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ON THE LEAST SIGNLESS LAPLACIAN EIGENVALUE OF SOME GRAPHS*

GUANGLONG YU†, SHUGUANG GUO†, AND MEILING XU‡

Abstract. For a graph, the least signless Laplacian eigenvalue is the least eigenvalue of its signless Laplacian matrix. This paper investigates how the least signless Laplacian eigenvalue of a graph changes under some perturbations, and minimizes the least signless Laplacian eigenvalue among all the nonbipartite graphs with given matching number and edge cover number, respectively.

Key words. Least eigenvalue, Signless Laplacian, Perturbation, Matching number, Edge cover number.

AMS subject classifications. 05C50.

1. Introduction. All graphs considered in this paper are connected, undirected and simple, i.e., no loops or multiple edges are allowed. We use standard terminology and notation. We denote by $G = [V(G),\ E(G)]$ a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$, where $|V(G)| = n$ is the order and $|E(G)| = m$ is the size of G. Recall that $Q(G) = D(G) + A(G)$ is called the *signless Laplacian matrix* of G, where $D(G) = \mathrm{diag}(d_1, d_2, \dots, d_n)$ with $d_i = d_G(v_i)$ being the degree of vertex v_i of G $(1 \le i \le n)$, and $A(G)$ is the adjacency matrix of G. The least eigenvalue of $Q(G)$, denote by $q_{min}(G)$, is called the *least signless Laplacian eigenvalue* of G. Noting that $Q(G)$ is positive semi-definite, we have $q_{min}(G) \ge 0$.

The signless Laplacian matrix has received a lot of attention in recent years, especially after D. Cvetković et al. put forward the study of this matrix in [2–5]. From [5], we know that, for a connected graph G, $q_{min}(G) = 0$ if and only if G is bipartite. Consequently, in [8], the least signless Laplacian eigenvalue was studied as a measure of nonbipartiteness of a graph. One can note that there are quite a few results about the least signless Laplacian eigenvalue. In [1], D.M. Cardoso et al. determined the graphs with the minimum least signless Laplacian eigenvalue among all the connected nonbipartite graphs with a prescribed number of vertices. In [7],

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†Department of Mathematics, Yancheng Teachers University, Yancheng, 224002, Jiangsu, China (yglong01@163.com, yctusgguo@gmail.com). Supported by NSFC (11271315, 11171290, 11201417) and BK2012245.

‡Department of Mathematics, Jiangsu Normal University, Xuzhou, 221116, Jiangsu, China. Supported by NSFC (11171290).

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L. de Lima et al. surveyed some known results about q_{min} and also proved some new ones. In [9], S. Fallat and Y. Fan investigated the relations between the least signless Laplacian eigenvalue and some parameters reflecting the graph bipartiteness. In [10], Y. Wang and Y. Fan investigated the least signless Laplacian eigenvalue of a graph under some perturbations, and minimized the least eigenvalue of the signless Laplacian among the class of connected graphs with fixed order which contains a given nonbipartite graph as an induced subgraph.

In this paper, we investigate how the least signless Laplacian eigenvalue of a graph changes under some perturbations, and consider the relation between it and the matching number, and the relation between it and the edge cover number. We recall some notions as follows:

- A *matching* of a graph G is a set of edges such that any two edges of the set are not incident. The *matching number* of G, denoted by $\alpha^{'}(G)$, is the maximum of the cardinalities of all matchings. A *maximal matching* of G is a matching of G with cardinality $\alpha^{'}(G)$.

- An *edge cover* of a graph G is a set of edges such that each vertex of G is incident with at least one edge of the set. The *edge cover number* of G, denoted by $\beta^{'}(G)$, is the minimum of the cardinalities of all edge covers.

It is known that for a connected graph G of order n, $\alpha^{'}(G)+\beta^{'}(G)=n$. With the results about the least signless Laplacian eigenvalue of a graph under some perturbations shown in this paper, we determine the graphs which have the minimum least signless Laplacian eigenvalue among the nonbipartite graphs with both given order, and given matching number or edge cover number, respectively.

2. Perturbation. We first introduce some notation. We denote by C_n and P_n the cycle and the path of order n respectively. In a graph G, we let $N_G(u)$ denote the neighbor set of a vertex u. The *distance* between two vertices u and v in a graph G, denoted by $d_G(u,v)$, is the length of one of the shortest paths from u to v. Let $G-uv$ denote the graph that arises from G by deleting the edge $uv \in E(G)$. Similarly, $G+uv$ is the graph that arises from G by adding an edge uv between its two nonadjacent vertices u and v. A *pendant vertex* is a vertex of degree 1. A vertex is called a *pendant neighbor* if it is adjacent to a pendant vertex. A connected graph G of order n is called a *unicyclic* graph if $|E(G)|=n$. The *union* of two simple graphs H and G is the simple graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

Let G_1 and G_2 be two disjoint graphs, and let $v_1 \in V(G_1)$, $v_2 \in V(G_2)$. The *coalescence* of G_1 and G_2, denoted by $G_1(v_1) \diamond G_2(v_2)$, is obtained from G_1, G_2 by identifying v_1 with v_2 and forming a new vertex u (see [10] for detail). The graph $G_1(v_1) \diamond G_2(v_2)$ is also written as $G_1(u) \diamond G_2(u)$. If a connected graph G can be

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expressed in the form $G = G_1(u) \diamond G_2(u)$, where G_1 and G_2 are both nontrivial and connected, then for $i = 1, 2$, G_i is called a *branch* of G with root u.

Let G be a graph of order n, and let $X = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$. Then X can be considered as a function defined on $V(G)$, that is, each vertex v_i is mapped to $x_i = x(v_i)$. One can find that

$$X^T Q(G) X = \sum_{uv \in E(G)} [x(u) + x(v)]^2.$$

In addition, for an arbitrary unit vector $X \in \mathbb{R}^n$, $q_{min}(G) \leq X^T Q(G) X$, with equality if and only if X is an eigenvector corresponding to $q_{min}(G)$. A branch H of G is called a *zero branch* with respect to X if $x(v) = 0$ for all $v \in V(H)$; otherwise, it is called a *nonzero branch* with respect to X.

LEMMA 2.1. [10] *Let G be a connected graph which contains a bipartite branch H with root u. Let X be an eigenvector of G corresponding to $q_{min}(G)$.*

(i) *If $x(u) = 0$, then H is a zero branch of G with respect to X;*

(ii) *If $x(u) \neq 0$, then $x(p) \neq 0$ for every vertex $p \in V(H)$. Furthermore, for every vertex $p \in V(H)$, $x(p)x(u)$ is either positive or negative, depending on whether p is or is not in the same part of the bipartite graph H as u; consequently, $x(p)x(q) < 0$ for each edge $pq \in E(H)$.*

LEMMA 2.2. [10] *Let G be a connected nonbipartite graph of order n, and let X be an eigenvector of G corresponding to $q_{min}(G)$. Let T be a tree, which is a nonzero branch of G with respect to X and with root u. Then $|x(q)| < |x(p)|$ whenever p, q are vertices of T such that q lies on the unique path from u to p.*

LEMMA 2.3. [10] *Let $G = G_1(v_2) \diamond G_2(u)$ and $G^* = G_1(v_1) \diamond G_2(u)$ be two graphs of order n, where G_1 is a connected graph containing two distinct vertices v_1, v_2, and G_2 is a connected bipartite graph containing a vertex u. If there exists an eigenvector $X = (\, x(v_1),\, x(v_2),\, \ldots,\, x(v_k),\, \ldots)^T$ of G corresponding to $q_{min}(G)$ such that $|x(v_1)| \geq |x(v_2)|$, then $q_{min}(G^*) \leq q_{min}(G)$, with equality only if $|x(v_1)| = |x(v_2)|$ and $d_{G_2}(u)x(u) = -\sum_{v \in N_{G_2}(u)} x(v)$, where $d_{G_2}(u) = |N_{G_2}(u)|$.*

LEMMA 2.4. *Let $G = G_1(v_2) \diamond T(u)$ and $G^* = G_1(v_1) \diamond T(u)$, where G_1 is a connected nonbipartite graph containing two distinct vertices v_1, v_2, and T is a nontrivial tree. If there exists an eigenvector $X = (\, x(v_1),\, x(v_2),\, \ldots,\, x(v_k),\, \ldots)^T$ of G corresponding to $q_{min}(G)$ such that $|x(v_1)| > |x(v_2)|$ or $|x(v_1)| = |x(v_2)| > 0$, then $q_{min}(G^*) < q_{min}(G)$.*

Proof. In G, we denote by v_2 the new vertex obtained by identifying v_2 and u. In G^*, we denote by v_1 the new vertex obtained by identifying v_1 and u. Suppose an eigenvector $X = (\, x(v_1),\, x(v_2),\, \ldots,\, x(v_k),\, \ldots)^T$ of G corresponding to $q_{min}(G)$

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satisfies that $|x(v_1)| > |x(v_2)|$ or $|x(v_1)| = |x(v_2)| > 0$. Next we prove $q_{min}(G^*) < q_{min}(G)$.

Note that $|x(v_1)| > 0$. For convenience, we suppose that $x(v_1) > 0$. Let the vector $Z = (z(v_1), z(v_2), \ldots, z(v_k), \ldots)^T \in \mathbb{R}^n$ defined on $V(G^*)$ satisfy that

$$z(w) = \begin{cases} (-1)^{d_T(u,w)}(|x(w)| + x(v_1) - |x(v_2)|), & w \in V(T-u); \\ x(w), & \text{otherwise.} \end{cases}$$

If $x(v_2) = 0$, then in G, T is a zero branch with respect to the eigenvector X. Noting the supposition that $x(v_1) > 0$, we get that $|z(w)| > 0$ for each $w \in V(T-u)$. Then $Z^TZ > X^TX$ now. Note that $Z^TQ(G^*)Z = X^TQ(G)X$ and $\frac{Z^TQ(G^*)Z}{Z^TZ} \geq q_{min}(G^*) > 0$. Then

$$q_{min}(G^*) \leq \frac{Z^TQ(G^*)Z}{Z^TZ} < \frac{X^TQ(G)X}{X^TX} = q_{min}(G).$$

If $x(v_2) \neq 0$, by Lemma 2.1, then in G, $x_px_q < 0$ for each edge $pq \in E(T)$. Noting that in G^*, for each edge $pq \in E(T)$, $z_pz_q < 0$ and $(z_p + z_q)^2 = (|x(p)| - |x(q)|)^2 = (x(p) + x(q))^2$, we get that $Z^TQ(G^*)Z = X^TQ(G)X$. Note that

$$\begin{cases} |z(w)| \geq |x(w)|, & w \in V(T-u); \\ |z(w)| = |x(w)|, & \text{otherwise.} \end{cases}$$

Then $Z^TZ \geq X^TX$. As a result,

$$q_{min}(G^*) \leq \frac{Z^TQ(G^*)Z}{Z^TZ} \leq \frac{X^TQ(G)X}{X^TX} = q_{min}(G).$$

Note that if $q_{min}(G^*) = q_{min}(G)$, then Z is an eigenvector corresponding to $q_{min}(G^*)$, and then $x(v_1) = z(v_1) > 0$. By Lemma 2.2, in G^*, for each $w \in N_T(u)$, we have $z(v_1) + z(w) < 0$. Then we have

$$\begin{aligned} q_{min}(G^*)z(v_1) &= d_G(v_1)z(v_1) + \sum_{w \in N_G(v_1)} z(w) + d_T(u)z(v_1) + \sum_{w \in N_T(u)} z(w) \\ &= q_{min}(G)z(v_1) + \sum_{w \in N_T(u)} (z(v_1) + z(w)) \\ &< q_{min}(G)z(v_1). \end{aligned}$$

This means that $q_{min}(G^*) \neq q_{min}(G)$. Consequently, $q_{min}(G^*) < q_{min}(G)$. This completes the proof. $\square$

Remark 2.5. In [10], Y. Wang and Y.Z. Fan proved that Lemma 2.4 holds for $G = G_1(v_2) \diamond S(u)$, where $S(u)$ is a nontrivial star with center u, and holds for

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$G = G_1(v_2) \diamond P(u)$, where P is a nontrivial path with an end vertex u. Hence, Lemma 2.4 is a generalization of their results.

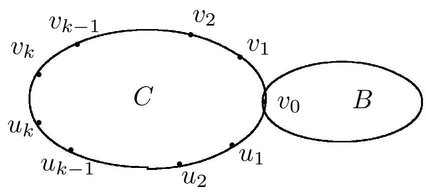

Fig. 2.1. G.

LEMMA 2.6. *Let $G = C(v_0) \diamond B(v_0)$ be a graph of order n, where $C = v_0v_1v_2 \cdots v_ku_ku_{k-1} \cdots u_1v_0$ is a cycle of length $2k+1$, and B is a bipartite graph of order $n-2k$ (see* Fig. 2.1*). Then there exists an eigenvector $X = (\,x(v_0),\, x(v_1),\, x(v_2),\, \ldots,\, x(v_k),\, x(u_1),\, x(u_2),\, \ldots,\, x(u_k),\, \ldots)^T$ corresponding to $q_{min}(G)$ satisfying the following:*

(i) $|x(v_0)| = \max\{|x(w)| \,|\, w \in V(C)\} > 0$;

(ii) $x(v_i) = x(u_i)$ *for* $i = 1, 2, \ldots, k$;

(iii) $x(v_i)x(v_{i-1}) \leq 0$ *and* $x(u_i)x(u_{i-1}) \leq 0$ *for* $i = 1, 2, \ldots, k$.

Moreover, if $2k + 1 < n$, then the multiplicity of $q_{min}(G)$ is one, and then any eigenvector corresponding to $q_{min}(G)$ satisfies (i), (ii), (iii).

Proof. Suppose $Y = (y(v_0),\, y(v_1),\, \ldots,\, y(v_k),\, y(u_1),\, \ldots,\, y(u_k),\, \ldots)^T \in \mathbb{R}^n$ is an eigenvector corresponding to $q_{min}(G)$. Then Y is a nonzero vector.

If $|V(B)| = 1$, then $2k + 1 = n$. Without loss of generality, we may assume that $|y(v_0)| = \max\{|y(w)| \,|\, w \in V(C)\}$. Note that if $|y(v_0)| = 0$, then Y is a zero vector, which contradicts that Y is nonzero. This means that $|y(v_0)| > 0$.

If $|V(B)| > 1$, then $2k+1 < n$. We claim that $|y(v_0)| = \max\{|y(w)| \,|\, w \in V(C)\}$. Otherwise, suppose that there exists some i $(1 \leq i \leq k)$ such that $|y(v_i)| > |y(v_0)|$. Let

$$G^{'} = G - \sum_{w \in N_B(v_0)} v_0w + \sum_{w \in N_B(v_0)} v_iw.$$

By Lemma 2.3, then $q_{min}(G^{'}) < q_{min}(G)$, which is a contradiction because $G^{'} \cong G$. Then our claim holds. Note that if $y(v_0) = 0$, then by Lemma 2.1, B is a zero branch with respect to Y. Simultaneously, if $y(v_0) = 0$, then $y(w) = 0$ for every $w \in V(C)$. Therefore, $Y = 0$, which contradicts that Y is nonzero. This means that $|y(v_0)| > 0$. Moreover, we conclude that if $2k + 1 < n$, any eigenvector corresponding to $q_{min}(G)$ satisfies (i).

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Let the vector $Y' = (y^{'}(v_0),\ y^{'}(v_1),\ \ldots,\ y^{'}(v_k),\ y^{'}(u_1),\ \ldots,\ y^{'}(u_k),\ \ldots)^T \in \mathbb{R}^n$ defined on $V(G)$ satisfy that

$$y^{'}(w) = \begin{cases} y(u_i), & w = v_i \ for\ i = 1,\ 2,\ \ldots,\ k; \\ y(v_i), & w = u_i \ for\ i = 1,\ 2,\ \ldots,\ k; \\ y(w), & \text{otherwise.} \end{cases}$$

Then

$$\frac{Y^{'T} Q(G) Y^{'}}{Y^{'T} Y^{'}} = \frac{Y^T Q(G) Y}{Y^T Y} = q_{min}(G).$$

Hence Y' is also an eigenvector of G corresponding to $q_{min}(G)$. Let $Z = (z(v_0), z(v_1), \ldots, z(v_k), z(u_1), \ldots, z(u_k), \ldots)^T = Y + Y'$. Then $Z \neq 0$, and Z is also an eigenvector of G corresponding to $q_{min}(G)$ which satisfies both (i) and (ii).

Let the vector $X = (x(v_0),\ x(v_1),\ \ldots,\ x(v_k),\ x(u_1),\ \ldots,\ x(u_k),\ \ldots)^T$ defined on $V(G)$ satisfy that $x(w) = (-1)^{d_G(v_0,w)}\text{sign}(z(v_0))|z(w)|$. Note that $|x(w)| = |z(w)|$ for any vertex $w \in V(G)$. Then $X^T X = Z^T Z$. Noting that $\text{sign}x(\omega) = -\text{sign}x(\gamma)$ for each edge $\omega\gamma \in (E(G) \setminus \{u_k v_k\})$ and $(x(u_k) + x(v_k))^2 = (z(u_k) + z(v_k))^2$, we have

$$(x(\omega) + x(\gamma))^2 \leq (z(\omega) + z(\gamma))^2$$

for each edge $\omega\gamma \in E(G)$. Consequently, we get that $X^T Q(G) X \leq Z^T Q(G) Z$, and

$$q_{min}(G) \leq \frac{X^T Q(G) X}{X^T X} \leq \frac{Z^T Q(G) Z}{Z^T Z} = q_{min}(G).$$

As a result, X is also an eigenvector of G corresponding to $q_{min}(G)$, which satisfies (i), (ii) and (iii).

Assume that the multiplicity of $q_{min}(G)$ is greater than one. Let $F = (f(v_0), f(v_1), \ldots, f(v_k), f(u_1), \ldots, f(u_k))^T$ be an eigenvector of G corresponding to $q_{min}(G)$ which is orthogonal to X. If $2k+1 < n$, then $|f(v_0)| > 0$. Suppose that $f(v_0) = \varepsilon x(v_0)$. Then $\Gamma = F - \varepsilon X = (\tau(v_0),\ \tau(v_1),\ \ldots,\ \tau(v_k),\ \tau(u_1),\ \ldots,\ \tau(u_k))^T$ is also an eigenvector of G corresponding to $q_{min}(G)$, but $\tau(v_0) = 0$, which contradicts our above conclusion that any eigenvector of G corresponding to $q_{min}(G)$ satisfies (i) if $2k+1 < n$. Now, we conclude that if $2k+1 < n$, then for any eigenvector F of G corresponding to $q_{min}(G)$, there exists a real number $k \neq 0$ such that $F = kX$. This means that the multiplicity of $q_{min}(G)$ is one if $2k+1 < n$, and means that any eigenvector corresponding to $q_{min}(G)$ satisfies (i), (ii), (iii). The proof is completed. ∎

LEMMA 2.7. *Suppose that v is a pedant vertex and uv is a pendant edge in a graph G. Then there must be a maximal matching of G containing the edge uv.*

Proof. Suppose that M_1 is a maximal matching of G. The lemma is trivial if $uv \in M_1$. If $uv \notin M_1$, then in M_1, there must be an edge incident with u. Otherwise,

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let $M_2 = M_1 \cup \{uv\}$. Then $|M_2| = |M_1| + 1$ and M_2 is also a matching of G, which contradicts that M_1 is maximal. Suppose $uz \in M_1$, where $uz \neq uv$. Let $M = (M_1 \backslash \{uz\}) \cup \{uv\}$. Then M is also a maximal matching of G, which contains the edge uv. The result follows. □

Let $k \geq 3$ be odd. Let $C^*_{k,\,l}$ be the graph of order n obtained by attaching a cycle C_k to an end vertex of a path P_{l+1} and attaching $n-k-l$ pendant edges to the other end vertex of the path P_{l+1} (see Fig. 2.2). In particular, $l = 0$ means attaching $n-k$ pendant edges to a vertex of C_k.

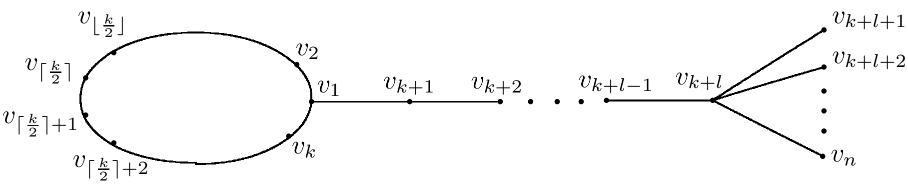

Fig. 2.2. $C^*_{k,\,l}$.

LEMMA 2.8. *Let $3 \leq k \leq n-2$ be odd, and let both $C^*_{k,\,l}$ and $C^*_{k,\,l+1}$ be of order n. Then we have*

(i) $\alpha^{'}(C^*_{k,\,l}) \leq \alpha^{'}(C^*_{k,\,l+1})$;

(ii) $q_{min}(C^*_{k,\,l+1}) < q_{min}(C^*_{k,\,l})$.

Proof. Let the vertices of $C^*_{k,\,l}$ be indexed as in Fig. 2.2. Note that

$$C^*_{k,\,l+1} = C^*_{k,\,l} - \sum_{i=k+l+2}^{n} v_i v_{k+l} + \sum_{i=k+l+2}^{n} v_i v_{k+l+1}.$$

By Lemma 2.7, we know that there exists a maximal matching M_1 of $C^*_{k,\,l}$ which contains $v_{k+l}v_{k+l+1}$. Note that M_1 is also a matching of $C^*_{k,\,l+1}$. Hence, $\alpha^{'}(C^*_{k,\,l}) \leq \alpha^{'}(C^*_{k,\,l+1})$. Then (i) follows.

Let $Y = (y(v_1), y(v_2), \ldots, y(v_k), y(v_{k+1}), y(v_{k+2}), \ldots, y(v_n))^T$ be an eigenvector corresponding to $q_{min}(C^*_{k,\,l})$. By Lemma 2.6, we know that $|y(v_1)| > 0$. Combining with Lemma 2.2, we have $0 < |y(v_1)| < |y(v_{k+l})| < |y(v_{k+l+1})|$. By Lemma 2.4, we get that $q_{min}(C^*_{k,\,l+1}) < q_{min}(C^*_{k,\,l})$. Then (ii) follows. □

Let $k \geq 3$ be odd, and let $\mathcal{C} = v_1 v_2 \cdots v_k v_1$ be a cycle of length k. For $j = 1, 2, \ldots, t$, each T_j is a nontrivial tree. Let $\mathcal{C}_k^{(T_1,T_2,\ldots,T_t;i_1,i_2,\ldots,i_t)}$ denote the graph obtained by identifying the vertex u_j of T_j and the vertex v_{i_j} of $\mathcal{C}$, where $1 \leq j \leq t$ and for $l \neq j$, $i_l = i_j$ possibly. Here, in $\mathcal{C}_k^{(T_1,T_2,\ldots,T_t;i_1,i_2,\ldots,i_t)}$, for any $1 \leq j \leq t$, we denote by v_{i_j} the new vertex obtained by identifying v_{i_j} and u_j. Let $\mathcal{C}_{(k,n)}^{T_1,T_2,\ldots,T_t} = \{\mathcal{C}_k^{(T_1,T_2,\ldots,T_t;i_1,i_2,\ldots,i_t)} | \, \mathcal{C}_k^{(T_1,T_2,\ldots,T_t;i_1,i_2,\ldots,i_t)}$ be of order $n\}$ and let $\mathcal{C}_k^{(T_1,T_2,\ldots,T_t;1)} =$

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$\mathcal{C}_k^{(T_1,T_2,\ldots,T_t;1,1,\ldots,1)}$. For understanding easily, we show three examples in Fig. 2.3.

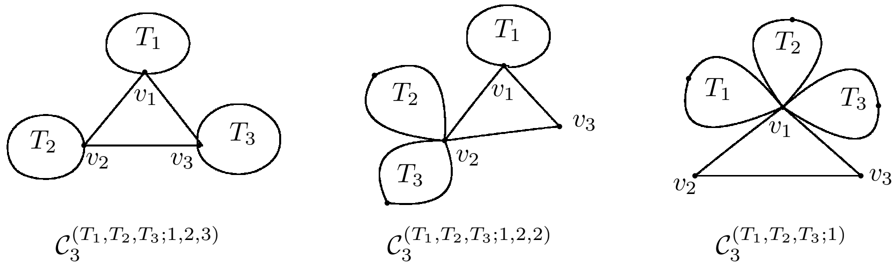

Fig. 2.3. $\mathcal{C}_3^{(T_1,T_2,T_3;1,2,3)}$, $\mathcal{C}_3^{(T_1,T_2,T_3;1,2,2)}$, and $\mathcal{C}_3^{(T_1,T_2,T_3;1)}$.

LEMMA 2.9. *Let $3 \le k < n$ be odd. Then for any $U \in \mathcal{C}_{(k,n)}^{T_1,T_2,\ldots,T_t}$, we have*

(i) $\alpha^{'}(\mathcal{C}_k^{(T_1,T_2,\ldots,T_t;1)}) \le \alpha^{'}(U)$;

(ii) $q_{min}(\mathcal{C}_k^{(T_1,T_2,\ldots,T_t;1)}) \le q_{min}(U)$, *with equality if and only if* $\mathcal{C}_k^{(T_1,T_2,\ldots,T_t;1)}$.

Proof. Let $U^{'} \in \mathcal{C}_{(k,n)}^{T_1,T_2,\ldots,T_t}$, and let the vertices of $U^{'}$ be indexed by v_1, v_2, $\ldots$, v_k, v_{k+1}, v_{k+2}, $\ldots$, v_n. Suppose that $q_{min}(U^{'}) = \min\{q_{min}(U) | \, U \in \mathcal{C}_{(k,n)}^{T_1,T_2,\ldots,T_t}\}$. Let $Y = (y(v_1), y(v_2), \ldots, y(v_k), y(v_{k+1}), y(v_{k+2}), \ldots, y(v_n))^T$ be an eigenvector corresponding to $q_{min}(U^{'})$. Then Y is nonzero. Suppose that $|y(v_1)| = \max\{|y(w)| \, | \, w \in V(\mathcal{C})\}$. Note that if $|y(v_1)| = 0$, by Lemma 2.1, then for $j = 1, 2, \ldots, t$, each T_j is a zero branch, and then Y is a zero vector, which contradicts that Y is nonzero. As a result, it follows that $|y(v_1)| > 0$.

If $U^{'} \ncong \mathcal{C}_k^{(T_1,T_2,\ldots,T_t;1)}$, then there exists a nontrivial tree T_j attaching to v_{i_j} ($v_{i_j} \neq v_1$). Let

$$U^* = U^{'} - \sum_{w \in N_{T_j}(u_j)} u_j w + \sum_{w \in N_{T_j}(u_j)} v_1 w.$$

By Lemma 2.4, we have $q_{min}(U^*) < q_{min}(U^{'})$. This contradicts that $q_{min}(U^{'})$ is minimal. Consequently, if $q_{min}(U^{'})$ is minimal, then $U^{'} \cong \mathcal{C}_k^{(T_1,T_2,\ldots,T_t;1)}$. Then (ii) follows.

Let M be a maximal matching of $\mathcal{C}_k^{(T_1,T_2,\ldots,T_t;1)}$, and let $\mathcal{T} = T_1 \cup T_2 \cup \cdots \cup T_t$ here. Suppose that there is a vertex $v_\alpha \in N_{\mathcal{T}}(v_1)$ such that $v_1 v_\alpha \in M$. Then $v_\alpha \in V(T_j)$ for some j ($1 \le j \le t$). For convenience, we assume that $j = 1$, that is, $v_\alpha \in V(T_1)$. Let $\mathcal{T}' = T_2 \cup \cdots \cup T_t$. Then there is no vertex $w \in V(\mathcal{T}')$ such that $v_1 w \in M$. Note that for any graph $U \in \mathcal{C}_{(k,n)}^{T_1,T_2,\ldots,T_t}$, U is either isomorphic to $\mathcal{C}_k^{(T_1,T_2,\ldots,T_t;1)}$, or isomorphic to a graph $\mathcal{C}_k^{(T_1,T_2,\ldots,T_t;i_1,i_2,\ldots,i_t)}$. In the graph $\mathcal{C}_k^{(T_1,T_2,\ldots,T_t;i_1,i_2,\ldots,i_t)}$, for convenience, we let $v_{i_1} = v_1$. Then $\mathcal{C}_k^{(T_1,T_2,\ldots,T_t;i_1,i_2,\ldots,i_t)} = \mathcal{C}_k^{(T_1,T_2,\ldots,T_t;1,i_2,\ldots,i_t)}$. Note that M is

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also a matching of the graph $\mathcal{C}_k^{(T_1,T_2,\ldots,T_t;1,i_2,\ldots,i_t)}$. Hence, if U is isomorphic to the graph $\mathcal{C}_k^{(T_1,T_2,\ldots,T_t;1,i_2,\ldots,i_t)}$, then $\alpha'(\mathcal{C}_k^{(T_1,T_2,\ldots,T_t;1)}) \leq \alpha'(U)$. If $U \cong \mathcal{C}_k^{(T_1,T_2,\ldots,T_t;1)}$, then $\alpha'(\mathcal{C}_k^{(T_1,T_2,\ldots,T_t;1)}) \leq \alpha'(U)$ naturally.

In the same way, we can prove that if there is no vertex $v \in N_{\mathcal{T}}(v_1)$ such that $v_1v \in M$, then $\alpha'(\mathcal{C}_k^{(T_1,T_2,\ldots,T_t;1)}) \leq \alpha'(U)$. And Then (i) follows. This completes the proof. ◻

LEMMA 2.10. *Let $3 \leq k < n$ be odd, and let $U_0 = \mathcal{C}_k^{(T_1;1)}$ be of order n. If there exist at least 2 pendant neighbors in U_0, then there exists l such that*

(i) $\alpha'(C^*_{k,\,l}) \leq \alpha'(U_0)$;

(ii) $q_{min}(C^*_{k,\,l}) < q_{min}(U_0)$.

Proof. By Lemma 2.6, we know that there exists an eigenvector $X = (x(v_1), x(v_2), \ldots, x(v_k), \ldots)^T$ corresponding to $q_{min}(U_0)$, such that $|x(v_1)| = \max\{|x(w)| \mid w \in V(\mathcal{C})\} > 0$. Then T_1 is a nonzero branch by Lemma 2.1. Let $|x(v_c)| = \max\{|x(w)| \mid w \in V(T_1),\ w \text{ is not a pendant vertex}\}$. By Lemma 2.2, we know that v_c is a pendant neighbor. We suppose that v_b is another pendant neighbor different from v_c, and suppose that $v_{i_1}, \ldots, v_{i_t}$ are all the pendant vertices adjacent to v_b. Let

$$U_1 = U_0 - \sum_{j=1}^{t} v_b v_{i_j} + \sum_{j=1}^{t} v_c v_{i_j}.$$

By Lemma 2.4, we have $q_{min}(U_1) < q_{min}(U_0)$. Note that the number of pendant neighbors of U_1 is less than that of U_0. Proceeding like this, from U_0, we can get a nonbipartite unicyclic graph $\mathcal{K}$ which has only one pendant neighbor, such that $q_{min}(\mathcal{K}) < q_{min}(U_0)$. Here, $\mathcal{K}$ is isomorphic to a $C^*_{k,\,l}$ for some l.

For convenience, we let $\mathcal{K} = C^*_{k,\,l}$ and let the vertices of $\mathcal{K}$ be indexed as in Fig. 2.2. From the above proof, we see that in the original graph U_0, v_{k+l} is not a pendant vertex, and at least one of $v_{k+l+1}, v_{k+l+2}, \ldots, v_n$, say v_n for convenience, is adjacent to v_{k+l}. Then $\mathscr{S} = \mathcal{K} - \sum_{i=k+l+1}^{n-1} v_i$ is a subgraph of U_0. By Lemma 2.7, we know that there is a maximal matching M of $\mathcal{K}$ containing $v_{k+l}v_n$. In fact, M is also a maximal matching of $\mathscr{S}$. Consequently, $\alpha'(\mathcal{K}) = \alpha'(\mathscr{S})$. Note that $\mathscr{S}$ is a subgraph of U_0. Hence, $\alpha'(\mathscr{S}) \leq \alpha'(U_0)$, and then $\alpha'(\mathcal{K}) \leq \alpha'(U_0)$. This completes the proof. ◻

LEMMA 2.11. *Let $k \geq 3$ be odd and $t = l + \lfloor \frac{k}{2} \rfloor - 1$, and let both $C^*_{k,\,l}$ and $C^*_{3,\,t}$ be of order n. Then we have*

(i) $\alpha'(C^*_{3,\,t}) \leq \alpha'(C^*_{k,\,l})$;

(ii) $q_{min}(C^*_{3,\,t}) \leq q_{min}(C^*_{k,\,l})$, *with equality if and only if $k = 3$.*

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Proof. The lemma is trivial for $k = 3$. Next we suppose $k \geq 5$.

Let the vertices of $C^*_{k,\,l}$ be indexed as in Fig. 2.2, and we let $Y = (y(v_1), y(v_2), \ldots, y(v_k), y(v_{k+1}), y(v_{k+2}), \ldots, y(v_n))^T$ be an eigenvector corresponding to $q_{min}(C^*_{k,\,l})$ satisfying Lemma 2.6. Combining with Lemma 2.2, we have $y(v_1) \neq 0$, and for $1 \leq i \leq k$, we have $|y(v_i)| \leq |y(v_1)| \leq |y(v_{k+l})|$.

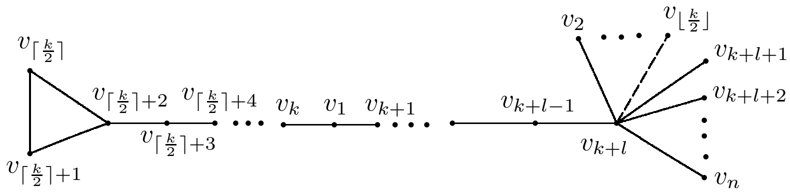

Fig. 2.4. $C^*_{3,\,t}$.

Let

$$C^*_{3,\,t} = C^*_{k,\,l} - \sum_{i=2}^{\lceil \frac{k}{2} \rceil} v_i v_{i-1} + \sum_{i=2}^{\lfloor \frac{k}{2} \rfloor} v_i v_{k+l} + v_{\lceil \frac{k}{2} \rceil} v_{\lceil \frac{k}{2} \rceil+2} \text{ (see Fig. 2.4)},$$

and let $Z = (z(v_1), z(v_2), \ldots, z(v_k), \ldots)^T \in \mathbb{R}^n$ defined on $V(C^*_{3,\,t})$ satisfy that

$$z(w) = \begin{cases} -\mathrm{sgn}(y(v_{k+l}))(|y(v_{k+l})| + |y(v_i) + y(v_{i-1})|), & w = v_i \ for \ i = 2,\ 3,\ \ldots,\ \lfloor \frac{k}{2} \rfloor; \\ y(w), & \text{otherwise.} \end{cases}$$

Note that

$$\begin{cases} |z(w)| \geq |y(w)|, & w = v_i \ for \ i = 2,\ 3,\ \ldots,\ \lfloor \frac{k}{2} \rfloor; \\ |z(w)| = |y(w)|, & \text{otherwise,} \end{cases}$$

and $y_{v_{\lfloor \frac{k}{2} \rfloor}} = y_{v_{\lceil \frac{k}{2} \rceil+2}}$, then $Z^T Z \geq Y^T Y$ and

$$Z^T Q(C^*_{3,\,t}) Z = Y^T Q(C^*_{k,\,l}) Y - (y_{v_{\lfloor \frac{k}{2} \rfloor}} + y_{v_{\lceil \frac{k}{2} \rceil}})^2 + (y_{v_{\lceil \frac{k}{2} \rceil}} + y_{v_{\lceil \frac{k}{2} \rceil+2}})^2 = Y^T Q(C^*_{k,\,l}) Y.$$

As a result, we get that

$$q_{min}(C^*_{3,\,t}) \leq \frac{Z^T Q(C^*_{3,\,t}) Z}{Z^T Z} \leq \frac{Y^T Q(C^*_{k,\,l}) Y}{Y^T Y} = q_{min}(C^*_{k,\,l}). \tag{2.1}$$

We claim that $q_{min}(C^*_{3,\,t}) < q_{min}(C^*_{k,\,l})$. Otherwise, suppose that $q_{min}(C^*_{3,\,t}) = q_{min}(C^*_{k,\,l})$. Then Z is an eigenvector corresponding to $q_{min}(C^*_{3,\,t})$. By Lemma 2.2, we have $|z(v_{k+l})| < |z(v_i)|$ for $i = 2, 3, \ldots, \lfloor \frac{k}{2} \rfloor$. By Lemmas 2.2 and 2.6, we know that in Y, $|y(v_i)| \leq |y(v_{k+l})|$ for $i = 2, 3, \ldots, \lfloor \frac{k}{2} \rfloor$. Note that $|y(v_{k+l})| = |z(v_{k+l})|$. Consequently, $|y(v_i)| < |z(v_i)|$ for $i = 2, 3, \ldots, \lfloor \frac{k}{2} \rfloor$. As a result, we get $Z^T Z > Y^T Y$.

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Since $q_{min}(C^*_{3,\,t}) \neq 0$, from inequality (2.1), we have $q_{min}(C^*_{3,\,t}) < q_{min}(C^*_{k,\,l})$, which contradicts that $q_{min}(C^*_{3,\,t}) = q_{min}(C^*_{k,\,l})$. So our claim holds.

We claim that there is a maximal matching of $C^*_{3,\,t}$ containing both $v_{k+l}v_n$ and $v_{\lceil \frac{k}{2}\rceil}v_{\lceil \frac{k}{2}\rceil+1}$. By Lemma 2.7, we know that there is a maximal matching $M_{C^*_{3,\,t}}$ of $C^*_{3,\,t}$ containing $v_{k+l}v_n$. Assume that $v_{\lceil \frac{k}{2}\rceil}v_{\lceil \frac{k}{2}\rceil+1} \notin M_{C^*_{3,\,t}}$. There must be one of $v_{\lceil \frac{k}{2}\rceil}v_{\lceil \frac{k}{2}\rceil+2}$, $v_{\lceil \frac{k}{2}\rceil+1}v_{\lceil \frac{k}{2}\rceil+2}$ in $M_{C^*_{3,\,t}}$. Suppose that $v_{\lceil \frac{k}{2}\rceil}v_{\lceil \frac{k}{2}\rceil+2} \in M_{C^*_{3,\,t}}$. We let $M^{'}_{C^*_{3,\,t}} = (M_{C^*_{3,\,t}} \backslash \{v_{\lceil \frac{k}{2}\rceil}v_{\lceil \frac{k}{2}\rceil+2}\}) \cup \{v_{\lceil \frac{k}{2}\rceil}v_{\lceil \frac{k}{2}\rceil+1}\}$. Note that $M^{'}_{C^*_{3,\,t}}$ is a matching of $C^*_{3,\,t}$ and $|M^{'}_{C^*_{3,\,t}}| = |M_{C^*_{3,\,t}}| = \alpha^{'}(C^*_{3,\,t})$. Then $M^{'}_{C^*_{3,\,t}}$ is also a maximal matching of $C^*_{3,\,t}$, which contains both $v_{k+l}v_n$ and $v_{\lceil \frac{k}{2}\rceil}v_{\lceil \frac{k}{2}\rceil+1}$. As a result, our claim holds.

We let $M^{\circ}_{C^*_{3,\,t}}$ be a maximal matching of $C^*_{3,\,t}$ containing both $v_{k+l}v_n$ and $v_{\lceil \frac{k}{2}\rceil}v_{\lceil \frac{k}{2}\rceil+1}$. Note that $M^{\circ}_{C^*_{3,\,t}}$ is also a matching of $C^*_{k,\,l}$. Hence, $\alpha^{'}(C^*_{3,\,t}) \leq \alpha^{'}(C^*_{k,\,l})$. Then the results follows. ∎

3. Minimizing graph.

3. Minimizing graph. A graph is called a *minimizing graph* in a class of graphs if its least signless Laplacian eigenvalue attains the minimum among all graphs in the class. In this section, we will apply the results in Section 2 to characterize the minimizing graphs among all the nonbipartite graphs with given matching number or edge cover number.

LEMMA 3.1. *Among all the nonbipartite unicyclic graphs with both given order n and given matching number $\alpha^{'} \geq 2$, the least signless Laplacian eigenvalue of a graph is minimized uniquely at $C^*_{3,\,2\alpha^{'}-3}$.*

Proof. Let G be a nonbipartite unicyclic graph with both given order n and given matching number $\alpha^{'}$, and let $C = v_1v_2\cdots v_kv_1$ (k is odd) be the unique cycle in G. Note that if $n = 3$, then $G \cong K_3$, and then $\alpha^{'}(K_3) = 1$. As a result, we have that if $\alpha^{'} \geq 2$, then $n \geq 4$. Note that $\alpha^{'}(C^*_{3,\,2\alpha^{'}-3}) = \alpha^{'}$. Then the result follows from Lemmas 2.8−2.11. This completes the proof. ∎

LEMMA 3.2. [6] *Let G be a graph with n vertices and m edges, and let e be an edge of G. Let $q_1 \geq q_2 \geq \cdots \geq q_{min}$ and $s_1 \geq s_2 \geq \cdots \geq s_n$ be the Q-eigenvalues of G and $G - e$ respectively. Then $0 \leq s_n \leq q_{min} \leq \cdots \leq s_2 \leq q_2 \leq s_1 \leq q_1$.*

LEMMA 3.3. *Among all the connected nonbipartite graphs with both given order n and given matching number $\alpha^{'} \geq 2$, the least signless Laplacian eigenvalue of a graph is minimized uniquely at $C^*_{3,\,2\alpha^{'}-3}$.*

Proof. Let G be a connected nonbipartite graph with both given order n and given matching number $\alpha^{'}$. Suppose C^o is an odd cycle in G. By deleting edges from G, we can get a connected unicyclic spanning subgraph of G, denoted by G', which contains C^o as the unique cycle. Obviously, $\alpha^{'}(G') \leq \alpha^{'}(G) = \alpha^{'}$. By Lemma 3.2, we

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know that $q_{min}(G') \leq q_{min}(G)$. By Lemmas 2.8, 3.1, we have

$$q_{min}(C^*_{3,\,2\alpha'-3}) \leq q_{min}(C^*_{3,\,2\alpha'(G')-3}) \leq q_{min}(G') \leq q_{min}(G).$$

Now, we begin to prove that $q_{min}(C^*_{3,\,2\alpha'-3}) = q_{min}(G)$ if and only if $G \cong C^*_{3,\,2\alpha'-3}$.

Assume that $q_{min}(G) = q_{min}(C^*_{3,\,2\alpha'-3})$.

Claim 1. $\alpha^{'}(G') = \alpha^{'}$. Otherwise, if $\alpha^{'}(G') < \alpha^{'}$, by Lemma 2.8, then

$$q_{min}(C^*_{3,\,2\alpha'-3}) < q_{min}(C^*_{3,\,2\alpha'(G')-3}),$$

and by Lemma 3.1, then

$$q_{min}(C^*_{3,\,2\alpha'(G')-3}) \leq q_{min}(G').$$

Noting that $q_{min}(G') \leq q_{min}(G)$, we have $q_{min}(C^*_{3,\,2\alpha'-3}) < q_{min}(G)$, which contradicts our assumption. Hence, our claim holds.

Claim 2. $G' \cong C^*_{3,\,2\alpha'-3}$. Otherwise, if $G' \ncong C^*_{3,\,2\alpha'-3}$, by Lemma 3.1, then

$$q_{min}(C^*_{3,\,2\alpha'-3}) < q_{min}(G').$$

Noting that $q_{min}(G') \leq q_{min}(G)$, we have $q_{min}(C^*_{3,\,2\alpha'-3}) < q_{min}(G)$, which contradicts our assumption. Hence, our claim holds.

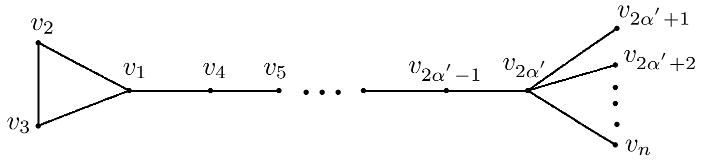

Fig. 3.1. G'.

From Claim 2 and our assumption, we conclude that $q_{min}(G) = q_{min}(G')$.

Assume that $E(G) \setminus E(G') \neq \emptyset$, and assume that the vertices of G' are indexed as in Fig. 3.1. Let $X = (x(v_1), x(v_2), \ldots, x(v_n))^T$ be a unit eigenvector corresponding to $q_{min}(G)$. Note that $E(G') \subseteq E(G)$. Then $X^TQ(G)X \geq X^TQ(G')X$, and then

$$q_{min}(G) = X^TQ(G)X \geq X^TQ(G')X \geq q_{min}(G').$$

From our above conclusion that $q_{min}(G) = q_{min}(G')$, we get

$$X^TQ(G)X = X^TQ(G')X = q_{min}(G').$$

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Therefore, X is also an eigenvector of G' corresponding to $q_{min}(G')$. Moreover, we conclude that for each edge $v_iv_j \in E(G) \setminus E(G')$, $x(v_i) + x(v_j) = 0$.

Because X is also an eigenvector of G' corresponding to $q_{min}(G')$, by Lemma 2.6, we know that $|x(v_1)| = \max\{|x(v_1)|, |x(v_2)|, |x(v_3)|\} > 0$. Combining with Lemmas 2.1, 2.2, we see that

(i) $|x(v_i)| < |x(v_j)|$ for $i = 1, 2, 3$, $j \geq 4$;

(ii) $|x(v_i)| < |x(v_j)|$ for $4 \leq i \leq 2\alpha'$, $4 < j \leq n$, $i < j$;

(iii) $x(v_i)x(v_j) > 0$ for $2\alpha' + 1 \leq i, j \leq n$.

Consequently, we conclude that for each edge $v_iv_j \in E(G) \setminus E(G')$, $x(v_i) + x(v_j) \neq 0$, which contradicts our conclusion that for each edge $v_iv_j \in E(G) \setminus E(G')$, $x(v_i) + x(v_j) = 0$. This means that the above assumption that $E(G) \setminus E(G') \neq \emptyset$ can not hold. This means $G = G'$. Then $G \cong C^*_{3,\,2\alpha'-3}$.

Conversely, if $G \cong C^*_{3,\,2\alpha'-3}$, then $q_{min}(C^*_{3,\,2\alpha'-3}) = q_{min}(G)$ naturally. This completes the proof. ∎

Theorem 3.4. *Among all the nonbipartite unicyclic graphs with both given order n and given matching number α', we have*

(i) *if $\alpha' = 1$, then the graphs are isomorphic to K_3;*

(ii) *if $\alpha' \geq 2$, then the least signless Laplacian eigenvalue of a graph is minimized uniquely at $C^*_{3,\,2\alpha'-3}$.*

Proof. Note that $\alpha'(P_4) \geq 2$. As a result, if a graph G contains P_4, then $\alpha'(G) \geq 2$. Note that for a nonbipartite unicyclic graph K of order $n \geq 4$, it can be checked that K contains P_4. Consequently, for a nonbipartite unicyclic graph K of order n, if $n \geq 4$, then $\alpha'(K) \geq 2$. Simultaneously, we get that if $\alpha' = 1$, then $n = 3$, and then (i) follows. (ii) follows from Lemma 3.1. ∎

Similar to Theorem 3.4, we get the following theorem.

Theorem 3.5. *Among all the connected nonbipartite graphs with both given order n and given matching number α', we have*

(i) *if $\alpha' = 1$, then the graphs are isomorphic to K_3;*

(ii) *if $\alpha' \geq 2$, then the least signless Laplacian eigenvalue of a graph is minimized uniquely at $C^*_{3,\,2\alpha'-3}$.*

Corollary 3.6. *Among all the connected nonbipartite graphs with both given order n and given edge cover number β', we have*

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(i) *if* $\beta^{'} = n - 1$*, then the graphs are isomorphic to* K_3*;*

(ii) *if* $\beta^{'} \leq n - 2$*, then the least signless Laplacian eigenvalue of a graph is minimized uniquely at* $C^{*}_{3,\,2n-2\beta'-3}$.

Proof. This corollary follows from the fact that $\alpha^{'}(G) + \beta^{'}(G) = n$ and Theorem 3.5. ◻

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