

ON THE ENERGY OF SINGULAR GRAPHS*

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Abstract. The nullity, $\eta(G)$, of a graph G is the algebraic multiplicity of the eigenvalue zero in the graph's spectrum. If $\eta(G) > 0$, then the graph G is said to be singular. The energy of a graph, $E(G)$, was first defined by I. Gutman (1985) as the sum of the absolute values of the eigenvalues of the graph's adjacency matrix $A(G)$. This paper considers the energy concept for singular graphs. In particular, it is proved that the change in energy upon the simple act of deleting a vertex is related to the type of vertices of the singular graph. Certain upper bounds are improved for the energy of the induced subgraph, $G - u$, which is obtained by deleting vertex u , with the aid of a parameter known as the null spread of u , $\eta_u(G) = \eta(G) - \eta(G - u)$. Also, some new bounds are given for the energy of the minimal configuration graphs, an important class of singular graphs of nullity one that are related to the graph's core. Furthermore, certain graphs that increase their energy when an edge is deleted are considered, such as the complete multipartite graphs and the hypercubes of even dimensions.

Key words. Nullity, Singular graphs, Energy of graphs, Null spread, Minimal configuration graphs, Complete multipartite graphs, Hypercube.

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1. Introduction and preliminaries. Let $G = (V, E)$ be a finite, undirected graph with nonempty vertex set V and edge set E . The *adjacency matrix*, $A(G)$, of a graph G on n vertices is the $n \times n$ matrix whose entries a_{ij} denote the number of edges from vertex u_i to vertex u_j . For a simple, undirected graph the adjacency matrix is a symmetric $(0, 1)$ -matrix. Thus, $A(G)$ has real eigenvalues and zeros on the diagonal, meaning that the sum of these eigenvalues equals to zero. A graph, G , is *singular* if the adjacency matrix, $A(G)$, is a singular matrix; that is, zero is an eigenvalue of G . The *nullity*, $\eta(G)$, of a singular graph G is the multiplicity of zero in the graph's spectrum. It is clear that there exist corresponding vectors x , such that $Ax = 0$. These vectors are defined as the *kernel eigenvectors* of a graph G . Let us consider a graph G of nullity one, with a kernel eigenvector $x = [x_1, x_2, \dots, x_m, 0, \dots, 0]^T$, where $x_i \neq 0$, $i = 1, 2, \dots, m$. The subgraph F of G , induced by the first m vertices corresponding to the first m entries, is called the *core* of G . The set of the remaining vertices, corresponding to the zero entries of the kernel eigenvector, is called the *periphery*.

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DEFINITION 1.1. [9] Let x be a kernel eigenvector of a singular graph on at least two vertices. If x has only non-zero entries, then G is referred to as a *core graph*.

DEFINITION 1.2. [10] A graph G , $|G| \geq 3$, is a *minimal configuration*, with core (F, x_F) of nullity $\eta(F)$, if it is a singular graph of nullity one, having $|F| + (\eta(F) - 1)$ vertices, with F as an induced subgraph, satisfying $|F| \geq 2$, $Fx_F = 0$, and $G \begin{bmatrix} x_F \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The vector x_F is said to be the non-zero part of the kernel eigenvector of G .

EXAMPLE 1.3. A path on $2k - 1$ vertices is a minimal configuration graph ($\eta(P_{2k-1}) = 1$) that has as a core the null graph N_k ($\eta(N_k) = k$).

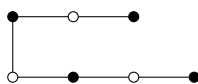


FIG. 1.1. The core of P_7 , N_4 colored black.

One of the most important theorems, considering the eigenvalues of a graph, is perhaps the interlacing theorem.

THEOREM 1.4. (Interlacing Theorem, [8]) Let G be a graph with spectrum $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, and let the spectrum of $G - u_1$ be $\mu_1 \geq \mu_2 \geq \dots \geq \mu_{n-1}$. Then, the spectrum of $G - u_1$ is “interlaced” with the spectrum of G , and $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \geq \mu_{n-1} \geq \lambda_n$.

It is clear from interlacing that the nullity of a graph can change, at most one, upon deleting (adding) a vertex. We will give next, the following definition:

DEFINITION 1.5. [4] Let $G - u$ be the induced subgraph of graph G obtained on deleting vertex u . The *null spread* of vertex u is: $n_u(G) = \eta(G) - \eta(G - u)$.

OBSERVATION 1.6. By interlacing: $-1 \leq n_u(G) \leq 1$.

OBSERVATION 1.7. Let G be a minimal configuration graph with core F of nullity $\eta(F)$. The $\eta(F) - 1$ vertices of G belong to the periphery of the graph, and their deletion increases the nullity of the graph, meaning that each vertex in the periphery has a null spread of -1 .

The energy of a graph G was first set by Gutman in 1978 as the sum of the absolute values of its eigenvalues:

$$E(G) = \sum_{i=1}^n |\lambda_i|.$$

The concept of energy of graphs originates from theoretical chemistry and has been studied rather intensively in the last decade. This paper focuses on the energy of

singular graphs, a subject first studied in [13]. In Section 2 of this paper, we improve some upper bounds for the energy of the induced subgraph of G , $G - u$, by identifying the vertices of the graph. In Section 3, we study the energy of certain classes of singular graphs, such as the minimal configurations and the r -partite graphs. We conclude this paper with some results on the energy change after deleting an edge of a complete multipartite graph or a hypercube.

2. Energy of subgraphs. Let G be a singular graph and $G - u$ an induced subgraph of G , obtained from G by deleting vertex u . In this section, we will improve the bound $E(G - u) \leq E(G)$, by identifying the vertices in G .

THEOREM 2.1. *Let $G = (V, E)$ be a graph and $u \in V$. If $n_u(G) = 1$, then*

$$E(G - u) \leq E(G).$$

The equality holds if and only if u is an isolated vertex.

Proof. If, upon deleting u , the nullity decreases by one, then the set of the non-zero eigenvalues in G and $G - u$ have the same cardinality. By the interlacing theorem, $E(G - u) \leq E(G)$. We will now prove that the equality holds.

Let u be an isolated vertex. Then, since u is associated with zero entries in the adjacency matrix and related to a zero eigenvalue in the spectrum of the graph, its removal has absolutely no effect to the sum of the absolute values of the non-zero eigenvalues of G . Thus, $E(G - u) = E(G)$.

Let $E(G) = E(G - u)$ and $n_u(G) = 1$, then $\sum_{i=1}^n |\lambda_i| = \sum_{i=1}^{n-1} |\mu_i|$, meaning that if we rearrange only the non-zero eigenvalues in non increasing order, $\lambda_i = \mu_i, \forall i$ and $\sum |\lambda_i|^2 = \sum |\mu_i|^2$. It is well known that $\sum |\lambda_i|^2 = 2m$, for a graph G with m edges, and thus, u is an isolated vertex. \square

THEOREM 2.2. *Let $G = (V, E)$ be a graph and $u \in V$. If $n_u(G) = -1$, then*

$$E(G - u) \leq E(G) - (|\lambda_l| + |\lambda_m|),$$

where λ_l and λ_m are the smallest non-negative and the largest non-positive eigenvalue, respectively.

In the case where G is a connected graph of nullity $\eta(G) = n - 2$, the equality holds if and only if G is a star graph and u is the center vertex of the graph.

Proof. If, upon deleting u , the nullity increases by one, then $G - u$ has two less non-zero eigenvalues than G . By interlacing, $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \dots \geq \lambda_l > 0 = \dots = 0 > \lambda_m \geq \mu_m \geq \dots \geq \lambda_n$, and since $E(G) = \sum_{i=1}^n |\lambda_i|$, we have $E(G - u) \leq E(G) - (|\lambda_l| + |\lambda_m|)$.

Now, let G be a star graph on n vertices. Then G has two non-zero eigenvalues, with multiplicity one, namely $-\sqrt{n-1}$ and $\sqrt{n-1}$, and $n-2$ zero eigenvalues. Upon deleting the center of the graph, the obtained graph is null and the nullity increases by one. Thus, $E(G-u) = 0 = E(G) - (|\sqrt{n-1}| + |-\sqrt{n-1}|)$.

Let us suppose that G is a graph of nullity $\eta(G) = n-2$. Then, if the nullity increases upon deleting a vertex, the obtained subgraph has $n-1$ vertices and a nullity of $\eta(G-u) = n-1$. It is well known that the only graph that is of the same nullity as its order is the null graph. Thus, G is a star graph, with vertex u as its center, and the equality $E(G-u) = E(G) - (|\lambda_l| + |\lambda_m|)$ holds. \square

OBSERVATION 2.3. Another example of a graph on n vertices that achieves the above equality is the graph that is a union of m complete graphs, K_2 , and $n-2m$ isolated vertices. The energy of $G = mK_2 \cup (n-2m)K_1$ is $E(G) = m(|-1| + |1|)$. Upon deleting a vertex u of K_2 , the nullity increases and the energy of $G-u$ is $E(G-u) = (m-1)(|-1| + |1|)$. Thus, $E(G-u) = E(G) - (|\lambda_l| + |\lambda_m|)$.

The interlacing inequalities imply the following.

THEOREM 2.4. Let $G = (V, E)$ be a graph and $u \in V$. If $n_u(G) = 0$, then

$$E(G-u) \leq E(G) - |\lambda_i|,$$

where λ_i is either the smallest non-negative or the largest non-positive eigenvalue of G .

EXAMPLE 2.5. We give an example of equality for Theorem 2.4. The spectrum of graph G , in Figure 2.1, is $\{2, 0, -1, -1\}$, and its energy is $E(G) = 4$. When we delete the white vertex u , the obtained subgraph has $\{1, 0, -1\}$ as its eigenvalues, and the energy of $G-u$ is $E(G-u) = 2$. Thus, $n_u(G) = 0$ and $E(G) - E(G-u) = 2 = \lambda_l$, where λ_l is the only non-negative eigenvalue of G .



FIG. 2.1. The graphs G , $G-u$.

It is clear, from the above theorems, that the type of vertices are important in determining the energy of a singular graph. Similar results, however, may be obtained for any λ , nonzero eigenvalue, which could also include non-singular graphs G .

Let $G = (V, E)$ be a graph and $u \in V$. Suppose $m(G)$ is the multiplicity of a non-negative eigenvalue λ for G , $m(G-u)$ is the multiplicity of λ for $G-u$, and $m_u(G) = m(G) - m(G-u)$ is the vertex spread of λ .

Then, in the case of $m_u(G) = 0$, it can be shown by using the interlacing theorem: $E(G - u) \leq \max \{E(G) - \lambda_l, E(G) - |\lambda_m|\}$, where λ_l (resp. λ_m) is the smallest positive (resp. largest negative) eigenvalue of G .

OBSERVATION 2.6. It is obvious that if H is an induced subgraph of a graph G , then $E(H) \leq E(G)$.

3. Energy of singular graphs. In this section, we study the energy of singular graphs. We give some new bounds for the energy of the minimal configuration graphs and improve some known bounds for energy, in the case that the graph is singular.

PROPOSITION 3.1. Let G be a minimal configuration with core of order at least three. Then, $E(G) > 2\sqrt{5}$.

Proof. A minimal configuration with core of order at least three has the path P_4 as an induced subgraph [11]. By Observation 2.6, and since P_4 is not singular, $E(G) > 2(\frac{1+\sqrt{5}+\sqrt{5}-1}{2})$. \square

THEOREM 3.2. Let G be a minimal configuration, with core F of nullity $\eta(F)$. Then,

$$E(G) \geq E(F) + (\eta(F) - 1)(|\lambda_l| + |\lambda_m|),$$

where λ_l and λ_m are the smallest non-negative and the largest non-positive eigenvalue of G , respectively.

Proof. Let w_i , $i = 1, \dots, \eta(F) - 1$ be the vertices of the periphery P . By Theorem 2.2, since $n_{w_i}(G) = -1$:

$$E(G - w_1) \leq E(G) - (|\lambda_l| + |\lambda_m|),$$

where λ_l and λ_m are the smallest non-negative and the largest non-positive eigenvalue of G , respectively. Then, by the same theorem:

$$E(G - w_1 - w_2) \leq E(G - w_1) - (|\mu_{l-1}| + |\mu_m|),$$

where μ_{l-1} and μ_m are the smallest non-negative and the largest non-positive eigenvalues of $G - w_1$, respectively. Since by interlacing, $|\mu_{l-1}| \geq |\lambda_l|$ and $|\mu_m| \geq |\lambda_m|$, $E(G - w_1 - w_2) \leq E(G - w_1) - (|\mu_{l-1}| + |\mu_m|) \leq E(G) - 2(|\lambda_l| + |\lambda_m|)$. It is then clear that

$$E(G - w_1 - w_2 - \dots - w_{\eta(F)-1}) \leq E(G) - (\eta(F) - 1)(|\lambda_l| + |\lambda_m|). \quad \square$$

LEMMA 3.3. [2] Let H be an induced subgraph of a graph G , with edge set m . Then,

$$E(G) - E(H) \leq E(G - m) \leq E(G) + E(H).$$

PROPOSITION 3.4. *Let G be a minimal configuration graph with core F , periphery P , and nullity $\eta(F)$. If n_F is the number of vertices of the core adjacent to some vertex of the periphery and $|\lambda_k|$ is the smallest absolute value of the graph's eigenvalues, then:*

$$|\lambda_k| < \frac{1}{2\sqrt{8}}(1 + \frac{n_F}{\eta(F)-1})(\sqrt{n_F + \eta(F) - 1} + \sqrt{2}).$$

Proof. Let the edge set of core F be m . Since the core F is an induced subgraph of the minimal configuration graph G , by Lemma 3.3: $E(G) - E(F) \leq E(G - m)$. When we remove the edges from F , the obtained graph consists only of edges between the core F and the periphery P . Since those two sets are independent, the $G - m$ graph is bipartite. Koolen and Moulton [7] proved that for a bipartite graph $K_{F,P}$: $E(K_{F,P}) \leq \frac{n}{\sqrt{8}}(\sqrt{n} + \sqrt{2})$. The vertices of $G - m$ are the $\eta(F) - 1$ vertices of the periphery and the vertices of the core, n_F , adjacent to those of the periphery. By Theorem 3.2: $E(G) - E(F) > (\eta(F) - 1)(|\lambda_l| + |\lambda_m|)$, where λ_l and λ_m are the smallest non-negative and the largest non-positive eigenvalues of G , respectively. Thus,

$$|\lambda_l| + |\lambda_m| < \frac{\eta(F)-1+n_F}{\sqrt{8}(\eta(F)-1)}(\sqrt{\eta(F) - 1} + \sqrt{2}),$$

or, if $|\lambda_k| = \min(|\lambda_l|, |\lambda_m|)$:

$$|\lambda_k| < \frac{1}{2\sqrt{8}}(1 + \frac{n_F}{\eta(F)-1})(\sqrt{n_F + \eta(F) - 1} + \sqrt{2}). \quad \square$$

OBSERVATION 3.5. Let us try to construct a minimal configuration graph G from a null graph N_p . It has been shown that the minimal configuration graph is a connected graph [9], which implies that all vertices of the null graph will be adjacent to some vertex of the periphery. Since the nullity of the null graph is equal to its order, $\eta(F) = n_F = p$. By Proposition 3.4, for the smallest absolute value of the eigenvalues of G , $|\lambda_k|$:

$$|\lambda_k| < \frac{1}{2\sqrt{8}}(1 + \frac{p}{p-1})(\sqrt{2p - 1} + \sqrt{2}).$$

For example, if we construct a graph from the null graph N_4 , its smallest, in absolute value, eigenvalue is not greater than 1.67465. In Figure 3.1, the minimal configuration graph has 1 as the smallest absolute value of eigenvalues.

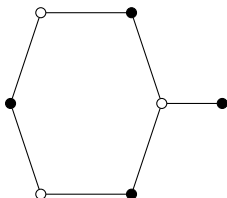


FIG. 3.1. A minimal configuration, constructed by the null graph N_4 .

McClelland's bounds (1971) for the energy of a $G(n, m)$ graph, containing the vertices and edges of the graph, are:

$$\sqrt{2m + n(n-1)|\det A|^{2/n}} \leq E(G) \leq \sqrt{2mn}.$$

The upper and lower bound of the above inequality can be improved for singular graphs, as shown in Propositions 3.6 and 3.7, respectively.

PROPOSITION 3.6. [5] *Let G be a graph on n vertices, and nullity $\eta(G)$. Then,*

$$E(G) \leq \sqrt{2(n - \eta(G))m}.$$

PROPOSITION 3.7. [1] *Let G be a graph on n vertices, and nullity $\eta(G)$. Then,*

$$E(G) \geq n - \eta(G).$$

We conclude this section with an upper bound for the energy of r -partite graphs.

LEMMA 3.8. *Let G be a complete r -partite graph, on n vertices and m edges. Then,*

$$E(G) \leq 2\sqrt{\frac{2(r-1)m}{r}}.$$

Proof. First, we rearrange the $n - \eta(G) = r$ non-zero eigenvalues of the graph in non increasing order ($\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-\eta(G)}$), after omitting the $\eta(G)$ zero eigenvalues.

We apply the Cauchy-Schwartz inequality to $(1, 1, \dots, 1)$ and $(\lambda_2, \lambda_3, \dots, \lambda_{n-\eta(G)})$,

$$(\sum_{i=2}^{n-\eta(G)} \lambda_i)^2 \leq \sum_{i=2}^{n-\eta(G)} \lambda_i^2 \cdot \sum_{i=2}^{n-\eta(G)} 1^2 = (n - \eta(G) - 1) \sum_{i=2}^{n-\eta(G)} \lambda_i^2.$$

Since, $-\lambda_1 = \sum_{i=2}^{n-\eta(G)} \lambda_i$: $\lambda_1^2 \leq (n - \eta(G) - 1) \sum_{i=2}^{n-\eta(G)} \lambda_i^2$ and $(n - \eta(G))\lambda_1^2 \leq (n - \eta(G) - 1) \sum_{i=1}^{n-\eta(G)} \lambda_i^2 = (n - \eta(G) - 1)2m$.

Since G is a complete r -partite graph, it has only one positive eigenvalue λ_1 [12] and $E(G) = 2\lambda_1$. Thus,

$$E(G) \leq 2\sqrt{\frac{2(n-\eta(G)-1)m}{n-\eta(G)}}.$$

It is clear that the equality holds if and only if G is a regular r -partite graph. \square

4. Deleting an edge. It has been shown that the energy of a graph may increase, decrease, or stay the same after deleting an edge [3]. In this section, we will study certain graphs that increase their energy when an edge is deleted, such as the complete multipartite graphs and the hypercube of even order.

PROPOSITION 4.1. *Let $K_{p,q}$ be a complete bipartite graph. Then, if we remove an edge e :*

$$E(K_{p,q} - e) = 2\sqrt{pq - 1 + 2\sqrt{(p-1)(q-1)}}.$$

Proof. Let the matrix of $K_{p,q} - e$ be:

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The matrix has four independent rows, and so $\eta(K_{p,q} - e) = p + q - 4$. Let $\mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4$ be the remaining non-zero eigenvalues. Then, since $K_{p,q} - e$ is bipartite, $\mu_1 = -\mu_3$ and $\mu_2 = -\mu_4$. The characteristic polynomial can now be written as: $x^{p+q-4}(x - \mu_1)(x + \mu_1)(x - \mu_2)(x + \mu_2) = x^{p+q-4}(x^4 - (\mu_1^2 + \mu_2^2)x^2 + \mu_1^2\mu_2^2)$. It is well known that, $\sum \mu_i^2 = 2m$, and so $\mu_1^2 + \mu_2^2 = pq - 1$. Also, $\sum \mu_i^k = \text{tr} A^k$. The diagonal entries of A^4 are:

$$A^4 = \begin{bmatrix} (q-1)^2 p & & & & & & & \\ & (q-1)^2 + q^2(p-1) & & & & & & \\ & & \ddots & & & & & \\ & & & (p-1)^2 q & & & & \\ & & & & (p-1)^2 + p^2(q-1) & & & \\ & & & & & \ddots & & \\ & & & & & & (p-1)^2 + p^2(q-1) & \end{bmatrix}$$

and $\text{tr} A^4 = (q^2 - 2q + 1)(p^2 + 2p - 1) + (p^2 - 2p + 1)(q^2 + 2q - 1)$. Since, $\mu_1^4 + \mu_2^4 = (\mu_1^2 + \mu_2^2)^2 - 2\mu_1^2\mu_2^2$, after some easy calculation $\mu_1^2\mu_2^2 = (p-1)(q-1)$. The energy of the graph is:

$$E(K_{p,q} - e) = 2(|\mu_1| + |\mu_2|) = 2\sqrt{\mu_1^2 + \mu_2^2 + 2|\mu_1||\mu_2|}$$

and thus,

$$E(K_{p,q} - e) = 2\sqrt{pq - 1 + 2\sqrt{(p-1)(q-1)}}. \quad \square$$

By Lemma 3.8, it is clear that:

$$E(K_{p,q} - e) - E(G) \geq 2(\sqrt{pq - 1 + 2\sqrt{(p-1)(q-1)}} - \sqrt{pq}).$$

PROPOSITION 4.2. Let $K_{t,t,\dots,t}$ be a complete r -partite graph, with $r \geq 3$, $t \geq 2$. If $K_{t,t,\dots,t} - e$ is its subgraph after removing edge e , then $E(K_{t,t,\dots,t} - e) \geq E(K_{t,t,\dots,t})$.

Proof. Let $K_{t,t,\dots,t} - e$ be the subgraph of a complete r -partite graph, $K_{t,t,\dots,t}$, after deleting an edge e between the first two sets of vertices. The graph's matrix will be of the form:

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

It is clear that $K_{t,t,\dots,t} - e$ has $r + 2$ independent rows. Working as in the proof of Proposition 4.1, we find that $-t$ is an eigenvalue of the graph, with multiplicity $r - 3$. Let $\mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4 \geq \mu_5$ be the remaining non-zero eigenvalues. Then, we find that:

$$\mu_2 = \frac{\sqrt{(t-1)^2 + 4(t-1)} - (t-1)}{2}, \mu_4 = -\frac{\sqrt{(t-1)^2 + 4(t-1)} + (t-1)}{2}.$$

Also $|\mu_2| + |\mu_4| \leq |\mu_3| + |\mu_5|$, where μ_3, μ_5 and μ_1 satisfy the polynomial:

$$\mu^3 - ((r-2)t-1)\mu^2 - (m - \frac{(r-2)(r-1)}{2}t^2 + (r-2)t)\mu - (r-1)(t-1)t = 0,$$

and m are the edges of $K_{t,t,\dots,t} - e$, ($m = \frac{r(r-1)}{2}t^2 - 1$).

Since $\sqrt{(t-1)^2 + 4(t-1)} \geq t + \frac{1}{rt}$ and $\mu_1 \geq \frac{2m}{rt} = (r-1)t - \frac{2}{rt}$,

$$\begin{aligned} E(K_{t,t,\dots,t} - e) &= \sum_{i=1}^5 |\mu_i| + (r-3)|-t| \\ &\geq |\mu_1| + 2(|\mu_2| + |\mu_4|) + (r-3)t \\ &\geq (r-1)t - \frac{2}{rt} + 2(t + \frac{1}{rt}) + (r-3)t \\ &= 2(r-1)t \\ &= E(K_{t,t,\dots,t}). \quad \square \end{aligned}$$

The hypercube Q_n is an undirected regular graph defined recursively, with reference to the cartesian product of two graphs, by $Q_1 = K_2$ and $Q_{n+1} = Q_n \times K_2$. The characteristic polynomial of the hypercube Q_n is $\varphi(Q_n) = \prod_{k=0}^n (x - n + 2k)^{\binom{n}{k}}$ [6]. It is straightforward that the hypercube Q_n is singular if and only if n is even. The adjacency matrix of the hypercube can be written as: $A(Q_n) = \begin{bmatrix} A(Q_{n-1}) & I_{2^{n-1}} \\ I_{2^{n-1}} & A(Q_{n-1}) \end{bmatrix}$, where $I_{2^{n-1}}$ denotes the identity matrix.

LEMMA 4.3. [3] For a partitioned matrix $C = \begin{bmatrix} A & X \\ Y & B \end{bmatrix}$, where both A and B are square matrices, we have:

$$\sum_j s_j(A) + \sum_j s_j(B) \leq \sum_j s_j(C),$$

where $s_j(\cdot)$ denote the singular values of a matrix.

THEOREM 4.4. Let Q_{2k} be a singular hypercube. If $Q_{2k} - e$ is its subgraph after removing edge e , then:

$$E(Q_{2k} - e) \geq E(Q_{2k}).$$

Proof. Let e be an edge corresponding to the identity matrix $I_{2^{2k-1}}$. The adjacency matrix of $Q_{2k} - e$ after deleting edge e is of the form:

$$A(Q_{2k-e}) = \begin{bmatrix} A(Q_{2k-1}) & J_{2^{2k-1}} \\ J_{2^{2k-1}} & A(Q_{2k-1}) \end{bmatrix},$$

where $J_{2^{2k-1}}$ is formed from the identity matrix by changing one diagonal entry to zero. By Lemma 4.3, $E(Q_{2k} - e) \geq 2E(Q_{2k-1})$.

The energy of Q_{2k} is:

$$\begin{aligned} E(Q_{2k}) &= \sum_{i=0}^{2k} \binom{2k}{i} |2k - 2i| \\ &= \sum_{i=0}^k \binom{2k}{i} (2k - 2i) - \sum_{i=k+1}^{2k} \binom{2k}{i} (2k - 2i) \\ &= \sum_{i=0}^k \binom{2k}{i} (2k - 2i) - \sum_{i=0}^{2k} \binom{2k}{i} (2k - 2i) + \sum_{i=0}^k \binom{2k}{i} (2k - 2i) \\ &= 2 \sum_{i=0}^k \binom{2k}{i} (2k - 2i) - \sum_{i=0}^{2k} \binom{2k}{i} (2k - 2i) \\ &= 4k \sum_{i=0}^k \binom{2k}{i} - 4 \sum_{i=0}^k i \binom{2k}{i} - \sum_{i=0}^{2k} 2k \binom{2k}{i} + \sum_{i=0}^{2k} 2i \binom{2k}{i} \\ &= 4k(2^{2k-1} + \frac{\binom{2k}{k}}{2}) - 4k2^{2k-1} - 2k2^{2k} + 2 \cdot 2k2^{2k-1} \\ &= 2k \binom{2k}{k}. \end{aligned}$$

In a similar way, we find that $E(Q_{2k-1}) = 2k \binom{2k-1}{k}$. Since,

$$\begin{aligned} E(Q_{2k}) &= 2k \binom{2k}{k} \\ &= 2k \binom{2k-1}{k} \frac{2k}{2k-k} \\ &= 4k \binom{2k-1}{k} \\ &= 2E(Q_{2k-1}), \end{aligned}$$

the proof is complete. \square

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