

EIGENVALUE PLACEMENT IN COMPLETIONS OF DAEs*

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Abstract. Differential algebraic equations (DAEs) are used to describe many physical processes. A completion of a DAE is an ordinary differential equation whose solutions include those of the DAE. Algorithms exist for designing stabilized completions of differential algebraic equations. Recent work on observers for DAEs has shown the need for more information on, and control of the placement of, the additional eigenvalues of the completion. This paper investigates this eigenvalue placement problem. Results are given relating the additional eigenvalues of the completion and the choice of stabilization matrix for certain important classes of linear DAEs.

Key words. Differential algebraic equations, Stability, Eigenvalue placement, Completions.

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1. Introduction. Differential algebraic equations (DAEs) are mixed systems of differential and algebraic equations $F(x', x, t) = 0$ for which the Jacobian $F_{x'}$ is identically singular. They have been extensively studied over the past 20 years because many problems in science and engineering are most naturally first formulated as DAEs [9, 21]. From the beginning of the consideration of DAEs, one natural approach was to imbed the solutions of the DAE into the solutions of an ODE which is called a completion of the DAE. The derivation of a completion always involves, in some way, the differentiation of some or all of the equations defining the DAE.

The first completions were found assuming explicit constraints and the primary use was in simulation of the DAE using ODE integrators. The completion always has additional dynamics (solutions) besides those of the DAE and it was observed that these additional dynamics could cause trouble for numerical simulations if their numerical solution lead to movement away from the solutions of the DAE. One remedy was the use of stabilized differentiation rather than just differentiation [3]. This proved very successful although there are theoretical concerns [1]. Subsequently there was work on finding completions for more general DAEs [10, 11, 22, 23, 24].

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DAEs occur in many control problems. An important tool in solving many control problems is the use of observers, which are dynamical systems used to asymptotically estimate the states of another system that are not directly accessible to measurement. With observers, you not only want to control the growth of error terms but you want to be able to specify the rate of convergence. If the convergence is too slow, the estimates will not be useful and if the convergence is too fast, the controllers will react to transient disturbances that should be ignored. Several authors have considered observer design and DAEs, for example, [4, 13, 14, 16, 17, 19, 25].

The goal of using completions in the design of observers lead first to additional research on stabilized completions and their rate of convergence [22, 23, 24] and then to research on the design of observers [5, 6, 7, 8]. This work considered both linear time invariant and nonlinear systems.

Two important problems in engineering are fault detection and fault identification. This is needed for both safety and performance reasons and is part of essentially all complex systems [20, 26]. One approach to fault detection and identification uses observers [30]. However, a new issue now arises. In a typical control problem, the system comes with its own eigenvalues. But with a completion additional eigenvalues are produced by the algorithm. If we want to detect a fault, it should be observable in some sense. But the ways previously given in the literature for designing stabilized completions tend to produce repeated eigenvalues and systems with repeated eigenvalues will have unobservable subspaces [2]. This leads to the need to be able to construct stabilized completions of DAEs for which the additional eigenvalues are distinct, have negative real part, and we have some control over where those eigenvalues are. This paper addresses the eigenvalue placement problem for completions of linear time invariant DAEs. The use of stabilized completions to build observers for fault detection is discussed in [28, 29].

This paper considers what is possible in terms of eigenvalue placement using the least squares completion. For a given application for which the results of this paper are not strong enough, alternative formulations or stabilization techniques may be needed.

2. Least squares stabilized completions. Two general approaches have been developed for computing stabilized completions of linear DAEs. We focus here on the one that is called the least squares completion. The other is called the alternative stabilized completion [23, 24] and is based on ideas from [21]. Linear time varying and nonlinear DAEs are of interest also, but as a first step we consider the linear time invariant case,

$$(2.1) \quad Ex' + Fx = f,$$

where E, F are square matrices, E is singular, there is a scalar s for which $sE + F$ is invertible, and x and f are functions of time t . Such DAEs are called solvable. E and F may be real or complex although in applications they are usually real. The vector x is an n dimensional vector which may also be real or complex. We shall use real notation in this paper. The complex case is covered by substituting the conjugate transpose for the transpose and unitary transformations for orthogonal transformations.

We know from DAE theory that the question is not whether to differentiate but rather where to differentiate and the best thing to differentiate are the known equations as opposed to computed quantities. Suppose that we apply the differential polynomial $D = \frac{d}{dt} + \lambda I$ to (2.1) k times. Usually it is assumed that λ is a real and positive scalar. Then putting each of these differentiated equations into a single equation, we get the system of equations called the derivative array. For convenience, we give here just the $k = 2$ case, but we consider more general cases in this paper. We have then

$$(2.2) \quad \mathcal{E}_\lambda z + \mathcal{F}_\lambda x = \hat{f}_\lambda,$$

where

$$\mathcal{E}_\lambda = \begin{bmatrix} E & 0 & 0 \\ F + \lambda E & E & 0 \\ 2\lambda F + \lambda^2 E & F + 2\lambda E & E \end{bmatrix}, \quad \mathcal{F}_\lambda = \begin{bmatrix} F \\ \lambda F \\ \lambda^2 F \end{bmatrix}$$

and

$$z = \begin{bmatrix} x' \\ x'' \\ x''' \end{bmatrix}, \quad \hat{f} = \begin{bmatrix} f \\ f' + \lambda f \\ f'' + 2\lambda f' + \lambda^2 f \end{bmatrix}.$$

If $\lambda = 0$, then we omit the subscript λ . To be fully consistent with our later notation we should write $\mathcal{E}_{\lambda I}$ but we omit the I .

There are several ways to define the index of a DAE. For the linear case, (2.1) with square coefficients the DAE is solvable if there is at least one value of parameter $\tilde{s}$ such that $\tilde{s}E + F$ is nonsingular. We assume our system is solvable. Then the index is the largest nilpotent block in the Kronecker form of the matrix pencil. Alternatively it is the number of differentiations needed to uniquely determine x' .

The matrix $\mathcal{E}_\lambda$ is rank deficient. If k is greater than or equal to the index of (2.1), then the first n entries of z can be determined from (2.2). But different answers are found depending on how the equations are solved [10]. Particularly in nonlinear or time varying systems where they have to be repeatedly solved, it is important to have some control over the solution. In the linear time invariant case, you still need

to know what the additional eigenvalues are. Accordingly we will use least squares solutions which have a number of nice properties. For the remainder of this paper, we assume k is greater or equal to the index of (2.2).

Solving (2.2) in the least squares sense we get the solution $\hat{z}$ is given by

$$(2.3) \quad \hat{z} = -\mathcal{E}_\lambda^\dagger \mathcal{F}_\lambda x + \mathcal{E}_\lambda^\dagger \hat{f}_\lambda.$$

$\mathcal{E}_\lambda^\dagger$ denotes the Moore-Penrose inverse of $\mathcal{E}_\lambda$ [12]. If x is a solution of the DAE, then the first n components of $\hat{z}$ are the derivative of x . The other components of $\hat{z}$ will usually not be the higher derivatives of x and will be ignored. Thus, taking the top n equations of (2.3), we get the completion

$$(2.4) \quad x' = \hat{A}_\lambda x + h_\lambda,$$

where $\hat{A}_\lambda$ is the top n rows of $-\mathcal{E}_\lambda^\dagger \mathcal{F}_\lambda$ and h_λ is the top n entries of $\mathcal{E}_\lambda^\dagger \hat{f}_\lambda$,

Let s be a scalar. Those s for which $\det(sE + F) = 0$ are called the finite generalized eigenvalues of the matrix pencil $sE + F$. The finite generalized eigenvalues play the same role for (2.1) that eigenvalues do for an ordinary differential equation. Let $\rho(\hat{A}_\lambda)$ be the eigenvalues of $\hat{A}_\lambda$ from (2.4) and $\rho(E, F)$ be the finite generalized eigenvalues of the matrix pencil $\{E, F\}$. Note that $\rho(I, -\hat{A}_\lambda) = \rho(\hat{A}_\lambda)$. Then

$$\rho(E, F) \subset \rho(\hat{A}_\lambda).$$

Let $\rho(E, F)^c$ be those complex numbers not in $\rho(E, F)$. Those $\hat{s} \in \rho(\hat{A}_\lambda) \cap \rho(E, F)^c$, are the additional eigenvalues from the completion and are what we are concerned with in this paper. The results in [22, 23] tell us that these additional eigenvalues will be $-\lambda$. They also tell us what their Jordan blocks will look like. Basically there will be Jordan blocks that are similar to the nilpotent blocks in the Kronecker form of $\{E, F\}$.

But this means the stabilized least squares completion will be producing repeated eigenvalues and that is something we would like to avoid if possible as pointed out in the introduction. Thus, we will modify the above procedure as follows. Instead of using $\frac{d}{dt} + \lambda I$ where $\operatorname{Re}(\lambda) > 0$, we shall use $\frac{d}{dt} + \Lambda$ where $\operatorname{Re}(s) > 0$ for all eigenvalues s of the $n \times n$ matrix Λ . Usually Λ will be chosen real but that is not necessary. Since we are only interested in the additional eigenvalues we will omit the nonhomogeneous terms f from (2.1) and h_λ from (2.4).

Let k be the index of (2.1) and define the $(k+1) \times (k+1)$ block matrix $\mathcal{E}$ by

$$\mathcal{E} = \begin{bmatrix} E & 0 & \ddots & \ddots & 0 \\ F & E & \ddots & \ddots & \vdots \\ 0 & F & E & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & F & E \end{bmatrix}$$

and the $(k+1) \times (k+1)$ block matrix $\mathcal{M}_\Lambda$, and $(k+1) \times 1$ block matrix $\mathcal{F}$ by

$$\mathcal{M}_\Lambda = \begin{bmatrix} I & 0 & \dots & \dots & 0 \\ \Lambda & I & \ddots & \dots & \vdots \\ \Lambda^2 & 2\Lambda & I & \ddots & \vdots \\ \Lambda^3 & 3\Lambda^2 & 3\Lambda & I & \vdots \\ * & * & * & * & \ddots \end{bmatrix}, \quad \mathcal{F} = \begin{bmatrix} F \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

In terms of previous notation, $\mathcal{E} = \mathcal{E}_0$ and $\mathcal{F} = \mathcal{F}_0$. Note that stabilized differentiation results in the coefficient matrices $\mathcal{E}_\Lambda = \mathcal{M}_\Lambda \mathcal{E}$ and $\mathcal{F}_\Lambda = \mathcal{M}_\Lambda \mathcal{F}$. Then the coefficient matrix of the Λ stabilized least squares completion is the top $n \times n$ block of

$$(2.5) \quad -\mathcal{E}_\Lambda^\dagger \mathcal{F}_\Lambda = -(\mathcal{M}_\Lambda \mathcal{E})^\dagger \mathcal{M}_\Lambda \mathcal{F},$$

which we denote $\hat{A}_\Lambda$. Whether Λ stabilized differentiation actually results in asymptotically stable additional dynamics is one question of interest in this paper.

Our interest here is only in the eigenvalues of $\hat{A}_\Lambda$. It is helpful to be able to simplify the matrices we are working with as long as the eigenvalues are not affected. Because we use a least squares in the calculation of our coefficient matrices we cannot use similarity to simplify our problems. We can however, use orthogonal transformations.

PROPOSITION 2.1. *If U, V are two $n \times n$ orthogonal matrices, then the eigenvalues of $\hat{A}_\Lambda$ for $\{E, F, \Lambda\}$ and $\hat{A}_{U\Lambda U^T}$ for $\{UEV, UFV, U\Lambda U^T\}$ are the same.*

Proof. Let $\mathcal{D}(Z)$ be a $(k+1)n \times (k+1)n$ block diagonal matrix with $n \times n$ matrix Z k times on the principle diagonal. Let $\tilde{\mathcal{E}}_{U\Lambda U^T}, \tilde{\mathcal{F}}_{U\Lambda U^T}$ be the large matrices of $\{UEV, UFV, U\Lambda U^T\}$.

Then from $\{UEV, UVF, U\Lambda U^T\}$ we get that (2.5) is $X = -\tilde{\mathcal{E}}_{U\Lambda U^T}^\dagger \tilde{\mathcal{F}}_{U\Lambda U^T}$, that is

$$\begin{aligned} X &= -(\mathcal{D}(U)\mathcal{M}_\Lambda\mathcal{D}(U^T)\mathcal{D}(U)\mathcal{E}\mathcal{D}(V))^\dagger (\mathcal{D}(U)\mathcal{M}_\Lambda\mathcal{D}(U^T)\mathcal{D}(U)\mathcal{F}\mathcal{D}(V)) \\ &= -(\mathcal{D}(U)\mathcal{M}_\Lambda\mathcal{E}\mathcal{D}(V))^\dagger (\mathcal{D}(U)\mathcal{M}_\Lambda\mathcal{F}\mathcal{D}(V)) \\ &= -\mathcal{D}(V^T)(\mathcal{M}_\Lambda\mathcal{E})^\dagger (\mathcal{M}_\Lambda\mathcal{F})\mathcal{D}(V) \end{aligned}$$

since $\mathcal{D}(U), \mathcal{D}(V)$ are orthogonal. Thus, the coefficient matrices for the two least squares completions are unitarily similar using V . $\square$

A key to using Proposition 2.1 is Theorem 3.1.

3. Explicit constraints. We have from Theorem 2 of [11] that adding extra equations to a stabilized derivative array by performing additional differentiation does not alter the completion. This gives us the following useful fact.

THEOREM 3.1. *Suppose that we have a solvable DAE*

$$(3.1) \quad \begin{bmatrix} E_1 \\ 0 \end{bmatrix} x' + \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} x = 0$$

with E_1 an $m \times n$ matrix. Let $\Lambda = \text{diag}\{P_1, P_2\}$ be an $n \times n$ block diagonal matrix with P_2 a $(n-m) \times (n-m)$ matrix. Then the additional eigenvalues of the Λ stabilized least squares completion of (3.1) include the eigenvalues of $-P_2$.

Proof. The proof consists of noting that if we form the $\Lambda = \text{diag}\{P_1, P_2\}$ stabilized derivative array for (3.1), then it is also a derivative array for

$$\begin{bmatrix} E_1 \\ F_2 \end{bmatrix} x' + \begin{bmatrix} F_1 \\ P_2 F_2 \end{bmatrix} x = 0.$$

Thus, the terms $w = F_2 x$ where x is from the completion, satisfy the differential equation $w' + P_2 w = 0$ and hence $\rho(-P_2)$ will be among the additional eigenvalues. $\square$

4. Index one systems. The first class of systems that we consider are the index one systems. They are of interest in their own right and also because they will serve as building blocks for our later results. There is a special case of an index one system which is a purely algebraic system. We consider it separately. For a purely algebraic homogeneous system, we have simply

$$Cx = 0$$

with C invertible. The augmented matrix for the Λ stabilized derivative array equations is then

$$\left[\begin{array}{cc|c} 0 & 0 & -C \\ C & 0 & -\Lambda C \end{array} \right]$$

and the Λ stabilized least squares completion is given by

$$x' = -C^{-1}\Lambda Cx.$$

In this simple case, the new eigenvalues are precisely those of $-\Lambda$ and to get a given set of eigenvalues, we only have to choose Λ accordingly. We shall see that the behavior of more complex systems can be different in several ways and this example is not indicative of the general case.

Suppose then that (2.1) is a general solvable homogeneous index one system with $E \neq 0$. There exist orthogonal transformations U, V which can be computed from the singular value decomposition (SVD) of E , so that letting $x = Vw$ and premultiplying by U gives

$$(4.1a) \quad Jw'_1 = Aw_1 + Bw_2$$

$$(4.1b) \quad 0 = Cw_1 + Dw_2$$

where J is invertible. In fact, J can be taken positive definite if the SVD is used. Since orthogonal transformations are used, by Proposition 2.1 it suffices to consider (4.1) instead of (2.1). D is invertible by the index one assumption. Note that solving the second equation for w_2 and substituting into the first equation gives $w'_1 = J^{-1}(A - BD^{-1}C)w_1$ so that the finite pencil eigenvalues are $\rho(J^{-1}A - J^{-1}BD^{-1}C)$.

Since orthogonal changes of coordinates were used, for a given Λ we can consider the $U\Lambda U^T$ stabilized completion of (4.1). This leads to the following fundamental result.

THEOREM 4.1. *Suppose that the index one DAE (2.1) with $E \neq 0$ is put into the form (4.1) by orthogonal transformations U, V . Suppose that*

$$U\Lambda U^T = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$$

and a Λ stabilized completion of the original DAE (2.1) was computed. Then the additional eigenvalues of the least squares completion stabilized by Λ are the eigenvalues of $-P_4$.

Proof. Since the DAE is index one we can take $k = 1$. The derivative array augmented matrix $[\mathcal{ME} \mid -\mathcal{MF}]$ is

$$(4.2) \quad \left[\begin{array}{cccc|cc} J & 0 & 0 & 0 & A & B \\ 0 & 0 & 0 & 0 & C & D \\ P_1J - A & -B & J & 0 & P_1A + P_2C & P_1B + P_2D \\ P_3J - C & -D & 0 & 0 & P_3A + P_4C & P_3B + P_4D \end{array} \right].$$

Note that we cannot just cancel the $\mathcal{M}$ in $[\mathcal{M}\mathcal{E} \mid \mathcal{M}\mathcal{F}]$ since that alters the least squares solution. We only care about the top two entries of the solution so we can drop the fourth column of (4.2). Switching the second row to the bottom is allowed since that is also an orthogonal operation. Thus, we only have to solve

$$(4.3) \quad \left[\begin{array}{ccc|cc} J & 0 & 0 & A & B \\ P_1J - A & -B & J & P_1A + P_2C & P_1B + P_2D \\ P_3J - C & -D & 0 & P_3A + P_4C & P_3B + P_4D \end{array} \right]$$

in the least squares sense. But (4.3) has a unique solution since the left side is invertible. The invertibility follows since it is known that if k is greater or equal to the index of the DAE, then the nullity of $\mathcal{E}$ is the same as the number of algebraic constraints. For an index one DAE that is the same as the number of zero rows, we have deleted to get an invertible left hand side. We have reduced our least squares problem to a standard nonsingular problem which frees us to use a wider variety of operations. The second equation in (4.3) only gives the x_1'' term which we do not care about. So we now have to solve

$$\left[\begin{array}{cc|cc} J & 0 & A & B \\ P_3J - C & -D & P_3A + P_4C & P_3B + P_4D \end{array} \right]$$

or

$$(4.4) \quad \left[\begin{array}{cc|cc} I & 0 & J^{-1}A & J^{-1}B \\ P_3J - C & -D & P_3A + P_4C & P_3B + P_4D \end{array} \right].$$

Adding $C - P_3J$ times row 1 to row 2 in (4.4) gives

$$\left[\begin{array}{cc|cc} I & 0 & J^{-1}A & J^{-1}B \\ 0 & -D & P_3A + P_4C + (C - P_3J)J^{-1}A & P_3B + P_4D + (C - P_3J)J^{-1}B \end{array} \right],$$

or equivalently,

$$\left[\begin{array}{cc|cc} I & 0 & J^{-1}A & J^{-1}B \\ 0 & -D & P_4C + CJ^{-1}A & P_4D + CJ^{-1}B \end{array} \right].$$

Thus, the eigenvalues of the completion are given by the eigenvalues of

$$(4.5) \quad \left[\begin{array}{cc} J^{-1}A & J^{-1}B \\ -D^{-1}(P_4C + CJ^{-1}A) & -D^{-1}P_4D - D^{-1}CJ^{-1}B \end{array} \right].$$

One way to reveal the eigenvalues of (4.5) is to do a similarity which makes (4.5) block upper triangular. We see a $-D^{-1}P_4D$ in the lower right corner of (4.5). If we had that as a diagonal block that would be great since the new eigenvalues then

would be $\rho(-P_4)$. Thus, we add $D^{-1}C$ times the top row to the second. We need this to be part of a similarity so we compute

$$\begin{bmatrix} J^{-1}A & J^{-1}B \\ -D^{-1}P_4C & -D^{-1}P_4D \end{bmatrix} \begin{bmatrix} I & 0 \\ -D^{-1}C & I \end{bmatrix}$$

which is

$$(4.6) \quad \begin{bmatrix} J^{-1}A - J^{-1}BD^{-1}C & J^{-1}B \\ 0 & -D^{-1}P_4D \end{bmatrix}.$$

Matrix (4.6) is upper triangular which proves that

$$\rho(\hat{A}_\Lambda) = \rho(J^{-1}A - J^{-1}BD^{-1}C) \cup \rho(-P_4)$$

as proposed. $\square$

From Theorem 4.1 we get several corollaries. The first recovers a special case of the classical result already known in the literature [22, 23].

COROLLARY 4.2. *Suppose the DAE is index one and Λ is a multiple λ of the identity. Then the additional eigenvalues of the Λ stabilized least squares completion are just $-\lambda$.*

Proof. This follows since $U\lambda IU^T = \lambda I$ for any orthogonal matrix U . $\square$

COROLLARY 4.3. *If the DAE is index one and Λ is positive definite, then all the additional eigenvalues of the Λ stabilized least squares completion will have negative real part. It is not possible to say what the eigenvalues will be without additional computation but one can say that their absolute value will be less than the largest eigenvalue of Λ .*

Proof. If Λ is positive definite, then P_4 will also be positive definite and a principle minor of $U\Lambda U^T$. The conclusion now follows. $\square$

COROLLARY 4.4. *Let (2.1) be any index one system which has at least one finite eigenvalue. That is, it is not purely algebraic. Then there is a matrix Λ , all of whose eigenvalues are positive, such that the Λ stabilized additional dynamics are not stable.*

Proof. Construct a matrix Λ with the P_4 having an eigenvalue which is negative but the matrix Λ has positive eigenvalues. $\square$

EXAMPLE 4.5. As a simple illustration of Corollary 4.4 take

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, F = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \Lambda = \begin{bmatrix} 4 & -2 \\ 5 & -2 \end{bmatrix}.$$

Then the pencil $\{E, F\}$ has one finite eigenvalue 0.5, Λ has eigenvalues $1 \pm i$ which have positive real part and the Λ stabilized least squares completion has eigenvalues

$\{0.5, 2.0\}$ as predicted by Theorem 4.1. The additional eigenvalue 2 does not have negative real part.

5. Hessenberg index two. The previous section shows that in the index one case the additional eigenvalues are determined by a specific submatrix and that submatrix is based on the structure of the coefficients. This suggests that in more general cases we should look at various representations of the structure that the coefficients of a DAE can have. One such structure is the Hessenberg DAE which arises in mechanics and other areas [9, 21]. A linear homogeneous Hessenberg index two DAE has the form

$$(5.1a) \quad Jx'_1 = Ax_1 + Bx_2$$

$$(5.1b) \quad 0 = Cx_1$$

where it is assumed that $CJ^{-1}B$ is invertible. In particular, we also have B is full column rank and C is full row rank.

The finite pencil eigenvalues in the index two case are a bit harder to describe than in the index one case. Note that if C, B are invertible, then $x_1 = x_2 = 0$ and there are no finite eigenvalues. So suppose that C is just full row rank. Then doing an orthogonal change of coordinates on x_1 we can get $CU = [C_1 \ 0]$ and the DAE (5.1) becomes (5.2) after multiplying by J^{-1} (we reuse the name x for our variables)

$$(5.2a) \quad x'_1 = A_{11}x_1 + A_{12}x_2 + B_1x_3$$

$$(5.2b) \quad x'_2 = A_{21}x_1 + A_{22}x_2 + B_2x_3$$

$$(5.2c) \quad 0 = C_1x_1$$

and C_1, B_1 are invertible. Furthermore, we may assume that C_1 is positive definite. This will be convenient later.

But this just says that $x_1 = 0$. Then our system of equations (5.2) becomes

$$(5.3a) \quad 0 = A_{12}x_2 + B_1x_3$$

$$(5.3b) \quad x'_2 = A_{22}x_2 + B_2x_3.$$

But since B_1 is invertible, this is a semi-explicit index one DAE written in the reverse order and we know the finite eigenvalues are $\rho(A_{22} - B_2B_1^{-1}A_{12})$.

Suppose that we take

$$(5.4) \quad \Lambda = \text{diag}\{P_1, P_2\},$$

where Λ is partitioned conformally with (5.1). We know that the eigenvalues of $-P_2$ will be included in the eigenvalues of the stabilized completion by Theorem 3.1. The

augmented matrix of the derivative array equations for (5.1) is,

$$(5.5) \quad \left[\begin{array}{cccccc|cc} J & 0 & 0 & 0 & 0 & 0 & A & B \\ 0 & 0 & 0 & 0 & 0 & 0 & C & 0 \\ -A + P_1 J & -B & J & 0 & 0 & 0 & P_1 A & P_1 B \\ -C & 0 & 0 & 0 & 0 & 0 & P_2 C & 0 \\ -2P_1 A + P_1^2 J & -2P_1 B & -A + 2P_1 J & -B & J & 0 & P_1^2 A & P_1^2 B \\ -2P_2 C & 0 & -C & 0 & 0 & 0 & P_2^2 C & 0 \end{array} \right].$$

In taking a least squares solution, we may ignore zero rows or columns of the left side of (5.5). Thus, we can drop the second row and sixth column of (5.5). Reordering of rows is an orthogonal operation. Thus, the first, fourth, third, fifth, and sixth rows give us

$$(5.6) \quad \left[\begin{array}{cccc|cc} J & 0 & 0 & 0 & A & B \\ -C & 0 & 0 & 0 & P_2 C & 0 \\ -A + P_1 J & -B & J & 0 & P_1 A & P_1 B \\ -2P_2 C & 0 & -C & 0 & P_2^2 C & 0 \\ -2P_1 A + P_1^2 J & -2P_1 B & -A + 2P_1 J & -B & P_1^2 A & P_1^2 B \end{array} \right].$$

But

$$\left[\begin{array}{ccc|cc} J & 0 & 0 & 0 & 0 \\ -C & 0 & 0 & 0 & 0 \\ -A + P_1 J & -B & J & 0 & 0 \\ -2P_2 C & 0 & -C & 0 & 0 \\ \hline -2P_1 A + P_1^2 J & -2P_1 B & -A + 2P_1 J & -B & J \end{array} \right]$$

is a block lower triangular matrix whose (1,1) block is full column rank by the Hessenberg assumption and whose (2,2) block is full row rank since J is invertible. Then from Theorem 3.4.1 of [12], we have its Moore-Penrose generalized inverse is also block lower triangular. Since we are only concerned with the first two block variables of (5.6), we may consider just the augmented matrix derivative array

$$(5.7) \quad \left[\begin{array}{cccc|cc} J & 0 & 0 & 0 & A & B \\ -C & 0 & 0 & 0 & P_2 C & 0 \\ -A + P_1 J & -B & J & 0 & P_1 A & P_1 B \\ -2P_2 C & 0 & -C & 0 & P_2^2 C & 0 \end{array} \right].$$

But (5.7) is the augmented matrix for the Λ derivative array of

$$\begin{bmatrix} J & 0 \\ -C & 0 \end{bmatrix} x' + \begin{bmatrix} -A & -B \\ -P_2 C & 0 \end{bmatrix} x = 0$$

which is index one by the index two Hessenberg assumption.

If C is a full row rank matrix, let

$$(5.8) \quad \Delta = \sqrt{J^T J + C^T C}, \quad \hat{\Delta} = \sqrt{I + C J^{-1} J^{-T} C^T}$$

and define U_C by

$$U_C = \begin{bmatrix} \Delta^{-1} J^T & -\Delta^{-1} C^T \\ \hat{\Delta}^{-1} C J^{-1} & \hat{\Delta}^{-1} \end{bmatrix}.$$

Then U_C is an orthogonal matrix since $U_C U_C^T = I$ and

$$(5.9) \quad U_C \begin{bmatrix} J \\ C \end{bmatrix} = \begin{bmatrix} \Delta \\ 0 \end{bmatrix}.$$

Now we can use (5.9) and (5.4) and apply Theorem 4.1 to get that the key entry is the (2,2) block of

$$U_C \Lambda U_C^T = \begin{bmatrix} \Delta^{-1} J^T & -\Delta^{-1} C^T \\ \hat{\Delta}^{-1} C J^{-1} & \hat{\Delta}^{-1} \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} J \Delta^{-1} & J^{-T} C^T \hat{\Delta}^{-1} \\ -C \Delta^{-1} & \hat{\Delta}^{-1} \end{bmatrix}.$$

Thus, we have proved the following.

THEOREM 5.1. *Suppose that we have the Hessenberg index two DAE (5.1) where C is full row rank. Let Λ be given by (5.4) and let $\hat{\Delta}$ be given by (5.8). Then the additional eigenvalues of the Λ stabilized least squares completion consist of the eigenvalues of $-P_2$, and the eigenvalues of*

$$-\hat{\Delta}^{-1} [C J^{-1} P_1 J^{-T} C^T + P_2] \hat{\Delta}^{-1}.$$

COROLLARY 5.2. *If Λ is positive definite, then the Λ stabilized least squares completion of the homogeneous index two Hessenberg DAE of Theorem 5.1 has stabilized additional dynamics.*

6. Hessenberg index three. The Hessenberg system of index three is very important because of its frequent appearance in constrained mechanics [15, 18, 27]. The linear homogeneous version is

$$(6.1a) \quad x'_1 = A_{11}x_1 + A_{12}x_2 + A_{13}x_3$$

$$(6.1b) \quad x'_2 = A_{21}x_1 + A_{22}x_2$$

$$(6.1c) \quad 0 = A_{32}x_2$$

where $A_{32}A_{21}A_{13}$ is nonsingular.

We ran a number of tests computing the Λ stabilized completion of (6.1) where Λ was block diagonal with randomly generated positive definite diagonal blocks and the A_{ij} were randomly generated. While many of the eigenvalues of $\hat{A}$ had negative real part, we frequently saw an eigenvalue with positive real part. Thus, the analogue of Corollary 4.3 does not hold for index three Hessenberg systems. When we generated random diagonal positive definite P_i , we frequently saw stabilization. However with enough experimentation, we did find examples where stabilization did not occur.

EXAMPLE 6.1. Let the coefficients of the index three Hessenberg DAE be

$$F = \begin{bmatrix} -0.8005 & -0.8214 & -0.3545 & -0.5722 & -0.7425 & -0.3891 \\ -0.1051 & -0.8411 & -0.4301 & -0.7008 & -0.7579 & -0.4293 \\ -0.9563 & -0.8497 & -0.6223 & -0.9635 & 0 & 0 \\ -0.5730 & -0.2763 & -0.5884 & -0.0859 & 0 & 0 \\ 0 & 0 & -0.5005 & -0.0902 & 0 & 0 \\ 0 & 0 & -0.5216 & -0.9047 & 0 & 0 \end{bmatrix},$$

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Take $\Lambda = \text{diag}(P_1, P_2, P_3)$ $P_1 = \text{diag}(0.2211, 0.4390)$, $P_2 = \text{diag}(0.0156, 0.0045)$, and $P_3 = \text{diag}(0.0018, 0.3909)$. There are no finite eigenvalues. Then the eigenvalues of $\hat{A}_\Lambda$ are $\{-0.4715, -0.3909, -0.0562, 0.0321, 0.0192, -0.0018\}$ of which two are positive.

There remains a way to get distinct additional eigenvalues with negative real part. We know that if we take Λ to be a multiple of the identity, then the additional eigenvalues will all be minus this multiple. We also know the size of the Jordan blocks. For an index three Hessenberg system they are 3×3 Jordan blocks. The eigenvalues of $\hat{A}$ are continuous in Λ . Thus, if we perturb the diagonal entries a small amount from λ the additional eigenvalues will still have negative real part. Most, but not all perturbations will produce distinct additional eigenvalues.

EXAMPLE 6.2. Returning to Example 6.1, let each P_i be $\text{diag}(0.3, 0.3)$. Then the eigenvalues of $\hat{A}_\Lambda$ are all -0.3 . Suppose that we then perturb the diagonals, for example, to give $P_1 = \text{diag}(0.2, 0.3)$, $P_2 = \text{diag}(0.25, 0.35)$, and $P_3 = \text{diag}(0.32, 0.40)$. Then the eigenvalues of $\hat{A}_\Lambda$ turn out to be $\{-0.2387, -0.4366, -0.4000, -0.3200, -0.3391 \pm 0.0259i\}$.

Of course, it is not enough just to get distinct eigenvalues since one must also worry about conditioning of the observability matrix.

7. Conclusion. This paper has examined the relationship between the stabilizing weight Λ which occurs in the algorithm for computing a least squares completion of a DAE and the additional eigenvalues of the computed completions. This relationship is explicitly developed for the important classes of general index one DAEs and for Hessenberg index two DAEs. It is shown for index three Hessenberg DAEs that having positive definite weights is no longer sufficient to guarantee that the additional eigenvalues have negative real part. A way around this that could be helpful on some problems is presented.

REFERENCES

- [1] U. Ascher, H. Chin, and S. Reich. Stabilization of DAEs and invariant manifolds. *Numerische Mathematik*, 67:131–149, 1994.
- [2] F.L. Lewis and V. Syrmos. *Optimal Control*, second edition. John Wiley & Sons, New York, 1995.
- [3] J. Baumgarte. Stabilization of constraints and integrals of motion in dynamical systems. *Computer Methods in Applied Mechanics and Engineering*, 1:1–16, 1972.
- [4] F. Blanchini. Eigenvalue assignment via state observer for descriptor systems. *Kybernetika*, 27:384–392, 1991.
- [5] K. Bobinyec, S.L. Campbell, and P. Kunkel. Stabilized completions of differential algebraic equations and the design of observers. *Proceedings of Neural, Parallel, and Scientific Computation*, Atlanta, pp. 59–64, 2010.
- [6] K. Bobinyec, S.L. Campbell, and P. Kunkel. Maximally reduced observers for linear time varying DAEs. *Proceedings of the IEEE Multi-Conference on Systems and Control*, Denver, pp. 1373–1378, 2011.
- [7] K. Bobinyec, S.L. Campbell, and P. Kunkel. Full order observers for linear DAEs. *Proceedings of the IEEE Conference on Decision and Control and European Control Conference*, Orlando, pp. 4011–4016, 2011.
- [8] K. Bobinyec, S.L. Campbell, and P. Kunkel. Constructing observers for linear time varying DAEs. *Proceedings of the IEEE Conference on Decision and Control*, Maui, pp. 5749–5754, 2012.
- [9] K.E. Brennan, S.L. Campbell, and L.R. Petzold. *Numerical Solution of Initial Value Problems in Differential-Algebraic Equations*. SIAM, Philadelphia, 1996.
- [10] S.L. Campbell. Uniqueness of completions for linear time varying differential algebraic equations. *Linear Algebra and its Applications*, 161:55–67, 1992.
- [11] S.L. Campbell. Least squares completions for nonlinear differential algebraic equations. *Numerische Mathematik*, 65:77–94, 1993.
- [12] S.L. Campbell and C.D. Meyer, Jr. *Generalized Inverses of Linear Transformations*. SIAM Classics, Philadelphia, 2009.
- [13] L. Dai. *Singular Control Systems*. Lecture Notes in Control and Information Sciences, Springer-Verlag, Berlin 1991.
- [14] M. Darouach and M. Boutayeb. Design of observers for descriptor systems. *IEEE Transactions on Automatic Control*, 40:1323–1327, 1995.
- [15] E. Eich-Soellner and C. Führer. *Numerical Methods in Multibody Dynamics*, B. G. Tuebner, Stuttgart, 1998, reprint Lund, 2008.
- [16] M.M. Fahmy and J. O'Reilly. Observers for descriptor systems. *International Journal of Control*, 49:2013–2028, 1989.

- [17] Z. Gao and H. Wang. Descriptor observer approaches for multivariable systems with measurement noises and application in fault detection and diagnosis. *Systems & Control Letters*, 4:304–313, 2006.
- [18] E.J. Haug and R.C. Deyo (eds.). *Real-time Integration Methods for Mechanical System Simulation*. Springer-Verlag, Berlin, 1991.
- [19] M. Hou and P.C. Müller. Observer design for descriptor systems. *IEEE Transactions on Automatic Control*, 44:164–168, 1999.
- [20] R. Iserman. *Fault Diagnosis Systems: An Introduction from Fault Detection to Fault Tolerance*. Springer, Berlin 2006.
- [21] P. Kunkel and V. Mehrmann. *Differential-Algebraic Equations. Analysis and Numerical Solution*. European Mathematical Society, Zürich, 2006.
- [22] I. Okay, S.L. Campbell, and P. Kunkel. The additional dynamics of least squares completions for linear differential algebraic equations. *Linear Algebra and its Applications*, 425:471–485, 2007.
- [23] I. Okay, S.L. Campbell, and P. Kunkel. Completions of implicitly defined vector fields and their applications. *Proceedings of the International Symposium on Mathematical Theory of Networks and Systems (MTNS 08)*, Blacksburg, 2008.
- [24] I. Okay, S.L. Campbell, and P. Kunkel. Completions of implicitly defined linear time varying vector fields. *Linear Algebra and its Applications*, 431:1422–1438, 2009.
- [25] J. O'Reilly. *Observers for Linear Systems*. Academic Press, Inc., Orlando, 1983.
- [26] R.J. Patton, P.M. Frank, and R.N. Clark (eds.). *Issues of Fault Diagnosis for Dynamic Systems*. Springer-Verlag, London, 2000.
- [27] P.J. Rabier and W.C. Rheinboldt. *Nonholonomic Motion of Rigid Mechanical Systems from a DAE Viewpoint*. SIAM, Philadelphia, 2000.
- [28] J.R. Scott and S.L. Campbell. Observer based fault detection in differential algebraic equations. *Proceedings of the SIAM Conference on Control and Its Applications*, San Diego, pp. 176–183, 2013.
- [29] J.R. Scott and S.L. Campbell. Observer based fault detection and identification in differential algebraic equations. *Proceedings of the ASME 2013 Dynamic Systems and Control Conference*, Palo Alto, 2013, to appear.
- [30] M. Staroswiecki and B. Jiang. Fault identification for a class of linear systems based on adaptive observer. *Proceedings of 40th IEEE Conference on Decision and Control*, 2001.