p_2 &= b^2 - 2b + c + d + e \\ p_3 &= b(c - b + e + c + d). \end{aligned}$$

First note that $2b - 1 < 0$ and so $b < \frac{1}{2}$. Since $p_3 < 0$, it follows that $b > c + d + e + c$, and in particular $b > c + d + e$. But this implies that $p_2 = b(b - 2) + c + d + e < -b + c + d + e < 0$, which is a contradiction. Thus $b \neq a$ if M has refined inertia $(2, 0, 0, 2)$.

Case 2 $a > b$, i.e., $a = b + \epsilon$ for $\epsilon > 0$. As before consider the coefficients

$$\begin{aligned} p_1 &= 2b + \epsilon - 1 \\ p_2 &= b(b + \epsilon) - b - \epsilon - b + c + d + e \\ p_3 &= bc + \epsilon c - b^2 - b\epsilon + be + \epsilon e + bc + bd. \end{aligned}$$

Since $p_1 < 0$, $2b + \epsilon - 1 < 0$ and so $2b + \epsilon < 1$. Since $p_3 < 0$, it follows that $-(b+\epsilon)b + (b+\epsilon)c + (b+\epsilon)e + bc + bd < 0$ and so $b + \epsilon > c + d + e + c + \frac{\epsilon}{b}(c + e)$, and in particular $b + \epsilon > c + d + e$. Finally, this gives $p_2 = -b(2 - \epsilon - b) - \epsilon + c + d + e < -b - \epsilon + c + d + e < 0$, which is a contradiction. Therefore, if $a > b$, then M does not have refined inertia $(2, 0, 0, 2)$.

Case 3 $a < b$. The proof of this case follows from that of Case 2 by interchanging e and d in the expression for p_3, and by replacing b with a throughout.

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Therefore, these three cases imply that $\mathcal{S}_5$ does not allow refined inertia $(2,0,0,2)$. By Observation 2.2 and Lemma 2.7, $\mathcal{S}_5$ requires $\mathbb{H}_4$. ◻

In order to obtain a characterization of the 4×4 tree sign patterns that require $\mathbb{H}_4$, we started with the list of potentially stable path and star sign patterns in [6, 8]. Elimination of those sign patterns that are sign stable or not sign nonsingular resulted in 21 tree sign patterns $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$ and $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{S}_4$, $\mathcal{S}_5$ above together with $\mathcal{P}_6$, $\mathcal{P}_7$, $\mathcal{P}_8$, $\mathcal{P}_9$, $\mathcal{P}_{10}$, $\mathcal{P}_{11}$ and $\mathcal{S}_6$, $\mathcal{S}_7$, $\mathcal{S}_8$, $\mathcal{S}_9$, $\mathcal{S}_{10}$ in the Appendix. Each of the patterns in the Appendix is shown to allow refined inertia $(4,0,0,0)$, i.e., its negative is potentially stable, leading to the following characterization.

Theorem 2.9. *A 4×4 tree sign pattern requires $\mathbb{H}_4$ if and only if it is potentially stable, sign nonsingular, not sign stable, and its negative is not potentially stable.*

3. Star sign patterns of order n.

3.1. Extending a star sign pattern. Using the star sign patterns $\mathcal{S}_1$ and $\mathcal{S}_2$, we construct $n\times n$ star sign patterns that allow $\mathbb{H}_n$. Note that these sign patterns are potentially stable by [5, Theorems 4.3 and 3.5].

Theorem 3.1. *With $\pm$ taken to be either $+$ or $-$, the $n\times n$ star sign patterns*

$$\mathcal{S}=\begin{bmatrix} - & + & + & + & + & + & \dots & + \\ + & - & 0 & 0 & 0 & 0 & \dots & 0 \\ - & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \pm & 0 & 0 & - & 0 & 0 & \dots & 0 \\ \pm & 0 & 0 & 0 & - & 0 & \dots & 0 \\ \vdots & & & & & \ddots & & \vdots \\ \pm & 0 & 0 & 0 & \dots & 0 & - & 0 \\ \pm & 0 & 0 & 0 & \dots & 0 & 0 & - \end{bmatrix}$$

require $\mathbb{H}_n$ for $n=3$ and 4, and allow $\mathbb{H}_n$ for $n\geq 5$.

Proof. For $n=3$, the sign pattern $\mathcal{S}$ is equivalent to a pattern in the Appendix of [1], and thus requires $\mathbb{H}_3$. For $n=4$, sign pattern $\mathcal{S}$ with the $(4,1)$ entry equal to $-$ is equivalent to $\mathcal{S}_1$ listed above. Taking the $(4,1)$ entry to be $+$, sign pattern $\mathcal{S}$ is equivalent to $\mathcal{S}_2$ listed above. Thus for $n=4$, $\mathcal{S}$ requires $\mathbb{H}_4$.

For $n\geq 5$, we show that sign patterns $\mathcal{S}$ allow each refined inertia in $\mathbb{H}_n$. Consider sign pattern $\widetilde{\mathcal{S}}$ obtained from $\mathcal{S}$ by replacing the $(j,1)$ and $(1,j)$ entries with zero for $j=3,4,\dots,n$. The eigenvalues of $\widetilde{\mathcal{S}}$ are the eigenvalues of the leading principal 2×2

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submatrix, one 0, and $n-3$ negative real numbers. Consider

$$\widetilde{S}=\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ a & -1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & -10 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & -10 & 0 & \dots & 0 \\ \vdots & & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -10 \end{bmatrix} \in Q(\widetilde{\mathcal{S}})$$

and

$$S(a,\epsilon)=\begin{bmatrix} -1 & 1 & \frac{\epsilon}{n} & \frac{\epsilon}{n} & \frac{\epsilon}{n} & \frac{\epsilon}{n} & \dots & \frac{\epsilon}{n} \\ a & -1 & 0 & 0 & 0 & 0 & \dots & 0 \\ -\epsilon & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \pm\epsilon & 0 & 0 & -10 & 0 & 0 & \dots & 0 \\ \pm\epsilon & 0 & 0 & 0 & -10 & 0 & \dots & 0 \\ \vdots & & & & & \ddots & & \vdots \\ \pm\epsilon & 0 & 0 & 0 & \dots & 0 & -10 & 0 \\ \pm\epsilon & 0 & 0 & 0 & \dots & 0 & 0 & -10 \end{bmatrix} \in Q(\mathcal{S})$$

with $a, \epsilon > 0$.

If $a > 1$, then the determinant of the leading 2×2 principal submatrix of $\widetilde{S}$ is negative, and thus the eigenvalues of $\widetilde{S}$ are one negative real number, one positive real number, 0, and -10 with multiplicity $n-3$. For $\epsilon > 0$ sufficiently small, the refined inertia of $S(a,\epsilon)$ is $(2, n-2, 0, 0)$, since the sign of the determinant of $S(a,\epsilon)$ is $(-1)^n$ and the eigenvalues of $S(a,\epsilon)$ are small perturbations of those of $\widetilde{S}$.

Now if $a < 1$, then $n-1$ of the eigenvalues of $\widetilde{S}$ are negative and one is zero. From the properties above, for sufficiently small $\epsilon > 0$, the refined inertia of $S(a,\epsilon)$ is $(0, n, 0, 0)$.

Fix a_1 such that $1 < a_1 < 8$ and $\epsilon_1 > 0$ sufficiently small so that $S(a_1, \epsilon_1)$ has refined inertia $(2, n-2, 0, 0)$, and fix a_2 such that $0 < a_2 < 1$ and $\epsilon_2 > 0$ sufficiently small so that $S(a_2, \epsilon_2)$ has refined inertia $(0, n, 0, 0)$. Let $\epsilon_3 = \min\{\epsilon_1, \epsilon_2, \frac{1}{2}\}$ and note that $S(a_1, \epsilon_3)$ has refined inertia $(2, n-2, 0, 0)$ and $S(a_2, \epsilon_3)$ has refined inertia $(0, n, 0, 0)$. Now consider the matrices $S(a, \epsilon_3)$ for $a_2 < a < a_1$. By Geršgorin's disc theorem [9, p. 14-5], each of these matrices has $n-3$ eigenvalues that lie within a closed disc of radius $\epsilon_3 \le \frac{1}{2}$ centered at -10 in the complex plane. Furthermore, using Geršgorin's theorem, it follows that the other three eigenvalues lie within a disjoint closed disc of radius 9 centered at the origin. Since the sign of the determinant of

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$S(a, \epsilon_3)$ is $(-1)^n$, as a decreases continuously from a_1 to a_2 there must be a value $\hat{a}$ in the interval (a_2, a_1) for which the refined inertia of $S(\hat{a}, \epsilon_3)$ is $(0, n-2, 0, 2)$. Therefore, the $n \times n$ star sign patterns $\mathcal{S}$ allow $\mathbb{H}_n$ for $n \geq 5$ and require $\mathbb{H}_n$ for $n = 3$ and 4. $\square$

The next example gives one instance of the above sign patterns $\mathcal{S}$ that does not require $\mathbb{H}_n$ for $n \geq 5$.

EXAMPLE 3.2. The sign pattern

$$\mathcal{S} = \begin{bmatrix} - & + & + & + & + \\ + & - & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 \\ - & 0 & 0 & - & 0 \\ + & 0 & 0 & 0 & - \end{bmatrix}$$

allows $\mathbb{H}_5$ by Theorem 3.1. Consider the following realization

$$S = \begin{bmatrix} -0.1 & 1 & 1 & 1 & 1 \\ 1000 & -100 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ -100 & 0 & 0 & -1 & 0 \\ 50 & 0 & 0 & 0 & -0.1 \end{bmatrix} \in Q(\mathcal{S}).$$

Since the eigenvalues of S are approximately $-109.1318, 3.3189 \pm 4.1492i, 0.0025$, and 1.2914, the refined inertia of S is $(4, 1, 0, 0)$. Hence, the sign pattern $\mathcal{S}$ does not require $\mathbb{H}_5$.

It follows by continuity that any $n \times n$ sign pattern with $\mathcal{S}$ as a 5×5 principal subpattern allows at least four eigenvalues with positive real part, and thus does not require $\mathbb{H}_n$.

3.2. A necessary condition for requiring $\mathbb{H}_n$. If a star sign pattern requires $\mathbb{H}_n$, then its digraph has some additional structure, namely that exactly one leaf vertex does not have a loop. The next result follows immediately from [5, Theorems 3.5 and 4.2].

LEMMA 3.3. [5] *Let $\mathcal{S} = [\sigma_{ij}]$ be an $n \times n$ potentially stable star sign pattern with 1 as the center vertex in $D(\mathcal{S})$ and without loss of generality $\sigma_{1i} = +$ for $i = 2, \ldots, n$. Then*
(i) if $\sigma_{11} \in \{+, 0\}$ then there exists i such that $\sigma_{i1} = -$ and $\sigma_{ii} = -$, and
(ii) for $i = 2, \ldots, n$,

$$|\{i|\sigma_{i1} = + \text{ and } \sigma_{ii} = +\}| = \left\lfloor \frac{|\{\sigma_{ii} = +\}|}{2} \right\rfloor. \tag{3.1}$$

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THEOREM 3.4. *For $n \geq 3$, let $\mathcal{S} = [\sigma_{ij}]$ be an $n \times n$ star sign pattern with 1 as the center vertex in $D(\mathcal{S})$. If $\mathcal{S}$ is sign nonsingular, potentially stable and not sign stable, then there exists a unique i such that $2 \leq i \leq n$ and $\sigma_{ii} = 0$.*

Proof. Since all $S \in Q(\mathcal{S})$ are nonsingular, at most one σ_{ii} can be zero. We now show by contradiction that at least one σ_{ii} must be zero. Recall that without loss of generality $\sigma_{1i} = +$ for $i = 2, \ldots, n$. If $\sigma_{ii} \neq 0$ for all $i = 2, \ldots, n$, then one of the following cases must occur.

Case 1. Let $\sigma_{ii} = -$ for $i = 2, \ldots, n$. Then σ_{i1} has the same sign for $i = 2, \ldots, n$; otherwise if say $\sigma_{i1}\sigma_{k1} = -$, then any $S = [s_{ij}] \in Q(\mathcal{S})$ has terms in $\det(S)$

$$-s_{kk}s_{1i}s_{i1} \prod_{j \neq 1,i,k} s_{jj} \qquad \text{and} \qquad -s_{ii}s_{1k}s_{k1} \prod_{j \neq 1,i,k} s_{jj}$$

of opposite sign, violating the sign nonsingularity of $\mathcal{S}$. If $\sigma_{i1} = +$ for all $i = 2, \ldots, n$, then $S \in Q(\mathcal{S})$ is symmetrizable by a positive diagonal similarity. Thus since $\mathcal{S}$ is potentially stable, sign nonsingular and symmetrizable, it must also be sign stable (since for any $S \in Q(\mathcal{S})$, all eigenvalues of S are real and thus negative). On the other hand, if $\sigma_{i1} = -$ for $i = 2, \ldots, n$, then $\mathcal{S}$ is sign stable if $\sigma_{11} = -$ or 0 [3, Corollary 10.2.3], and $\mathcal{S}$ is not sign nonsingular if $\sigma_{11} = +$. Thus each case gives a contradiction.

Case 2. Let $\sigma_{11} = -$ and $\sigma_{ii} = +$ for some i such that $2 \leq i \leq n$. Since $\mathcal{S}$ is potentially stable and sign nonsingular with $\operatorname{sgn}(\det(S)) = \operatorname{sgn}((-1)^n)$ for all $S \in Q(\mathcal{S})$, there exists $k \neq i$ such that $\sigma_{kk} = +$. Therefore, by Lemma 3.3 (ii), since the right hand side of (3.1) is at least one, the equality in (3.1) implies that i and k can be chosen, without loss of generality, so that $\sigma_{i1} = -$ and $\sigma_{k1} = +$. Therefore, as in Case 1, any $S \in Q(\mathcal{S})$ has two terms in $\det(S)$ of opposite sign, violating the sign nonsingularity of $\mathcal{S}$.

Case 3. Let $\sigma_{11} \in \{+, 0\}$ and $\sigma_{ii} = +$ for some i such that $2 \leq i \leq n$. By Lemma 3.3 (ii), i can be chosen such that $\sigma_{i1} = -$. By Lemma 3.3 (i), there exists a k such that $\sigma_{k1} = -$ and $\sigma_{kk} = -$. Thus, as in Case 1, the sign nonsingularity of $\mathcal{S}$ is violated. □

The next result follows immediately from Theorem 3.4, since if $\mathcal{S}$ requires $\mathbb{H}_n$ then it is potentially stable, sign nonsingular and not sign stable.

COROLLARY 3.5. *For $n \geq 3$, let $\mathcal{S} = [\sigma_{ij}]$ be an $n \times n$ star sign pattern with 1 as the center vertex in $D(\mathcal{S})$. If $\mathcal{S}$ requires $\mathbb{H}_n$, then there exists a unique i such that $2 \leq i \leq n$ and $\sigma_{ii} = 0$.*

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4. Reducible sign patterns. Reducible sign patterns that either require or allow $\mathbb{H}_n$ are considered in [1]. The following result is an extension of [1, Observation 1.5] for the requires problem.

THEOREM 4.1. *Suppose* $\mathcal{A} = \left[\begin{array}{c|c} \mathcal{A}_1 & \sharp \\ \hline O & \mathcal{A}_2 \end{array}\right]$*, where* $\mathcal{A}_1$ *is a sign pattern of order* n_1*,* $\mathcal{A}_2$ *is a sign pattern of order* n_2 *and* $\sharp$ *denotes an arbitrary* $n_1 \times n_2$ *sign pattern. Then* $\mathcal{A}$ *requires* $\mathbb{H}_{n_1+n_2}$ *if and only if exactly one sign pattern* $\mathcal{A}_i$ *requires* $\mathbb{H}_{n_i}$ *and the other sign pattern* $\mathcal{A}_j$ *is sign stable.*

Proof. Suppose first without loss of generality that $\mathcal{A}_1$ requires $\mathbb{H}_{n_1}$ and $\mathcal{A}_2$ is sign stable. Therefore, any realization of $\mathcal{A}$ necessarily has refined inertia in $\mathbb{H}_{n_1+n_2}$ and $\mathcal{A}$ requires $\mathbb{H}_{n_1+n_2}$.

Conversely, if $\mathcal{A}$ requires $\mathbb{H}_{n_1+n_2}$, then exactly one sign pattern $\mathcal{A}_i$ requires $\mathbb{H}_{n_i}$, in which case the other sign pattern $\mathcal{A}_j$ must be sign stable. ☐

Now consider the allows problem for the reducible sign pattern $\mathcal{A}$ in Theorem 4.1. From [1, Observation 1.5], if $\mathcal{A}_i$ allows $\mathbb{H}_{n_i}$ and $\mathcal{A}_j$ is potentially stable with distinct $i, j \in \{1, 2\}$, then $\mathcal{A} = \mathcal{A}_i \oplus \mathcal{A}_j$ allows $\mathbb{H}_{n_i+n_j}$. However the following proposition and example show that the converse is false.

PROPOSITION 4.2. *Let*

$$\mathcal{P} = \begin{bmatrix} 0 & + & 0 & 0 & 0 \\ - & 0 & + & 0 & 0 \\ 0 & - & - & + & 0 \\ 0 & 0 & - & 0 & + \\ 0 & 0 & 0 & - & 0 \end{bmatrix}.$$

The path sign pattern $\mathcal{P}$ *allows only two refined inertias, namely* $(0, 5, 0, 0)$ *and* $(0, 3, 0, 2)$.

Proof. First notice that $\mathcal{P}$ satisfies the hypotheses of [3, Theorem 10.2.1] and so it is sign semi-stable, i.e., does not allow any eigenvalues with positive real part. Thus since $\mathcal{P}$ is sign nonsingular with negative trace and sign semi-stable, the only possible refined inertias are $(0, 1, 0, 4)$, $(0, 3, 0, 2)$ and $(0, 5, 0, 0)$. To eliminate refined inertia $(0, 1, 0, 4)$ consider

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -b & 0 & 1 & 0 & 0 \\ 0 & -c & -a & 1 & 0 \\ 0 & 0 & -d & 0 & 1 \\ 0 & 0 & 0 & -e & 0 \end{bmatrix} \in Q(\mathcal{P})$$

where $a, b, c, d, e \in \mathbb{R}^+$, which has characteristic polynomial $c_A(x) = x^5 + ax^4 +$

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$(b+c+d+e)x^3 + a(b+e)x^2 + (bd+be+ce)x + abe$. If A has refined inertia $(0,1,0,4)$, then the characteristic polynomial of A is $(x+\alpha)(x^2+\beta)(x^2+\gamma) = x^5 + \alpha x^4 + (\beta+\gamma)x^3 + \alpha(\beta+\gamma)x^2 + \beta\gamma x + \alpha\beta\gamma$ with $\alpha,\beta,\gamma \in \mathbb{R}^+$. Equating these polynomials gives

$$\begin{aligned} a &= \alpha \\ a(b+e) &= \alpha(\beta+\gamma) \Rightarrow b+e = \beta+\gamma \\ b+c+d+e &= \beta+\gamma \Rightarrow c+d = 0. \end{aligned}$$

This is a contradiction and so $\mathcal{P}$ does not allow refined inertia $(0,1,0,4)$. If P is a realization of $\mathcal{P}$ with all nonzero entries having magnitude 1, then $ri(P) = (0,3,0,2)$. If $\widetilde{P}$ is obtained from P by changing the $(2,1)$ entry to -2, then $ri(\widetilde{P}) = (0,5,0,0)$. Therefore, the only refined inertias allowed by $\mathcal{P}$ are $(0,3,0,2)$ and $(0,5,0,0)$. □

Example 4.3. The path sign patterns

$$\mathcal{P}_1 = \begin{bmatrix} 0 & + & 0 & 0 & 0 \\ - & 0 & + & 0 & 0 \\ 0 & - & - & + & 0 \\ 0 & 0 & - & 0 & + \\ 0 & 0 & 0 & - & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{P}_2 = \begin{bmatrix} - & + & 0 & 0 \\ + & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & + & - \end{bmatrix}$$

do not allow $\mathbb{H}_5$ and $\mathbb{H}_4$, respectively, however $\mathcal{A} = \mathcal{P}_1 \oplus \mathcal{P}_2$ allows, but does not require, $\mathbb{H}_9$. To see this, first note that $\mathcal{P}_1$ is sign semi-stable by Proposition 4.2. Hence, $\mathcal{P}_1$ does not allow refined inertia $(2,3,0,0)$ and consequently does not allow $\mathbb{H}_5$. Next notice that since any realization of $\mathcal{P}_2$ is symmetrizable, $\mathcal{P}_2$ does not allow refined inertia $(0,2,0,2)$. Therefore, $\mathcal{P}_2$ does not allow $\mathbb{H}_4$. However, $\mathcal{P}_2$ is potentially stable and so it allows refined inertia $(0,4,0,0)$. Using a realization of $\mathcal{P}_1$ that is stable and a realization of $\mathcal{P}_2$ that is stable, $\mathcal{A}$ allows refined inertia $(0,9,0,0)$. By Observation 4.2 there is a realization $P_1 \in Q(\mathcal{P}_1)$ that has refined inertia $(0,3,0,2)$. Using a realization of $\mathcal{P}_2$ that is stable, $\mathcal{A}$ allows refined inertia $(0,7,0,2)$. Finally if $\widetilde{\mathcal{P}}_2$ is obtained from $\mathcal{P}_2$ by replacing the diagonal entries with zero, then the refined inertia of any realization $\widetilde{P}_2 \in Q(\widetilde{\mathcal{P}}_2)$ is $(2,2,0,0)$, since all eigenvalues of $\widetilde{P}_2$ are real, nonzero and (from the characteristic polynomial) $-\alpha$ is an eigenvalue if and only if α is an eigenvalue. Thus there exists an $\epsilon > 0$ sufficiently small so that $\widetilde{P}_2 - \epsilon I$ has refined inertia $(2,2,0,0)$. Using this realization of $\mathcal{P}_2$ and a realization of $\mathcal{P}_1$ that is stable gives a realization of $\mathcal{A}$ with refined inertia $(2,7,0,0)$. Therefore, $\mathcal{A}$ allows $\mathbb{H}_9$. Finally, using the realizations P_1 and $\widetilde{P}_2 - \epsilon I$ above gives a realization of $\mathcal{A}$ that has refined inertia $(2,5,0,2)$ and so $\mathcal{A}$ does not require $\mathbb{H}_9$.

5. Concluding remarks. Each path sign pattern with $n = 3$ (listed in [1, Appendix]) and $n = 4$ (listed in Section 2.1 above) that requires $\mathbb{H}_n$ has a zero in the $(1,1)$ entry, the (n,n) entry or both, i.e., at at least one leaf in its digraph. By

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Corollary 3.5, each star sign pattern of order $n \geq 3$ that requires $\mathbb{H}_n$ has a zero at a unique leaf vertex in its digraph, but the question of whether or not a path sign pattern $\mathcal{P}$ of order $n \geq 5$ that requires $\mathbb{H}_n$ must have a zero at a leaf vertex in $D(\mathcal{P})$ remains open. The question also remains open as to whether or not this is true for every tree sign pattern $\mathcal{A}$ with order $n \geq 5$ that is potentially stable, sign nonsingular and not sign stable.

Necessary and sufficient conditions for a tree sign pattern to require $\mathbb{H}_3$ are given by [1, Theorem 2.1] and to require $\mathbb{H}_4$ by Theorem 2.9. The requires problem for $\mathbb{H}_n$ with $n \geq 5$ remains open.

6. Appendix. In addition to $\mathcal{P}_1, \ldots, \mathcal{P}_5$ and $\mathcal{S}_1, \ldots, \mathcal{S}_5$, up to equivalence there are eleven 4×4 tree sign patterns from [6] and [8] that are sign nonsingular, potentially stable and not sign stable. We now list these sign patterns and show below that they allow refined inertia $(4, 0, 0, 0)$, and thus do not require $\mathbb{H}_4$.

Let

$$\mathcal{P}_6 = \begin{bmatrix} 0 & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & - \end{bmatrix}, \ \mathcal{P}_7 = \begin{bmatrix} 0 & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & - \end{bmatrix}, \ \mathcal{P}_8 = \begin{bmatrix} + & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & 0 \end{bmatrix},$$

$$\mathcal{P}_9 = \begin{bmatrix} + & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & 0 \end{bmatrix}, \ \mathcal{P}_{10} = \begin{bmatrix} 0 & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & 0 \end{bmatrix}, \ \mathcal{P}_{11} = \begin{bmatrix} 0 & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & + & + \\ 0 & 0 & + & 0 \end{bmatrix},$$

$$\mathcal{S}_6 = \begin{bmatrix} - & + & + & + \\ - & - & 0 & 0 \\ - & 0 & + & 0 \\ + & 0 & 0 & 0 \end{bmatrix}, \ \mathcal{S}_7 = \begin{bmatrix} - & + & + & + \\ + & + & 0 & 0 \\ - & 0 & + & 0 \\ - & 0 & 0 & 0 \end{bmatrix}, \ \mathcal{S}_8 = \begin{bmatrix} 0 & + & + & + \\ + & 0 & 0 & 0 \\ - & 0 & + & 0 \\ - & 0 & 0 & - \end{bmatrix},$$

$$\mathcal{S}_9 = \begin{bmatrix} + & + & + & + \\ - & 0 & 0 & 0 \\ + & 0 & - & 0 \\ - & 0 & 0 & - \end{bmatrix}, \ \mathcal{S}_{10} = \begin{bmatrix} + & + & + & + \\ - & + & 0 & 0 \\ + & 0 & 0 & 0 \\ - & 0 & 0 & - \end{bmatrix}.$$

Each of these sign patterns is equivalent to the negative of one of these sign patterns, as the following table specifies.

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Sign pattern	Negative is equivalent to
$\mathcal{P}_6$	$\mathcal{P}_8$
$\mathcal{P}_7$	$\mathcal{P}_9$
$\mathcal{P}_{10}$	$\mathcal{P}_{10}$
$\mathcal{P}_{11}$	$\mathcal{P}_{11}$
$\mathcal{S}_6$	$\mathcal{S}_{10}$
$\mathcal{S}_7$	$\mathcal{S}_9$
$\mathcal{S}_8$	$\mathcal{S}_8$

Considering for example $\mathcal{S}_6$ and $\mathcal{S}_{10}$, if all entries in $\mathcal{S}_6$ are negated, then a sign pattern that is equivalent to $\mathcal{S}_{10}$ is obtained and vice versa. Since these two sign patterns are potentially stable [6], taking a stable realization of $\mathcal{S}_6$ and negating it gives a matrix that has four eigenvalues with positive real part and a sign pattern equivalent to $\mathcal{S}_{10}$. Therefore, $\mathcal{S}_{10}$ does not require $\mathbb{H}_4$. Similarly $\mathcal{S}_6$ does not require $\mathbb{H}_4$. By a similar argument, these 11 sign patterns all allow refined inertia $(4, 0, 0, 0)$ and hence do not require $\mathbb{H}_4$. However, it can be shown with numerical examples that each sign pattern allows $\mathbb{H}_4$.

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REFERENCES

[1] E. Bodine, L. Deaett, J.J. McDonald, D.D. Olesky, and P. van den Driessche. Sign patterns that require or allow particular refined inertias. *Linear Algebra Appl.*, 437:2228–2242, 2012.

[2] T. Britz, J.J. McDonald, D.D. Olesky, and P. van den Driessche. Minimal spectrally arbitrary sign-patterns. *SIAM J. Matrix Anal. Appl.*, 26:257–271, 2004.

[3] R.A. Brualdi and B.L. Shader. *Matrices of Sign Solvable Linear Systems.* Cambridge University Press, Cambridge, 1995.

[4] M.S. Cavers, C. Garnett, I.-J. Kim, D.D. Olesky, P. van den Driessche, and K.N. Vander Meulen. Techniques for identifying inertially arbitrary patterns. *Electron. J. Linear Algebra*, 26:71–89, 2013.

[5] Y. Gao and J. Li. On the potential stability of star sign pattern matrices. *Linear Algebra Appl.*, 327:61–68, 2001.

[6] C.R. Johnson and T.A. Summers. The potentially stable tree sign patterns for dimensions less than five. *Linear Algebra Appl.*, 126:1–13, 1989.

[7] I.-J. Kim, D.D. Olesky, B.L. Shader, P. van den Driessche, H. van der Holst, and K.N. Vander Meulen. Generating potentially nilpotent full sign patterns. *Electron. J. Linear Algebra*, 18:162–175, 2009.

[8] Q. Lin, D.D. Olesky, and P. van den Driessche. The distance of potentially stable sign patterns to the unstable matrices. *SIAM J. Matrix Anal. Appl.*, 24:356–367, 2002.

[9] M. Tsatsomeros. Matrix Equalities and Inequalities. In: L. Hogben (editor), *Handbook of Linear Algebra*, Chapman & Hall/CRC, New York, 2007.