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ON THE FEICHTINGER CONJECTURE*

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Abstract. The Feichtinger Conjecture is proved for a class of Bessel sequences of unit norm vectors in a Hilbert space. Also, it is proved that every Bessel sequence of unit vectors in a Hilbert space can be partitioned into finitely many uniformly separated sequences.

Key words. Bessel sequence, Riesz sequence, Feichtinger Conjecture.

AMS subject classifications. 46C05, 42C15.

1. Introduction. There are many variations of the Feichtinger Conjecture, all equivalent with the following:

Every Bessel sequence of unit vectors in a Hilbert space can be partitioned into finitely many Riesz sequences.

For details on the Feichtinger Conjecture and the connection with other problems, see [2], [3], [4], [9] and [10], and references in these papers.

We denote by $\mathcal{H}$ a Hilbert space and $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$. We say that $\mathcal{F}$ is a *Bessel sequence* if there exists $B > 0$ so that

$$\sum_{n=0}^{\infty} |\langle x, f_n\rangle|^2 \leq B\|x\|^2, \quad \forall\, x \in \mathcal{H}.$$

B is called Bessel constant for $\mathcal{F}$.

We say that $\mathcal{F}$ is a *frame* for $\mathcal{H}$ if it is a Bessel sequence and there exists $A > 0$ so that

$$A\|x\|^2 \leq \sum_{n=0}^{\infty} |\langle x, f_n\rangle|^2, \quad \forall\, x \in \mathcal{H}.$$

For important applications of frames, see the references of the paper [7].

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We say that $\mathcal{F}$ is a *Riesz sequence (or Riesz basic sequence)* if there are $A, B > 0$ such that

$$A \sum |c_k|^2 \leq \| \sum c_k f_k \| \leq B \sum |c_k|^2$$

for any finite sequence (c_k). Riesz sequences are particular cases of frames (see [5]).

Let be $I \subset \mathbb{N}$. If $\mathcal{F}$ is a Bessel sequences in $\mathcal{H}$, then $\mathcal{F}_I = \{f_n\}_{n \in I}$ is clearly also a Bessel sequence in $\mathcal{H}$.

In [5], O. Christensen, using Schur's test, give conditions on a sequence $\{f_n\}_{n=0}^{\infty}$ to be a Bessel sequence, that it only involves inner products between the elements $\{f_n\}_{n=0}^{\infty}$:

Proposition 1.1. [5] *Let $\{f_n\}_{n=0}^{\infty}$ be a sequence in $\mathcal{H}$ and assume that there exists a constant $B > 0$ such that*

$$\sum_{k=0}^{\infty} |\langle f_j, f_k \rangle| \leq B, \quad \forall j \in \mathbb{N}.$$

Then $\{f_n\}_{n=0}^{\infty}$ is a Bessel sequence with bound B.

We call this sequences *Bessel-Schur sequences*.

The intrinsically localized sequences, introduced by K. Gröchenig in [8], are particular cases of Bessel-Schur sequences. In the same paper, he proves that every localized frame is a finite union of Riesz sequences. Another type of localized sequences was introduced by R. Balan, P.G. Casazza, C. Heil, and Z. Landau in [1]. They show that the Feichtinger Conjecture is true for l^1-self-localized frames which are norm-bounded below. l^1-self-localized Bessel sequences are also Bessel-Schur sequences.

On the other hand, we recall the following definition:

Definition 1.2. [4] A sequence $\{f_n\}_{n \in I}$ of unit vectors in $\mathcal{H}$ is called *separated* if there exists a constant $\gamma < 1$ such that

$$|\langle f_n, f_k \rangle| \leq \gamma$$

for any $n, k \in \mathbb{N}$, $n \neq k$.

In [4], the authors, among others, give the following result:

Theorem 1.3. *Let $\mathcal{H}$ be a Hilbert space and let $\{f_n\}_{n \in I}$ be a Bessel sequence of unit vectors in $\mathcal{H}$. Then $\{f_n\}_{n \in I}$ can be partitioned into finitely many separated Bessel sequences.*

In the following, we prove that the Bessel-Schur sequences satisfies the Feichtinger Conjecture. Also, we prove that every Bessel sequence of unit vectors in a Hilbert space can be partitioned into finitely many uniformly separated sequences.

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2. The results. First, we give a condition for a Bessel sequence of unit vectors to be a Riesz sequence.

THEOREM 2.1. *Let $\mathcal{F}_I = \{f_n\}_{n\in I}$ be a Bessel sequence of unit vectors. We suppose that*

$$\sigma := \sup_{j\in I} \sum_{\substack{i\in I\\ i\neq j}} |\langle f_i, f_j\rangle| < 1.$$

Then, $\mathcal{F}_I$ is a Riesz sequence.

Proof. If $\mathcal{F}_I$ is a Bessel sequence in $\mathcal{H}$, then the following operators are linear and bounded:

$$T : l^2(I) \to \mathcal{H}, \quad T(c_i) = \sum_{i\in I} c_i f_i \quad \text{(synthesis operator)},$$
$$\Theta : \mathcal{H} \to l^2(I), \quad \Theta x = \{\langle x, f_i\rangle\}_{i\in I} \quad \text{(analysis operator)}.$$

Moreover, Θ is the adjoint of T (see [5]).

For $c = (c_n)_{n\in I} \in l^2(I)$, we have

$$\begin{aligned}(\Theta T)(c) &= \left\{ \langle \sum_{k\in I} c_k f_k, f_j\rangle \right\}_{j\in I}\\ &= \left\{ \sum_{k\in I} c_k \langle f_k, f_j\rangle \right\}_{j\in I}\end{aligned}$$

and hence,

$$(\Theta T)(c) - c = \left\{ \sum_{k\neq j} c_k \langle f_k, f_j\rangle \right\}_{j\in I}.$$

By Cauchy-Schwartz inequality, it follows

$$\begin{aligned}\|(\Theta T)(c) - c\|_2^2 &= \sum_{j\in I} \left| \sum_{k\neq j} c_k \langle f_k, f_j\rangle \right|^2\\ &\le \sum_{j\in I} \left(\sum_{\substack{k\in I\\ k\neq j}} |c_k| |\langle f_k, f_j\rangle|^{1/2} \cdot |\langle f_k, f_j\rangle|^{1/2} \right)^2\\ &\le \sum_{j\in I} \left(\sum_{\substack{k\in I\\ k\neq j}} |c_k|^2 |\langle f_k, f_j\rangle| \right) \left(\sum_{\substack{k\in I\\ k\neq j}} |\langle f_k, f_j\rangle| \right)\\ &\le \sigma \sum_{j\in I} \left(\sum_{\substack{k\in I\\ k\neq j}} |c_k|^2 |\langle f_k, f_j\rangle| \right).\end{aligned}$$

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Changing the order of summation, we obtain

$$\begin{aligned}\|(\Theta T)(c)-c\|_2^2 &\le \sigma\sum_{k\in I}|c_k|^2\sum_{\substack{j\in I\\ j\neq k}}|\langle f_k,f_j\rangle| \\ &\le \sigma^2\|c\|_2^2,\end{aligned}$$

and so, $\|\Theta T-I\|\le\sigma<1$.

Therefore, ΘT is invertible, thus Θ is surjective. It follows that $\mathcal{F}_I$ is a Riesz-Fischer sequence. From Theorem 3 in [13, Ch. 4, Sec. 2], we have that there exists $A>0$ so that

$$A\sum|c_k|^2\le\|\sum c_kf_k\|^2$$

for every finite sequence (c_k). Since $\mathcal{F}_I$ is a Bessel sequence, we have

$$\|\sum c_kf_k\|^2\le B\sum|c_k|^2$$

for (c_k) finite sequence (see [5]). So, $\mathcal{F}_I$ is a Riesz sequence. □

THEOREM 2.2. *Every Bessel-Schur sequence of unit vectors is union of finite Riesz sequences.*

Proof. Let $j\in\mathbb{N}$ fixed. We have:

$$\sum_{i=0}^{\infty}|\langle f_j,f_i\rangle|\le B,$$

and hence,

$$\sum_{\substack{i=0\\ i\neq j}}|\langle f_j,f_i\rangle|\le B-1,\quad\text{for any } j\in\mathbb{N}. \tag{2.1}$$

We denote

$$a_{ij}=\begin{cases}|\langle f_j,f_i\rangle|, & j\neq i,\\ 0, & j=i.\end{cases}$$

We have $a_{ij}=a_{ji}\ge 0$ and $a_{ii}=0$.

The relation (2.1) is equivalent with

$$\sup_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}}a_{ij}\le B-1.$$

By Mills' Lemma (see [6, Ch. X] or [12]) there is a partition $\mathbb{N}=I_1\cup I_2$ such that

$$\sup_{j\in I_p}\sum_{i\in I_p}a_{ij}\le\frac{B-1}{2};\quad p=1,2.$$

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By iteration, for any $m \geq 1$, there is a partition $\mathbb{N} = I_1 \cup I_2 \cup \ldots \cup I_{2^m}$ such that

$$\sup_{j \in I_p} \sum_{i \in I_p} a_{ij} \leq \frac{B-1}{2^m}, \quad \forall\, p = 1, 2, \ldots, 2^m.$$

We take m so that $\dfrac{B-1}{2^m} < 1$, and apply Theorem 2.1. □

3. An equivalent form of the Feichtinger conjecture. We consider the following class of sequences.

DEFINITION 3.1. Let $\mathcal{F}_I = \{f_n\}_{n \in I}$ be a sequence of unit vectors. We say that this sequence is *uniformly separated* if the following condition holds:

$$\eta := \sup_{j \in I} \sum_{\substack{i \in I \\ i \neq j}} |\langle f_i, f_j \rangle|^2 < 1.$$

The following result is a refinement of a result from [4].

THEOREM 3.2. *Every Bessel sequence of unit vectors is union of finite uniformly separated sequences.*

Proof. Let $\mathcal{F}$ be a Bessel sequence of unit vectors:

$$\sum_{i=0}^{\infty} |\langle x, f_i \rangle|^2 \leq B\|x\|^2, \quad \forall\, x \in \mathcal{H}.$$

Let $j \in \mathbb{N}$ fixed. We take $x = f_j$:

$$\sum_{i=0}^{\infty} |\langle f_j, f_i \rangle|^2 \leq B\|f_j\|^2 = B,$$

and hence,

$$\sum_{\substack{i=0 \\ i \neq j}} |\langle f_j, f_i \rangle|^2 \leq B - 1, \quad \text{for any } j \in \mathbb{N}. \tag{3.1}$$

It is clear that $B \geq 1$. We denote

$$b_{ij} = \begin{cases} |\langle f_j, f_i \rangle|^2, & j \neq i, \\ 0, & j = i. \end{cases}$$

We have $b_{ij} = b_{ji} \geq 0$ and $b_{ii} = 0$.

The relation (3.1) is equivalent with

$$\sup_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} b_{ij} \leq B - 1.$$

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By Mills' Lemma (see [6, Ch. X] or [12]), there is a partition $\mathbb{N} = I_1 \cup I_2$ such that

$$\sup_{j \in I_p} \sum_{i \in I_p} b_{ij} \leq \frac{B-1}{2}; \quad p = 1, 2.$$

By iteration, for any $m \geq 1$, there is a partition $\mathbb{N} = I_1 \cup I_2 \cup \ldots \cup I_{2^m}$ such that

$$\sup_{j \in I_p} \sum_{i \in I_p} b_{ij} \leq \frac{B-1}{2^m}, \quad \forall\, p = 1, 2, \ldots 2^m.$$

We take m so that $\dfrac{B-1}{2^m} < 1$ and apply Definition 3.1. □

From the above Theorem, we obtain the following equivalent form of the Feichtinger Conjecture:

Every uniformly separated Bessel sequence of unit norm vectors can be partitioned into finitely many Riesz sequences.

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(1) Theorem 2.1 can also be proven using the Schur test and the spectral mapping theorem.

(2) There is a credible claim by A. Marcus, D.A. Spielman, and N. Srivastava [11] (26 June 2013) to have solved the Kadison-Singer problem.

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