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ON PROPERTIES OF THE GENERALIZED MAJORIZATION*

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Abstract. In this paper, a complete solution of a problem involving generalized majorization of partitions is given: for two pairs of partitions $(\mathbf{d},\mathbf{a})$ and $(\mathbf{c},\mathbf{b})$ necessary and sufficient conditions for the existence of a partition $\mathbf{g}$ that is majorized by both pairs is determined. The obtained conditions are explicit, the solution is constructive and it uses novel techniques and indices. Although the problem is motivated by the applications in matrix pencil completions problems, all results are purely combinatorial and they give a new perspective on comparison of partitions.

Key words. Partitions, Generalized majorization, Matrix completion.

AMS subject classifications. 05A17, 15A83.

1. Introduction. The concept of majorization of partitions turned out to be a powerful tool in matrix and matrix pencils completion problems.

Let $\mathbf{a} := (a_1, \ldots, a_s)$ and $\mathbf{w} := (w_1, \ldots, w_s)$ be two partitions, i.e., two finite non-increasing sequences of integers. *If*

$$\sum_{i=1}^{j} w_i \leq \sum_{i=1}^{j} a_i, \text{ for all } j = 1, \ldots, s,$$

and

$$\sum_{i=1}^{s} w_i = \sum_{i=1}^{s} a_i,$$

then we say that $\mathbf{w}$ *is majorized by* $\mathbf{a}$*, and write*

$$\mathbf{w} \prec \mathbf{a}.$$

This, *classical*, majorization is well studied and there are many results concerning its properties [3, 12, 13, 16]. It also appears in matrix completion problems, see e.g., [2, 11, 17, 18], in particular, to express conditions for matrix row completion problem up to a square matrix.

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However, by using classical majorization one cannot compare more than two partitions. For that reason in [4] the notion *generalized majorization* was introduced. It deals with triples of partitions $\mathbf{a}$, $\mathbf{d}$ and $\mathbf{g}$, such that the length of $\mathbf{g}$ is equal to the sum of the lengths of $\mathbf{a}$ and $\mathbf{d}$. The generalized majorization is denoted by

$$\mathbf{g} \prec' (\mathbf{d}, \mathbf{a}).$$

In the case when $\mathbf{d}$ is a partition of length zero, the generalized majorization becomes the classical majorization between partitions $\mathbf{g}$ and $\mathbf{a}$.

This majorization was motivated by results in matrix completion problems in the case of column or row completion of rectangular matrices. As to our knowledge, it first appeared in [1, 15], and later on in [5, 8, 9]. In [9] some combinatorial properties of the generalized majorization have been obtained, including the generalization of the elementary operations for classical majorization.

The generalized majorization turns out to be a very convenient way of writing the conditions of the results in completion problems involving both row and column minimal indices.

In this paper, we give a complete solution to the following problem:

Problem 1. *Let* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{c}$ *and* $\mathbf{d}$ *be partitions such that the sum of the lengths of* $\mathbf{a}$ *and* $\mathbf{d}$ *is equal to the sum of the lengths of* $\mathbf{b}$ *and* $\mathbf{c}$*.*

Find necessary and sufficient conditions for the existence of a partition $\mathbf{g}$*, such that*

$$\mathbf{g} \prec' (\mathbf{d}, \mathbf{a})$$
$$\mathbf{g} \prec' (\mathbf{c}, \mathbf{b}).$$

Problem 1 is a hard and challenging combinatorial task. The obtained conditions involve novel indices and labels on the partitions $\mathbf{c}$ and $\mathbf{d}$ introduced in Section 3. In fact, Problem 1 was inspired by matrix and matrix pencils completion problems. Its solution enables new approach to those problems, since it allows studying relations between partitions made of column (or row) minimal indices of the pencils involved. For instance, a solution of a particular case of Problem 1 when $\mathbf{b} = 0$ was obtained in [5, 8].

We expect more applications of this result. In particular, its impact on the general rank distance problem [7], as well as to the general matrix pencils completion problem [14] and its particular cases [6, 7, 10] is crucial.

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1.1. Notation. For any sequence of integers satisfying $c_1 \geq \cdots \geq c_m$, by $\sum_{c_i \leq a} c_i$ we mean the sum of all the integers c_i that are less than or equal to a. We put $\sum_{i \in W} c_i = 0$ whenever W is an empty set. Also, we assume that $\sum_{i=a}^{b} c_i = 0$ if $a > b$. Finally, we assume $c_0 = +\infty$ and $c_{m+1} = c_{m+2} = \cdots = -\infty$.

Let x be an integer such that $c_j \geq x \geq c_{j+1}$, for some $j \in \{0, \ldots, m\}$. And let $\mathbf{c} = (c_1, \ldots, c_m)$. Then by $\mathbf{c} \cup \{x\}$ we mean the partition $\bar{\mathbf{c}} = (c_1, \ldots, c_j, x, c_{j+1}, \ldots, c_m)$. Also, $\bar{\mathbf{c}} \setminus \{x\} := \mathbf{c}$.

2. Generalized majorization. Generalized majorization presents a generalization of the majorization in Hardy-Littlewood-Pólya sense [12]. It deals with three partitions such that the length of one of them is equal to the sum of the lengths of another two.

Let m and s be nonnegative integers, and let $d_1 \geq \cdots \geq d_m$, $g_1 \geq \cdots \geq g_{m+s}$, and $a_1 \geq \cdots \geq a_s$ be integers.

Consider partitions

$$\mathbf{a} = (a_1, \ldots, a_s), \tag{2.1}$$

$$\mathbf{d} = (d_1, \ldots, d_m) \tag{2.2}$$

and

$$\mathbf{g} = (g_1, \ldots, g_{m+s}). \tag{2.3}$$

Definition 2.1. If

$$d_i \geq g_{i+s}, \qquad i = 1, \ldots, m, \tag{2.4}$$

$$\sum_{i=1}^{h_j} g_i - \sum_{i=1}^{h_j - j} d_i \leq \sum_{i=1}^{j} a_i, \qquad j = 1, \ldots, s, \tag{2.5}$$

$$\sum_{i=1}^{m+s} g_i = \sum_{i=1}^{m} d_i + \sum_{i=1}^{s} a_i, \tag{2.6}$$

where $h_j := \min\{i | d_{i-j+1} < g_i\}$, $j = 1, \ldots, s$, then we say that $\mathbf{g}$ is *majorized* by $\mathbf{d}$ and $\mathbf{a}$. This type of majorization we call *the generalized majorization*, and we write

$$\mathbf{g} \prec' (\mathbf{d}, \mathbf{a}).$$

As to our knowledge, this type of majorization was first considered in [1, 15], and later on under this name in [4, 5, 6, 8, 9].

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If $m = 0$, i.e., if $\mathbf{d}$ is an empty partition, then the generalized majorization between $\mathbf{g}$ and $(\mathbf{d}, \mathbf{a})$, becomes a classical majorization between $\mathbf{g}$ and $\mathbf{a}$.

We note that, if (2.6) is satisfied, then (2.5) is equivalent to the following:

$$\sum_{i=h_j+1}^{m+s} g_i \geq \sum_{i=h_j-j+1}^{m} d_i + \sum_{i=j+1}^{s} a_i, \quad j = 1, \ldots, s. \tag{2.7}$$

Also, we note that from the definition of h_j's we have:

$$0 < h_1 < h_2 < \cdots < h_s < m + s + 1,$$

and we set $h_0 := 0$ and $h_{s+1} := m + s + 1$.

There is an additional property given in [8, Lemma 4.2]:

LEMMA 2.2. *Suppose that $d_1 \geq \cdots \geq d_m$, $g_1 \geq \cdots \geq g_{m+s}$ and $a_1 \geq \cdots \geq a_s$ satisfy (2.4) and (2.7). Let u be such that $h_j < u \leq h_{j+1}$, for some $j \in \{0, \ldots, s\}$. Then the following is also valid:*

$$\sum_{i=u}^{m+s} g_i \geq \sum_{i=u-j}^{m} d_i + \sum_{i=j+1}^{s} a_i.$$

Various combinatorial properties of the generalized majorization have been obtained in [9]. In this paper, we are focusing on different properties and aspects of generalized majorization.

2.1. Basic properties of the generalized majorization. In this subsection, we show some properties of generalized majorization that enables simplifying partitions involved. More precisely, the results of the following lemmas will allow that without loss of generality Problem 1 can be considered in the case when the partitions $\mathbf{c}$ and $\mathbf{d}$ do not have the same elements.

LEMMA 2.3. *Let $\mathbf{a}$, $\mathbf{d}$ and $\mathbf{g}$ be partitions as given in (2.1)–(2.3). Let x be an integer and let $\mathbf{g}' := \mathbf{g} \cup \{x\}$ and $\mathbf{d}' := \mathbf{d} \cup \{x\}$. Then*

$$\mathbf{g} \prec' (\mathbf{d}, \mathbf{a}) \tag{2.8}$$

if and only if

$$\mathbf{g}' \prec' (\mathbf{d}', \mathbf{a}). \tag{2.9}$$

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Proof. Let $u = \max\{i | g_i > x\}$. Then

$$g_u > x \geq g_{u+1}. \tag{2.10}$$

Let $l = \max\{i | d_i > x\}$. Then

$$d_l > x \geq d_{l+1}. \tag{2.11}$$

Let $\mathbf{g}' = (g'_1, \ldots, g'_{m+s+1})$ and $\mathbf{d}' = (d'_1, \ldots, d'_{m+1})$. From (2.10) and (2.11) we have

$$\begin{array}{llllllll} \mathbf{g}': & g_1 \geq & \cdots \geq & g_u > & x \geq & g_{u+1} \geq & \cdots & \cdots \geq & g_{m+s}, \\ \mathbf{d}': & d_1 \geq & \cdots \geq & d_l > & x \geq & d_{l+1} \geq & \cdots \geq & d_m, & \end{array}$$

i.e.,

$$\begin{array}{llll} g'_i = g_i, & i \leq u, & d'_i = d_i, & i \leq l, \\ g'_{u+1} = x, & & d'_{l+1} = x, & \\ g'_i = g_{i-1}, & i \geq u+2, & d'_i = d_{i-1}, & i \geq l+2. \end{array} \tag{2.12}$$

Let $h_j := \min\{i | d_{i-j+1} < g_i\}$, $j = 1, \ldots, s$ ($h_0 := 0, h_{s+1} := m+s+1$), and $h'_j := \min\{i | d'_{i-j+1} < g'_i\}$, $j = 1, \ldots, s$ ($h'_0 := 0, h'_{s+1} := m+s+2$).

Let $\alpha \in \{0, \ldots, s\}$ be such that $h_\alpha \leq u < h_{\alpha+1}$, i.e., such that $g_{h_\alpha} > x \geq g_{h_\alpha+1}$. Then by the definition of h_α and $h_{\alpha+1}$, we have

$$d_{h_\alpha-\alpha} \geq g_{h_\alpha-1} \geq g_{h_\alpha} > x \geq g_{h_{\alpha+1}} > d_{h_{\alpha+1}-\alpha}. \tag{2.13}$$

By (2.12), (2.13) implies $u \leq l + \alpha$, and thus we have

$$h'_j = h_j, \quad j \leq \alpha, \tag{2.14}$$

$$h'_j = h_j + 1, \quad j > \alpha. \tag{2.15}$$

Now, suppose that (2.8) is valid. From the definition of the generalized majorization, we have that (2.8) is equivalent to (2.4), (2.5) and (2.6).

In order to prove (2.9), we are left with proving

$$d'_i \geq g'_{i+s}, \qquad i = 1, \ldots, m+1, \tag{2.16}$$

$$\sum_{i=1}^{h'_j} g'_i - \sum_{i=1}^{h'_j-j} d'_i \leq \sum_{i=1}^{j} a_i, \qquad j = 1, \ldots, s, \tag{2.17}$$

$$\sum_{i=1}^{m+s+1} g'_i = \sum_{i=1}^{m+1} d'_i + \sum_{i=1}^{s} a_i. \tag{2.18}$$

Equation (2.18) follows directly by (2.6) and by the definition of $\mathbf{g}'$ and $\mathbf{d}'$.

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Let us prove (2.16). By (2.4) and (2.11), we have $x \geq d_{l+1} \geq g_{l+1+s}$. Hence, by (2.10) we obtain

$$u \leq l + s. \tag{2.19}$$

By (2.4), (2.19) and (2.12), for $i \geq l+2$ we have

$$d'_i = d_{i-1} \geq g_{i-1+s} = g'_{i+s}.$$

For $i \leq u - s$, we have

$$d'_i = d_i \geq g_{i+s} = g'_{i+s}.$$

Finally, for $u - s + 1 \leq i \leq l + 1$, we have

$$d'_i \geq d'_{l+1} = x = g'_{u+1} \geq g'_{i+s}.$$

Hence, we have proved that (2.16) is valid.

Now we are left with proving (2.17). Since $h_\alpha \leq u < h_{\alpha+1}$, (2.17) for $j \leq \alpha$, becomes the same inequality as in (2.5), while for $j > \alpha$, (2.17) has the same additional summand x added to the both sides of (2.5). Hence, (2.17) follows by (2.5).

Now suppose that (2.9) is valid.

In order to prove (2.8) we are left with proving (2.4), (2.5) and (2.6). Since (2.9) is valid, we have (2.16), (2.17) and (2.18). Hence, the definitions of $\mathbf{g}$ and $\mathbf{d}$ together with (2.18) give (2.6).

By (2.16), we have that $x = d'_{l+1} \geq g'_{l+s+1}$, and so

$$u \leq l + s.$$

Then, we have

$$\begin{array}{rc}
\text{for } i \geq l+1: & d_i = d'_{i+1} \geq g'_{i+1+s} = g_{i+s}, \\
\text{for } i \leq u-s: & d_i = d'_i \geq g'_{i+s} = g_{i+s}, \\
\text{for } u-s < i \leq l: & d_i \geq d_l > x \geq g_{u+1} \geq g_{i+s}
\end{array}$$

which proves (2.4).

Hence, we are left with proving (2.5). Now, for $j \leq \alpha$, from (2.14) the inequality (2.5) is the same as the corresponding inequality in (2.17). For $j > \alpha$, from (2.15), the inequality in (2.5) is the same as the corresponding inequality in (2.17) after subtracting the same summand, x, from both sides. Hence, (2.5) follows from (2.17), which finishes our proof. $\square$

LEMMA 2.4. *Let* $\mathbf{a} = (a_1, \ldots, a_s)$, $\mathbf{d} = (d_1, \ldots, d_m)$, *and* $\bar{\mathbf{g}} = (\bar{g}_1, \ldots, \bar{g}_{m+s})$ *be partitions such that*

$$d_i \geq \bar{g}_{i+s},$$

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$$\sum_{i=\bar{h}_j+1}^{m+s} \bar{g}_i \geq \sum_{i=\bar{h}_j-j+1}^{m} d_i + \sum_{i=j+1}^{s} a_i, \quad j=1,\ldots,s, \tag{2.20}$$

$$\sum_{i=1}^{m+s} \bar{g}_i \geq \sum_{i=1}^{m} d_i + \sum_{i=1}^{s} a_i,$$

where $\bar{h}_j := \min\{i | d_{i-j+1} < \bar{g}_i\}$, $j=1,\ldots,s$.

Let $f \in \{2,\ldots,m+s\}$, *and let* $\mathbf{g}=(g_1,\ldots,g_{m+s})$ *be a partition such that*

$$g_i = \bar{g}_i, \quad i \geq f,$$

$$g_i \leq \bar{g}_i, \quad i < f,$$

$$\bar{g}_{f-1} \geq g_1 \geq g_{f-1} \geq g_1 - 1,$$

$$\sum_{i=1}^{m+s} g_i \geq \sum_{i=1}^{m} d_i + \sum_{i=1}^{s} a_i. \tag{2.21}$$

Then

$$\sum_{i=h_j+1}^{m+s} g_i \geq \sum_{i=h_j-j+1}^{m} d_i + \sum_{i=j+1}^{s} a_i, \quad j=0,\ldots,s, \tag{2.22}$$

where $h_j := \min\{i | d_{i-j+1} < g_i\}$, $j=1,\ldots,s$, $h_0 := 0$.

Proof. From the definition of h_j and $\bar{h}_j$ we have $\bar{h}_j \leq h_j$, $j=1,\ldots,s$. Moreover, let $p \in \{0,\ldots,s\}$ be such that $\bar{h}_p < f \leq \bar{h}_{p+1}$. Then we have that $h_j = \bar{h}_j$, $j \geq p+1$, and so by (2.20), equation (2.22) is trivially satisfied for all $j \geq p+1$.

Since $f-1 < \bar{h}_{p+1}$, we have $d_{f-p-1} \geq \bar{g}_{f-1} \geq g_1$. Thus, $h_j \geq f-p+j-1$, for every $j=1,\ldots,p$.

We shall prove by induction on j that the condition (2.22) is satisfied for every $j=0,\ldots,p$, thus completing the proof of (2.22).

If $j=0$, (2.22) equals (2.21).

Now, let $1 \leq j \leq p$, and suppose that (2.22) is satisfied for $j-1$.

If $g_{h_j} \leq a_j$, then by the induction hypothesis, we have

$$\sum_{i=h_j+1}^{m+s} g_i = \sum_{i=h_{j-1}+1}^{m+s} g_i - \sum_{i=h_{j-1}+1}^{h_j} g_i \geq \sum_{i=h_{j-1}-j+2}^{m} d_i + \sum_{i=j}^{s} a_i - \sum_{i=h_{j-1}+1}^{h_j-1} g_i - g_{h_j} =$$

$$= \sum_{i=h_j-j+1}^{m} d_i + \sum_{i=j+1}^{s} a_i + \sum_{i=h_{j-1}-j+2}^{h_j-j} d_i - \sum_{i=h_{j-1}+1}^{h_j-1} g_i + a_j - g_{h_j}. \tag{2.23}$$

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By the definition of h_j, we have that for all $i < h_j$, $g_i \le d_{i-j+1}$. Hence, by (2.23) follows (2.22), as wanted.

So, in the rest of the proof of (2.22) we assume that $g_{h_j} > a_j$. Then since $g_{h_j} \le g_1 \le g_{f-1} + 1$, we have $g_{f-1} + 1 > a_j$, i.e., $g_{f-1} \ge a_j$.

If $h_j < f$, then by (2.20) and Lemma 2.2, we have

$$\sum_{i=h_j+1}^{m+s} g_i = \sum_{i=h_j+1}^{f-1} g_i + \sum_{i=f}^{m+s} \bar{g}_i \ge \sum_{i=h_j+1}^{f-1} g_i + \sum_{i=f-p}^{m} d_i + \sum_{i=p+1}^{s} a_i =$$

$$= \left(\sum_{i=h_j+1}^{f-1} g_i + \sum_{i=f-p}^{h_j-j} d_i - \sum_{i=j+1}^{p} a_i \right) + \left(\sum_{i=h_j-j+1}^{m} d_i + \sum_{i=j+1}^{s} a_i \right).$$

Let us see that $\sum_{i=h_j+1}^{f-1} g_i + \sum_{i=f-p}^{h_j-j} d_i - \sum_{i=j+1}^{p} a_i \ge 0$. Indeed, by the definition of h_j we have

$$\begin{aligned} \sum_{i=h_j+1}^{f-1} g_i + \sum_{i=f-p}^{h_j-j} d_i - \sum_{i=j+1}^{p} a_i &\ge \sum_{i=h_j+1}^{f-1} g_i + \sum_{i=f-p+j-1}^{h_j-1} g_i - \sum_{i=j+1}^{p} a_i \\ &\ge \sum_{i=f-p+j}^{f-1} g_i - \sum_{i=j+1}^{p} a_i, \end{aligned}$$

and since $g_{f-1} \ge a_j$, the last expression is nonnegative, as wanted.

If $h_j \ge f$, then $\bar{h}_p < f < h_j + 1 \le h_{p+1} = \bar{h}_{p+1}$. Thus, by (2.20) and Lemma 2.2, we have

$$\begin{aligned} \sum_{i=h_j+1}^{m+s} g_i = \sum_{i=h_j+1}^{m+s} \bar{g}_i &\ge \sum_{i=h_j+1-p}^{m} d_i + \sum_{i=p+1}^{s} a_i \\ &= \left(\sum_{i=h_j+1-p}^{h_j-j} d_i - \sum_{i=j+1}^{p} a_i \right) + \left(\sum_{i=h_j-j+1}^{m} d_i + \sum_{i=j+1}^{s} a_i \right). \end{aligned}$$

Finally, since $g_{h_j} > a_j$, from the definition of h_j, we have

$$\sum_{i=h_j+1-p}^{h_j-j} d_i - \sum_{i=j+1}^{p} a_i \ge \sum_{i=h_j-p+j}^{h_j-1} g_i - \sum_{i=j+1}^{p} a_i \ge 0,$$

as wanted. ∎

Lemma 2.5. *Let* $\mathbf{a}$*,* $\mathbf{d}$ *and* $\mathbf{g}$ *be partitions as given in (2.1)–(2.3), such that*

$$\mathbf{g} \prec' (\mathbf{d}, \mathbf{a}). \tag{2.24}$$

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Let x be an integer such that there exists $u \in \{1, \dots, m\}$ with $d_u = x$, and there does not exist $l \in \{1, \dots, m+s\}$ such that $g_l = x$.

Then there exists a partition $\mathbf{g}' = (g'_1, \dots, g'_{m+s})$, such that

$$\mathbf{g}' \prec' (\mathbf{d}, \mathbf{a}),$$

and such that there exists $l \in \{1, \dots, m+s\}$ with $g'_l = x$. Moreover, such a $\mathbf{g}'$ can be defined independently from $\mathbf{d}$ and $\mathbf{a}$, i.e., it depends only on $\mathbf{g}$ and x.

Proof. First suppose that $g_1 > x > g_{m+s}$. Then there exists $j \in \{1, \dots, m+s-1\}$ such that

$$g_j > x > g_{j+1}.$$

If $g_j + g_{j+1} \geq 2x$, then let $g'_j = g_j + g_{j+1} - x$ and $g'_{j+1} = x$.

If $g_j + g_{j+1} < 2x$, then let $g'_j = x$ and $g'_{j+1} = g_j + g_{j+1} - x$.

In both cases, we have $g_j \geq g'_j \geq g'_{j+1} \geq g_{j+1}$. Also, for $i \neq j, j+1$, we set $g'_i = g_i$. Thus, we have that g'_i's are nonincreasing. We claim that for such defined $\mathbf{g}' := (g'_1, \dots, g'_{m+s})$ we have

$$\mathbf{g}' \prec' (\mathbf{d}, \mathbf{a}). \tag{2.25}$$

By the definition of $\prec'$, (2.25) is equivalent to:

$$d_i \geq g'_{i+s}, \qquad i = 1, \dots, m, \tag{2.26}$$

$$\sum_{i=1}^{h'_w} g'_i - \sum_{i=1}^{h'_w - w} d_i \leq \sum_{i=1}^{w} a_i, \qquad w = 1, \dots, s, \tag{2.27}$$

$$\sum_{i=1}^{m+s} g'_i = \sum_{i=1}^{m} d_i + \sum_{i=1}^{s} a_i, \tag{2.28}$$

where $h'_w := \min\{i | d_{i-w+1} < g'_i\}$, $w = 1, \dots, s$. Since (2.24) is valid, we have (2.4), (2.5) and (2.6). Hence, the definition of $\mathbf{g}'$ together with (2.6), gives (2.28).

For all $i \neq j - s + 1$, we have $d_i \geq g_{i+s} \geq g'_{i+s}$.

Since $d_{j-s} \geq g_j > x = d_u$, we have $u \geq j - s + 1$. So

$$d_{j-s+1} \geq d_u = x \geq g'_{j+1},$$

as well. Hence, we have proved (2.26).

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Let $h_w := \min\{i | d_{i-w+1} < g_i\}$, $w = 1, \dots, s$ ($h_0 := 0$, $h_{s+1} := m + s + 1$). Now let $\alpha \in \{0, \dots, s\}$ be such that

$$h_\alpha \leq j < h_{\alpha+1}.$$

First of all, we have $j + 1 < h_{\alpha+1}$. Otherwise, we would have $j + 1 = h_{\alpha+1}$, and so $g_{j+1} = g_{h_{\alpha+1}} > d_{j+1-\alpha}$. However, since $j < h_{\alpha+1}$, we have $d_{j-\alpha} \geq g_j > x$, and so $u \geq j - \alpha + 1$. Hence, $d_{j-\alpha+1} \geq d_u = x > g_{j+1}$, which is a contradiction.

Now, if $h_\alpha < j < j+1 < h_{\alpha+1}$, then we have $d_{j-\alpha} \geq g_j > x$, and so $u \geq j-\alpha+1$. Hence, $d_{j-\alpha+1} \geq x \geq g'_{j+1}$. So, in this case we have $h'_i = h_i$, for all $i = 1, \dots, s$, and

$$\sum_{i=1}^{h_l} g_i = \sum_{i=1}^{h'_l} g'_i, \quad l = 1, \dots, s.$$

Hence, (2.27) follows from (2.5).

If $h_\alpha = j < j + 1 < h_{\alpha+1}$, then again we have $d_{j-\alpha} \geq g_{j-1} \geq g_j > x$, and so $u \geq j - \alpha + 1$, i.e., $d_{j-\alpha+1} \geq x \geq g'_{j+1}$. So $h'_i = h_i$, for $i \geq \alpha + 1$. Also, obviously $h'_i = h_i$, for $i \leq \alpha - 1$. Since, for all $l \neq \alpha$ we have

$$\sum_{i=1}^{h_l} g_i = \sum_{i=1}^{h'_l} g'_i,$$

for all such indices (2.27) follows from (2.5).

As for h'_α, we have $h_\alpha \leq h'_\alpha < h'_{\alpha+1} = h_{\alpha+1}$.

If $h'_\alpha = h_\alpha$, then

$$\sum_{i=1}^{h'_\alpha} g'_i = \sum_{i=1}^{h_\alpha} g'_i \leq \sum_{i=1}^{h_\alpha} g_i \leq \sum_{i=1}^{h_\alpha-\alpha} d_i + \sum_{i=1}^{\alpha} a_i = \sum_{i=1}^{h'_\alpha-\alpha} d_i + \sum_{i=1}^{\alpha} a_i.$$

If $h'_\alpha > h_\alpha$, then

$$\sum_{i=1}^{h'_\alpha} g'_i = \sum_{i=1}^{h'_\alpha} g_i = \sum_{i=1}^{h_\alpha} g_i + \sum_{i=h_\alpha+1}^{h'_\alpha} g_i \leq \sum_{i=1}^{h_\alpha-\alpha} d_i + \sum_{i=1}^{\alpha} a_i + \sum_{i=h_\alpha-\alpha+1}^{h'_\alpha-\alpha} d_i = \sum_{i=1}^{h'_\alpha-\alpha} d_i + \sum_{i=1}^{\alpha} a_i.$$

The middle inequality follows from (2.5) for $j = \alpha$ and the fact that $h'_\alpha < h_{\alpha+1}$. This ends the proof in the case $g_1 > x > g_{m+s}$.

Since $x = d_u \geq d_m \geq g_{m+s}$, the only remaining case is

$$x > g_1.$$

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In this case, let

$$g'_1 = x,$$

and we define the rest of g'_i's in the following way:

Let $f := \max\{v \in \{2,\dots,m+s\} | (v-1)g_v + x > \sum_{i=1}^{v} g_i\}$. Then we define $g'_i = g_i$ for $i > f$, and we define $g'_2 \geq \cdots \geq g'_f$ as the most homogeneous partition such that $\sum_{i=2}^{f} g'_i = \sum_{i=1}^{f} g_i - x$.

Such defined $g'_1 \geq \cdots \geq g'_{m+s}$ satisfy (2.28). Also, since $d_i \geq g_{i+s} \geq g'_{i+s}$ for all $i = 1,\dots,m$, we have (2.26). So we are left with proving (2.27). Since $d_1 \geq d_u = x = g'_1 \geq g_1$, we have that $h_1 \geq 2$,

$$\sum_{i=2}^{m+s} g_i \geq \sum_{i=2}^{m} d_i + \sum_{i=1}^{s} a_i$$

and

$$\sum_{i=2}^{m+s} g'_i \geq \sum_{i=2}^{m} d_i + \sum_{i=1}^{s} a_i.$$

If we denote $\tilde{\mathbf{g}} = \mathbf{g} \setminus \{g_1\}$, $\tilde{\mathbf{d}} = \mathbf{d} \setminus \{d_1\}$ and $\tilde{\mathbf{g}}' = \mathbf{g}' \setminus \{g'_1\}$, then by applying Lemma 2.4 for $\tilde{\mathbf{g}}$, $\tilde{\mathbf{d}}$, $\mathbf{a}$ and $\tilde{\mathbf{g}}'$, we obtain (2.27), as wanted. ∎

2.2. Relaxation on partitions c and d. Let m, n, k and s be nonnegative integers such that

$$m + s = n + k.$$

Let

$$\mathbf{a} = (a_1,\dots,a_s), \tag{2.29}$$

$$\mathbf{b} = (b_1,\dots,b_k), \tag{2.30}$$

$$\mathbf{c} = (c_1,\dots,c_n) \tag{2.31}$$

and

$$\mathbf{d} = (d_1,\dots,d_m) \tag{2.32}$$

be partitions of nonincreasing integers.

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Lemmas 2.3–2.5 enable to consider Problem 1 in the case when there do not exist $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$ such that $c_i = d_j$. More precisely, we have the following result:

PROPOSITION 2.6. *Consider the partitions* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *and* $\mathbf{d}$ *as given in (2.29)–(2.32).*

Let x *be such that there exist* $w \in \{1, \ldots, n\}$ *and* $u \in \{1, \ldots, m\}$ *such that*

$$c_w = d_u = x.$$

Let

$$\mathbf{d}' := (d_1, \ldots, d_{u-1}, d_{u+1}, \ldots, d_m),$$

and

$$\mathbf{c}' := (c_1, \ldots, c_{w-1}, c_{w+1}, \ldots, c_n).$$

Then there exists a partition $\mathbf{g} = (g_1, \ldots, g_{m+s})$ *such that*

$$\mathbf{g} \prec' (\mathbf{d}, \mathbf{a}) \tag{2.33}$$

$$\mathbf{g} \prec' (\mathbf{c}, \mathbf{b}) \tag{2.34}$$

if and only if there exists a partition $\mathbf{g}' = (g'_1, \ldots, g'_{m+s-1})$ *such that*

$$\mathbf{g}' \prec' (\mathbf{d}', \mathbf{a}) \tag{2.35}$$

$$\mathbf{g}' \prec' (\mathbf{c}', \mathbf{b}). \tag{2.36}$$

Proof. Let there exists $\mathbf{g}$ which satisfies (2.33) and (2.34). By Lemma 2.5, there exists a partition $\mathbf{g}'' = (g''_1, \ldots, g''_{m+s})$ such that there exists $l \in \{1, \ldots, m + s\}$ with $g''_l = x$, and such that

$$\mathbf{g}'' \prec' (\mathbf{d}, \mathbf{a})$$
$$\mathbf{g}'' \prec' (\mathbf{c}, \mathbf{b}).$$

Now, let $\mathbf{g}' = \mathbf{g}'' \setminus \{x\}$.

By Lemma 2.3, we have that (2.35) and (2.36) are satisfied, as wanted.

Conversely, let $\mathbf{g}'$ be such that (2.35) and (2.36) are satisfied. Let $\mathbf{g} := \mathbf{g}' \cup \{x\}$. Then by Lemma 2.3, (2.33) and (2.34) are valid, as wanted. ∎

Thus, from now on without loss of generality, we shall consider Problem 1 only for partitions $\mathbf{c}$ and $\mathbf{d}$ such that for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$, we have $c_i \neq d_j$.

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3. Sets S and Δ. Let **a**, **b**, **c** and **d** be partitions as given in (2.29)–(2.32).

Moreover, let $c_1 \geq \cdots \geq c_n$ and $d_1 \geq \cdots \geq d_m$ be such that there do not exist $i \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,m\}$ such that $c_i = d_j$.

In this section, we shall define certain labels on the sequences $c_1,\ldots,c_n$ and $d_1,\ldots,d_m$. We are going to define the set S of the indices of *chosen integers* among $c_1 \geq \cdots \geq c_n$ and the set Δ as the indices of *special ones* among $d_1 \geq \cdots \geq d_m$.

We note that the definition is similar to, but not the same as the one from Section 3.1. in [8]. It is given by recursion on $(c_1,\ldots,c_n) \cup (d_1,\ldots,d_m)$, starting from the smallest integer. More precisely:

Put $S := \emptyset$, $\Delta := \emptyset$.

The definitions are given recursively, so first we take the smallest integer from $(c_1,\ldots,c_n,d_1,\ldots,d_m)$. Then we have two possibilities (a) and (b), depending on whether the chosen integer belongs to $(d_1,\ldots,d_m)$ or it belongs to $(c_1,\ldots,c_n)$.

(a) If the observed integer is among $(d_1,\ldots,d_m)$, say d_j, then let

$$q_j = s - \sharp\{c_i < d_j | i \in S\} + \sharp\{i > j | i \notin \Delta\} + 1. \tag{3.1}$$

If $q_j > s$ or $q_j < 0$, we add j to Δ, i.e., $\Delta := \Delta \cup \{j\}$.

If $s \geq q_j \geq 0$, we check the following equation:

$$\sum_{\substack{c_i < d_j \\ i \in S}} c_i < \sum_{\substack{i > j \\ i \notin \Delta}} d_i + d_j + \sum_{i=q_j+1}^{s} a_i. \tag{3.2}$$

If d_j satisfies (3.2), then we add j to Δ, i.e., $\Delta := \Delta \cup \{j\}$.

(b) If the observed integer is among $(c_1,\ldots,c_n)$, say c_j, then let

$$q'_j = k - \sharp\{d_i < c_j | i \in \Delta\} + \sharp\{i > j | i \notin S\} + 1. \tag{3.3}$$

If $q'_j > k$ or $q'_j < 0$, we add j to the set S, i.e., $S := S \cup \{j\}$.

If $k \geq q'_j \geq 0$, we check the following inequality:

$$\sum_{\substack{d_i < c_j \\ i \in \Delta}} d_i < \sum_{\substack{i > j \\ i \notin S}} c_i + c_j + \sum_{i=q'_j+1}^{k} b_i. \tag{3.4}$$

If (3.4) is satisfied, then we add j to the set S, i.e., $S := S \cup \{j\}$.

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Now, we select the next smallest integer among $(c_1, \dots, c_n, d_1, \dots, d_m)$, and proceed.

In this way, we have defined the sets Δ and S. These sets will play an essential role in the rest of the paper. Also, it is trivial to see that these definitions are symmetric, i.e., one can be obtained from the another by simply exchanging Δ, d_i, a_i and q_i with S, c_i, b_i and q_i', respectively, and vice-versa.

3.1. Some additional notation. We also introduce a couple of definitions in order to simplify and clarify further notation.

Let $h := \sharp S$. We shall denote by $c^1 \geq \cdots \geq c^h$ the nonincreasing ordering of c_i's such that $i \in S$. Also, for each c^x, $x = 1, \dots, h$, we shall define

$$z_x := \max\{i | d_i > c^x\}$$

and

$$m_x := \max\{i | a_i > c^x\},$$

i.e.,

$$d_{z_x} > c^x > d_{z_x+1}, \qquad a_{m_x} > c^x \geq a_{m_x+1}.$$

We also set $c^0 := c_0$, $c^{h+1} := c_{n+1}$, $z_0 := 0$, $z_{h+1} := m$, $m_0 := 0$ and $m_{h+1} := s$.

Moreover, we define

$$t_x := s - (h - x) + \sharp\{i \notin \Delta | d_i < c^x\}, \quad x = 0, \dots, h, \tag{3.5}$$

and $t_{h+1} := s + 1$.

Analogously, let $h' := \sharp\Delta$. We shall denote by $d^1 \geq \cdots \geq d^{h'}$ the nonincreasing ordering of d_i's such that $i \in \Delta$. Also, for each d^x, $x = 1, \dots, h'$, we shall define

$$z_x' := \max\{i | c_i > d^x\}$$

and

$$m_x' := \max\{i | b_i > d^x\},$$

i.e.,

$$c_{z_x'} > d^x > c_{z_x'+1}, \qquad b_{m_x'} > d^x \geq b_{m_x'+1}.$$

We also set $d^0 := d_0$, $d^{h'+1} := d_{m+1}$, $z_0' := 0$, $z_{h'+1}' := n$, $m_0' := 0$ and $m_{h'+1}' := k$.

Moreover, we define

$$t_x' := k - (h' - x) + \sharp\{i \notin S | c_i < d^x\}, \quad x = 0, \dots, h', \tag{3.6}$$

and $t_{h'+1}' := k + 1$.

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4. Properties of the sets S and Δ . In this section, we present some auxiliary lemmas. Many of them are analogous to lemmas from Section 3.1 in [8]. However, since the definition of the sets S and Δ (D^C) are different from the analogous definition given in [8], we have to re-state and re-prove all of the lemmas given in [8].

LEMMA 4.1. *Let $j \in \{1, \ldots, m\}$ and $x \in \{0, \ldots, h\}$ be such that $c^x > d_j > c^{x+1}$, and $j \in \Delta$. Then*

$$z_x + 1, z_x + 2, \ldots, j - 1 \in \Delta. \tag{4.1}$$

Proof. If $x = h$, then $q_j > s$. Moreover, by the definition of q_i's we have that $q_{z_x+1} \geq \cdots \geq q_{j-1} \geq q_j > s$, thus proving (4.1).

If $x < h$, then $d_j > c^h$. Suppose on the contrary to (4.1) that among $\{z_x + 1, \ldots, j-1\}$ there are indices that are not from Δ. Denote by u the largest of them. Then since $q_j = q_u$ (by the definition), we have that $s \geq q_j = q_u \geq 0$, and thus d_j satisfies (3.2), i.e.,

$$\sum_{i=x+1}^{h} c^i < d_j + \sum_{\substack{i > j \\ i \notin \Delta}} d_i + \sum_{i=q_j+1}^{s} a_i,$$

while d_u does not satisfy (3.2), and so we have

$$\sum_{i=x+1}^{h} c^i \geq d_u + \sum_{\substack{i > j \\ i \notin \Delta}} d_i + \sum_{i=q_j+1}^{s} a_i.$$

Last two equations together give that $d_j > d_u$, which is a contradiction. This proves (4.1), as wanted. ∎

Now, as a corollary of Lemma 4.1, we have:

LEMMA 4.2. *Let $c^x > d_j > c^{x+1}$, and let $j \notin \Delta$. Then $j+1, \ldots, z_{x+1} \notin \Delta$.*

Denote by w_x the number of d_i's such that $i \notin \Delta$ and such that $c^x > d_i > c^{x+1}$. Then from the definition of z_x and w_x, by Lemmas 4.1 and 4.2, we have

$$d_{z_x} > c^x > \underbrace{d_{z_x+1} \geq d_{z_x+2} \geq \ldots \geq d_{z_{x+1}-w_x}}_{\in\Delta} \geq \underbrace{d_{z_{x+1}-w_x+1} \geq \ldots \geq d_{z_{x+1}}}_{\notin\Delta} > c^{x+1}. \tag{4.2}$$

In particular, we have

$$t_{x+1} = t_x - w_x + 1.$$

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Completely analogously, we obtain the following dual results:

LEMMA 4.3. *Let $j \in \{1,\dots,n\}$ and $x \in \{0,\dots,h'\}$ be such that $d^x > c_j > d^{x+1}$, and $j \in S$. Then*

$$z'_x + 1, z'_x + 2, \dots, j-1 \in S.$$

LEMMA 4.4. *Let $d^x > c_j > d^{x+1}$, and let $j \notin S$. Then $j+1,\dots,z'_{x+1} \notin S$.*

Denote by w'_x the number of c_i's such that $i \notin S$ and such that $d^x > c_i > d^{x+1}$. Then from the definition of z'_x and w'_x, by Lemmas 4.3 and 4.4, we have

$$c_{z'_x} > d^x > \underbrace{c_{z'_x+1} \geq c_{z'_x+2} \geq \dots \geq c_{z'_{x+1}-w'_x}}_{\in S} \geq \underbrace{c_{z'_{x+1}-w'_x+1} \geq \dots \geq c_{z'_{x+1}}}_{\notin S} > d^{x+1}. \tag{4.3}$$

The proofs of the following two lemmas follow directly by the definitions of z_i, z'_i, t_i and t'_i:

LEMMA 4.5. *Let $j \in S$. Let $i \in \{1,\dots,h\}$ be such that $c_j = c^i$ and let $x \in \{0,\dots,h'\}$ be such that $d^x > c_j > d^{x+1}$. Then*

$$z_i + t_i = j + t'_x.$$

LEMMA 4.6. *Let $j \in \Delta$. Let $i \in \{1,\dots,h'\}$ be such that $d_j = d^i$ and let $x \in \{0,\dots,h\}$ be such that $c^x > d_j > c^{x+1}$. Then*

$$z'_i + t'_i = j + t_x.$$

LEMMA 4.7. *Let $x \in \{1,\dots,h\}$, and let $m_x \geq t_x$. Let*

$$\sum_{i=z_u+t_u}^{z_u+m_u} e_i \leq \sum_{i=u}^{h} c^i - \sum_{\substack{i > z_u \\ i \notin \Delta}} d_i - \sum_{i=m_u+1}^{s} a_i, \tag{4.4}$$

for all $u \geq x$ such that $m_u \geq t_u$.

Here $(e_{z_u+t_u},\dots,e_{z_u+m_u})$ are defined as the smallest $m_u - t_u + 1$ elements among $(a_1,\dots,a_{m_u},d_1,\dots,d_{z_u})$.

Let $d_j \in (e_{z_x+t_x},\dots,e_{z_x+m_x})$. Then $j \notin \Delta$.

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Proof. The proof goes by induction on x. Let $x = h$. Then $m_h \geq s(= t_h)$. Hence, $a_s > c^h$. By (4.4) for $u = h$, we have that

$$c^h < \min(d_{z_h}, a_s) = e_{z_h+s} \leq c^h,$$

which is a contradiction. Hence, the condition $m_x \geq t_x$ implies that $x < h$. Thus, let $x < h$ and suppose that our claim is valid for all $i > x$, for which $m_i \geq t_i$.

If $m_x = t_x$, then $e_{z_x+t_x} = d_j$ and $j = z_x$. Thus $s \geq q_j = m_x \geq 0$. Moreover, by (4.4), d_j satisfies

$$d_j = d_{z_x} \leq \sum_{i=x}^{h} c^i - \sum_{\substack{i > j \\ i \notin \Delta}} d_i - \sum_{i=m_x+1}^{s} a_i,$$

i.e., by (3.2) $j \notin \Delta$, as wanted.

Now, consider the case $m_x > t_x$. First we shall prove that then $m_{x+1} \geq t_{x+1}$. Suppose on the contrary that $m_{x+1} < t_{x+1}$. Then $t_x < m_x \leq m_{x+1} < t_{x+1}$, i.e.,

$$t_x + 2 \leq t_{x+1}. \tag{4.5}$$

However, by the definition of t_x, t_{x+1} and w_x we have that $t_x = t_{x+1} + w_x - 1 \geq t_{x+1} - 1$. Thus, (4.5) is impossible.

Moreover, we shall prove that $m_x > t_x$ implies $z_x = z_{x+1} - w_x$, i.e., that all d_i's such that $c^x > d_i > c^{x+1}$, satisfy $i \notin \Delta$.

In order to prove this, suppose on the contrary that $z_x < z_{x+1} - w_x$. Then, $z_{x+1} - w_x \in \Delta$ and $d_{z_{x+1}-w_x} < c^x$ (see (4.2)). By the induction hypothesis, we have that $d_{z_{x+1}-w_x}$ is not among $(e_{z_{x+1}+t_{x+1}}, \dots, e_{z_{x+1}+m_{x+1}})$, i.e., all e_i's for $i = z_{x+1} + t_{x+1}, \dots, z_{x+1} + m_{x+1}$ are less than or equal to $d_{z_{x+1}-w_x}$. This implies that among $(e_{z_{x+1}+t_{x+1}}, \dots, e_{z_{x+1}+m_{x+1}})$, there are at most w_x d_i's, and at least $m_{x+1} - t_{x+1} + 1 - w_x = m_{x+1} - t_x$ a_i's. Thus a_{t_x+1} is among $(e_{z_{x+1}+t_{x+1}}, \dots, e_{z_{x+1}+m_{x+1}})$ and so $a_{t_x+1} \leq d_{z_{x+1}-w_x} < c^x$. This means that $m_x \leq t_x$, which contradicts $m_x > t_x$.

Hence, $z_x = z_{x+1} - w_x$, and so $z_x + t_x + 1 = z_{x+1} + t_{x+1}$.

Now we have that d_j is among $(e_{z_x+t_x+1}, \dots, e_{z_x+m_x})$ or d_j is $e_{z_x+t_x}$.

If d_j is among $(e_{z_x+t_x+1}, \dots, e_{z_x+m_x})$, then since $z_x + t_x + 1 = z_{x+1} + t_{x+1}$, d_j is among $(e_{z_{x+1}+t_{x+1}}, \dots, e_{z_{x+1}+m_{x+1}})$. Hence, by the induction hypothesis we have that $j \notin \Delta$.

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If d_j is $e_{z_x+t_x}$, then (4.4) gives

$$d_j + \sum_{i=z_x+t_x+1}^{z_x+m_x} e_i = \sum_{i=z_x+t_x}^{z_x+m_x} e_i \leq \sum_{i=x}^{h} c^i - \sum_{\substack{i > z_x \\ i \notin \Delta}} d_i - \sum_{i=m_x+1}^{s} a_i. \tag{4.6}$$

The sequence $(e_{z_x+t_x+1}, \ldots, e_{z_x+m_x})$ consists of $d_{j+1}, \ldots, d_{z_x}$ and $a_{q_j+1}, \ldots, a_{m_x}$. Hence,

$$q_j \leq m_x \leq s. \tag{4.7}$$

By the induction hypothesis $j+1, \ldots, z_x \notin \Delta$. Thus, (4.6) gives

$$d_j \leq \sum_{i=x}^{h} c^i - \sum_{\substack{i > j \\ i \notin \Delta}} d_i - \sum_{i=q_j+1}^{s} a_i. \tag{4.8}$$

Last implies that $q_j \geq 0$, which together with (4.7), (4.8) and (3.2), gives that $j \notin \Delta$, as wanted. ∎

Completely analogously, we have the dual result:

LEMMA 4.8. *Let $y \in \{1, \ldots, h'\}$, and let $m'_y \geq t'_y$. Let*

$$\sum_{i=z'_u+t'_u}^{z'_u+m'_u} e'_i \leq \sum_{i=u}^{h'} d^i - \sum_{\substack{i > z'_u \\ i \notin S}} c_i - \sum_{i=m'_u+1}^{k} b_i, \tag{4.9}$$

for all $u \geq y$ such that $m'_u \geq t'_u$. Here, $(e'_{z'_u+t'_u}, \ldots, e'_{z'_u+m'_u})$ are defined as the smallest $m'_u - t'_u + 1$ elements among $(b_1, \ldots, b_{m'_u}, c_1, \ldots, c_{z'_u})$.

Let $c_j \in (e'_{z'_y+t'_y}, \ldots, e'_{z'_y+m'_y})$. Then $j \notin S$.

LEMMA 4.9. *For every $x = 0, \ldots, h-1$, we have $z_{x+1} + t_{x+1} > z_x + t_x$.*

Proof. From the definition of w_x we have $w_x \leq z_{x+1} - z_x$, and so

$$z_{x+1} + t_{x+1} = z_{x+1} + t_x + 1 - w_x \geq z_x + t_x + 1 > z_x + t_x,$$

as wanted. ∎

Analogously, we have:

LEMMA 4.10. *For every $x = 0, \ldots, h'-1$, we have $z'_{x+1} + t'_{x+1} > z'_x + t'_x$.*

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From (4.2) and the definitions of q_j's and t_i's (see (3.1) and (3.5)), we have the following result:

LEMMA 4.11. *Let $x \in \{0, \ldots, h\}$. Then*

$$q_{z_x+1} = q_{z_x+2} = \cdots = q_{z_{x+1}-w_x} = t_x + 1,$$

and

$$q_{z_{x+1}-w_x+j} = t_x - j + 1, \quad j = 1, \ldots, w_x.$$

In particular, if $w_x > 0$ (i.e., if $t_{x+1} \leq t_x$), then for every i such that $t_{x+1} \leq i \leq t_x$ there exists $j \notin \Delta$ such that $c^x > d_j > c^{x+1}$ and $q_j = i$.

From (4.3) and the definitions of q'_j's and t'_i's (see (3.3) and (3.6)), we have the following result:

LEMMA 4.12. *Let $x \in \{0, \ldots, h'\}$. Then,*

$$q'_{z'_x+1} = q'_{z'_x+2} = \cdots = q'_{z'_{x+1}-w'_x} = t'_x + 1,$$

and

$$q'_{z'_{x+1}-w'_x+j} = t'_x - j + 1, \quad j = 1, \ldots, w'_x.$$

In particular, if $w'_x > 0$ (i.e., if $t'_{x+1} \leq t'_x$), then for every i such that $t'_{x+1} \leq i \leq t'_x$ there exists $j \notin S$ such that $d^x > c_j > d^{x+1}$ and $q'_j = i$.

LEMMA 4.13. *Let $x = 0, \ldots, h$. Suppose that the condition (4.4) is valid for all $u = x+1, \ldots, h$ such that $m_u \geq t_u$. Then*

$$t_x \geq 0.$$

Proof. The proof goes by induction on x.

For $x = h$, we have $t_h = s \geq 0$.

Now, let $x < h$. By the induction hypothesis we have that $t_{x+1} \geq 0$.

By the definition of t_i's we have

$$t_x = t_{x+1} - 1 + w_x.$$

Hence, if $t_{x+1} > 0$ or $w_x > 0$, we directly obtain $t_x \geq 0$, as wanted. So, the only remaining case is if $t_{x+1} = 0$ and $w_x = 0$ (and hence, $t_x = -1$). The set

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$\{e_{z_{x+1}}, \dots, e_{z_{x+1}+m_{x+1}}\}$ has $m_{x+1}+1$ elements, and since it consists of d_i's and a_i's strictly greater than c^{x+1}, there must be at least one d_j among them (since $a_{m_{x+1}} > c^{x+1} \geq a_{m_{x+1}+1}$). Then $d_{z_{x+1}} \in \{e_{z_{x+1}}, \dots, e_{z_{x+1}+m_{x+1}}\}$. So, by Lemma 4.7 we have that $z_{x+1} \notin \Delta$. By the definition of Δ this gives that $q_{z_{x+1}} \geq 0$. Let $v \geq 0$ be such that $c^{x-v-1} > d_{z_{x+1}} > c^{x-v}$. Then by the definition of q_i's and t_j's we have that $q_{z_{x+1}} = t_x - v$, and thus $t_x \geq v \geq 0$. Last contradicts our assumption that $t_x = -1$. ∎

Analogously, we obtain the dual result:

Lemma 4.14. *Let $y = 0, \dots, h'$. Suppose that the condition (4.9) is valid for all $u = y+1, \dots, h'$ such that $m'_u \geq t'_u$. Then*

$$t'_y \geq 0.$$

Lemma 4.15. *For every j such that $t_0 < j \leq s$, there exists $i \in \{1, \dots, h\}$, such that $t_i = j$.*

Proof. Suppose that for some j with $t_0 < j \leq s$, there are no $i \in \{1, \dots, h\}$, such that $t_i = j$. Then from $t_{i+1} \leq t_i + 1$, for $i = 0, \dots, h-1$, we have that $t_i < j$ implies $t_{i+1} < j$, for every $i = 0, \dots, h-1$. Since $t_0 < j$, this would imply that $t_h < j$, which is a contradiction since $t_h = s$. ∎

Analogously, we have the dual result:

Lemma 4.16. *For every j such that $t'_0 < j \leq k$, there exists $i \in \{1, \dots, h'\}$, such that $t'_i = j$.*

Lemma 4.17. *[8, Lemma 4.9] Let $u_1 \geq \cdots \geq u_k$ and $v_1 \geq \cdots \geq v_k$ be integers. If*

$$\sharp\{i | u_i > v_j\} \geq j, \quad \textit{for all} \quad j = 1, \dots, k,$$

then

$$\sum_{i=1}^{k} u_i \geq \sum_{i=1}^{k} v_i + k.$$

Lemma 4.18. *For every $i = 0, \dots, h-1$, we have $t_i < s$, while $t_h = s$.*

Proof. By the definition of t_h we directly obtain that $t_h = s$. Now, suppose that there exists $i \in \{0, \dots, h-1\}$ such that $t_i \geq s$. Let $y := \max\{i | t_i \geq s\}$. Note that

$$t_{h-1} = s - 1. \tag{4.10}$$

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Indeed, by the definition $t_{h-1} = s-1+w_{h-1}$. However, if $w_{h-1} > 0$, then $c^{h-1} > d_{z_h}$, with $z_h \notin \Delta$, and $q_{z_h} = s$. Hence, by (3.2) we would have $c^h \geq d_{z_h}$, which is impossible. Thus (4.10) is satisfied.

So $y < h-1$. Thus, $t_y \geq s > t_{y+1}$ and so $w_y \geq 2$. By Lemma 4.11 there exists $f \notin \Delta$ such that $c^y > d_f > c^{y+1}$ and $q_f = s$. Hence, d_f does not satisfy (3.2), i.e.,

$$\sum_{i=y+1}^{h} c^i \geq \sum_{\substack{i \geq f \\ i \notin \Delta}} d_i. \tag{4.11}$$

We note that the number of summands on both sides of (4.11) is equal.

For every $j = y+1,\ldots,h$, we have $t_j \leq s = q_f$. From the definitions of q_i's (3.1) and t_i's (3.5), we obtain $\sharp\{i \notin \Delta | i \geq f, d_i > c^j\} = \sharp\{i \notin \Delta | f \leq i \leq z_j\} \geq j-y$, $j = y+1,\ldots,h$. This together with Lemma 4.17 contradicts (4.11). Thus, $t_i < s$, for all $i = 0,\ldots,h-1$, as wanted. ☐

And analogously, we have the dual result:

Lemma 4.19. *For every $i = 0,\ldots,h'-1$, we have $t'_i < k$, while $t'_{h'} = k$.*

5. Main result.

Consider partitions $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ as given in (2.29)–(2.32).

In this section, we give a complete and explicit solution to Problem 1.

As it was proven in Subsection 2.2, it is enough to resolve the Problem 1 in the case when partitions $\mathbf{c}$ and $\mathbf{d}$ do not have same elements.

Thus we assume that $\mathbf{c}$ and $\mathbf{d}$ are such that there are no $i \in \{1,\ldots,n\}$ and no $j \in \{1,\ldots,m\}$ such that $c_i = d_j$. For such $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ we define the sets S and Δ, together with labels t_x, t'_y, c^x and d^y as in Section 3.

The solution of Problem 1 is given in the following theorem:

Theorem 5.1. *Let m, n, k and s be nonnegative integers such that $m+s = n+k$. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ be partitions as given in (2.29)– (2.32).*

Then there exists a partition $\mathbf{g} = (g_1,\ldots,g_{m+s})$, such that

$$\mathbf{g} \prec' (\mathbf{d},\mathbf{a}) \tag{5.1}$$

$$\mathbf{g} \prec' (\mathbf{c},\mathbf{b}) \tag{5.2}$$

if and only if the following conditions are valid:

$$(i)\ \sum_{i=1}^{n} c_i + \sum_{i=1}^{k} b_i = \sum_{i=1}^{m} d_i + \sum_{i=1}^{s} a_i,$$

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$$(ii)\ \sum_{i=x+1}^{h} c^i \geq \sum_{\substack{d_i\,<\,c^x\\ i\notin\Delta}} d_i + \sum_{i=t_x+1}^{s} a_i, \quad x=0,\dots,h,$$

$$(iii)\ \sum_{i=y+1}^{h'} d^i \geq \sum_{\substack{c_i\,<\,d^y\\ i\notin S}} c_i + \sum_{i=t'_y+1}^{k} b_i, \quad y=0,\dots,h'.$$

5.1. Auxiliary results. In this section, we give four crucial lemmas for the proof of the necessity part of Theorem 5.1.

LEMMA 5.2. *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *and* $\mathbf{d}$ *be partitions as given in (2.29)–(2.32). Let* $\mathbf{g}$ *be a partition such that*

$$\mathbf{g} \prec' (\mathbf{d},\mathbf{a}). \tag{5.3}$$

Let d_j, $j\in\{1,\dots,m\}$ *be such that* $j\in\Delta$.

Let $x\in\{0,\dots,h\}$ *be such that*

$$c^x > d_j > c^{x+1}.$$

Suppose that

$$c^i \geq g_{z_i+t_i}, \quad i\geq x+1, \tag{5.4}$$

$$\sum_{i=y+1}^{h} c^i \geq \sum_{\substack{d_i\,<\,c^y\\ i\notin\Delta}} d_i + \sum_{i=t_y+1}^{s} a_i, \quad \textit{for} \quad y\geq x, \tag{5.5}$$

and that for all $d_i < c^{x+1}$, $i\in\Delta$, *the following holds*

$$d_i \geq g_{i+t_w}, \quad \textit{where } w \textit{ is such that} \quad c^w > d_i > c^{w+1}. \tag{5.6}$$

Then

$$d_j \geq g_{j+t_x}. \tag{5.7}$$

Proof. The proof is split into two parts depending on the value of x.

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Let $x = h$. By (3.5), we have that $t_h = s$, hence (5.7) follows from (5.3).

Let $x < h$. Since $c^x > d_j > c^{x+1}$ and $j \in \Delta$, we have that $w_x < z_{x+1} - z_x$.

Let $h_u := \min\{i | d_{i-u+1} < g_i\}$, $u = 1, \dots, s$ ($h_0 := 0$, $h_{s+1} := m + s + 1$).

If $h_{t_x+1} \geq z_{x+1} + t_{x+1}$, then for all $i < z_{x+1} + t_{x+1} - t_x = z_{x+1} - w_x + 1$, we have $d_i \geq g_{i+t_x}$, as wanted. Thus it is sufficient to prove that $h_{t_x+1} \geq z_{x+1} + t_{x+1}$.

Suppose on the contrary that $h_{t_x+1} \leq z_{x+1} + t_{x+1} - 1$, and let u be such that $h_u \leq z_{x+1} + t_{x+1} - 1 < h_{u+1}$. Then $u \geq t_x + 1$. From (5.3) by Lemma 2.2, we have that the following is valid:

$$\sum_{i=z_{x+1}+t_{x+1}}^{m+s} g_i \geq \sum_{i=z_{x+1}+t_{x+1}-u}^{m} d_i + \sum_{i=u+1}^{s} a_i. \tag{5.8}$$

Moreover, since $q_{z_{x+1}-w_x} = t_x + 1$ and since (5.5) is valid, by Lemmas 4.13 and 4.18, we have $s \geq q_{z_{x+1}-w_x} \geq 0$. Hence $z_{x+1} - w_x \in \Delta$ implies that (3.2) is valid for $d_{z_{x+1}-w_x}$, i.e.,

$$\sum_{i=x+1}^{h} c^i < \sum_{\substack{i > z_{x+1} - w_x \\ i \notin \Delta}} d_i + d_{z_{x+1}-w_x} + \sum_{i=q_{z_{x+1}-w_x}+1}^{s} a_i. \tag{5.9}$$

By (5.4) and (5.6), we have

$$\sum_{i=z_{x+1}+t_{x+1}}^{m+s} g_i \leq \sum_{i=x+1}^{h} c^i + \sum_{i=z_{x+1}+1}^{m} d_i - \sum_{\substack{i > z_{x+1} \\ i \notin \Delta}} d_i. \tag{5.10}$$

Inequalities (5.9) and (5.10) together give

$$\sum_{i=z_{x+1}+t_{x+1}}^{m+s} g_i < \sum_{i=z_{x+1}-w_x}^{m} d_i + \sum_{i=t_x+2}^{s} a_i. \tag{5.11}$$

Since $w_x = t_x - t_{x+1} + 1$ and $u \geq t_x + 1$, (5.8) and (5.11) together give

$$\sum_{i=z_{x+1}+t_{x+1}-u}^{z_{x+1}-t_x+t_{x+1}-2} d_i < \sum_{i=t_x+2}^{u} a_i. \tag{5.12}$$

Note that the number of the summands on both sides of (5.12) is equal. We shall prove that the smallest summand on the left hand side, $d_{z_{x+1}-t_x+t_{x+1}-2}$, is larger than or equal to the largest summand on the right hand side, a_{t_x+2}, of (5.12), thus obtaining the contradiction.

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Equation (5.9), together with (5.5) for $y = x$, gives $d_{z_{x+1}-w_x} > a_{t_x+1}$, which finishes our proof. □

Now we can state the dual result:

LEMMA 5.3. *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *and* $\mathbf{d}$ *be partitions as given in (2.29)–(2.32). Let* $\mathbf{g}$ *be a partition such that*

$$\mathbf{g} \prec' (\mathbf{c}, \mathbf{b}).$$

Let c_j, $j \in \{1, \dots, n\}$ *be such that* $j \in S$.

Let $y \in \{0, \dots, h'\}$ *be such that*

$$d^y > c_j > d^{y+1}.$$

Suppose that

$$d^i \geq g_{z'_i + t'_i}, \quad i \geq y+1,$$

$$\sum_{i=x+1}^{h'} d^i \geq \sum_{\substack{c_i \, < \, d^x \\ i \notin S}} c_i + \sum_{i=t'_x+1}^{k} b_i, \quad \textit{for} \quad x \geq y,$$

and that for all $c_i < d^{y+1}$, $i \in S$, *the following holds*

$$c_i \geq g_{i+t'_w}, \quad \textit{where } w \textit{ is such that} \quad d^w > c_i > d^{w+1}.$$

Then

$$c_j \geq g_{j+t'_y}.$$

LEMMA 5.4. *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *and* $\mathbf{d}$ *be partitions as given in (2.29)–(2.32). Let* $\mathbf{g}$ *be a partition such that*

$$\mathbf{g} \prec' (\mathbf{d}, \mathbf{a}). \tag{5.13}$$

Let $x \in \{0, \dots, h-1\}$. *Suppose that*

$$c^i \geq g_{z_i + t_i}, \quad i \geq x+1, \tag{5.14}$$

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$$\sum_{i=y+1}^{h} c^i \geq \sum_{\substack{d_i < c^y \\ i \notin \Delta}} d_i + \sum_{i=t_y+1}^{s} a_i, \quad \textit{for} \quad y \geq x+1, \tag{5.15}$$

and that for all $d_i < c^{x+1}$, $i \in \Delta$, *the following holds*

$$d_i \geq g_{i+t_w}, \quad \textit{where } w \textit{ is such that} \quad c^w > d_i > c^{w+1}. \tag{5.16}$$

Then

$$\sum_{i=x+1}^{h} c^i \geq \sum_{\substack{d_i < c^x \\ i \notin \Delta}} d_i + \sum_{i=t_x+1}^{s} a_i. \tag{5.17}$$

Proof. The proof is split into two parts. First let suppose that $h = x+1$. Then (5.17) becomes

$$c^h \geq a_s. \tag{5.18}$$

Let $h_u := \min\{i | d_{i-u+1} < g_i\}$, $u = 1, \ldots, s$ ($h_0 := 0$, $h_{s+1} := m+s+1$). Let $j \in \{0, \ldots, s\}$ be such that

$$h_j < z_h + t_h = z_h + s \leq h_{j+1}.$$

Then (5.13) together with Lemma 2.2 gives

$$\sum_{i=z_h+s}^{m+s} g_i \geq \sum_{i=z_h+s-j}^{m} d_i + \sum_{i=j+1}^{s} a_i. \tag{5.19}$$

Also, (5.14) and (5.16) give

$$c^h + \sum_{i=z_h+1}^{m} d_i \geq \sum_{i=z_h+s}^{m+s} g_i. \tag{5.20}$$

Inequalities (5.19) and (5.20) give

$$c^h + \sum_{i=z_h+1}^{m} d_i \geq \sum_{i=z_h+s-j}^{m} d_i + \sum_{i=j+1}^{s} a_i. \tag{5.21}$$

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If $j = s$, equation (5.21) becomes $c^h \geq d_{z_h}$, which is impossible by the definition of z_h. Thus, we have $j < s$ and then (5.21) becomes

$$c^h + \sum_{i=z_h+1}^{z_h+s-j-1} d_i \geq \sum_{i=j+1}^{s} a_i.$$

The last implies

$$(s-j)c^h \geq c^h + \sum_{z_h+1}^{z_h+s-j-1} d_i \geq \sum_{i=j+1}^{s} a_i \geq (s-j)a_s,$$

i.e., we obtain (5.18), as wanted.

Now suppose that $x + 1 < h$.

If $w_x > 0$, then by the definition of z_{x+1}, w_x and by Lemmas 4.1 and 4.2, we have that $z_{x+1} - w_x + 1 \notin \Delta$. Hence, $d_{z_{x+1}-w_x+1}$ does not satisfy (3.2), i.e., we have

$$\sum_{i=x+1}^{h} c^i \geq d_{z_{x+1}-w_x+1} + \sum_{\substack{i > z_{x+1} - w_x + 1 \\ i \notin \Delta}} d_i + \sum_{i=q_{z_{x+1}-w_x+1}+1}^{s} a_i. \tag{5.22}$$

Moreover, we have

$$d_{z_{x+1}-w_x+1} + \sum_{\substack{i > z_{x+1} - w_x + 1 \\ i \notin \Delta}} d_i = \sum_{\substack{i > z_x \\ i \notin \Delta}} d_i,$$

and

$$q_{z_{x+1}-w_x+1} = t_x.$$

Hence, (5.22) implies (5.17), as wanted.

Thus, we are left with the case $w_x = 0$. Then

$$t_{x+1} = t_x + 1$$
$$\sum_{\substack{i > z_{x+1} \\ i \notin \Delta}} d_i = \sum_{\substack{i > z_x \\ i \notin \Delta}} d_i.$$

We shall consider two subcases. Let $m_{x+1} < t_{x+1}$. Then

$$c^{x+1} \geq a_{m_{x+1}+1} \geq a_{t_{x+1}} = a_{t_x+1}.$$

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Last, together with (5.15) for $y = x+1$, gives (5.17), as wanted.

Now, let $m_{x+1} \geq t_{x+1}$. Since (5.13) is valid, by Lemma 2.2, we have that for $u \in \{0,\dots,s\}$ such that $h_u < z_{x+1}+t_{x+1} \leq h_{u+1}$, we have

$$\sum_{i=z_{x+1}+t_{x+1}}^{m+s} g_i \geq \sum_{i=z_{x+1}+t_{x+1}-u}^{m} d_i + \sum_{i=u+1}^{s} a_i. \tag{5.23}$$

Now, by (5.14), (5.16) and (5.23), follow

$$\sum_{i=x+1}^{h} c^i + \sum_{\substack{i > z_{x+1} \\ i \in \Delta}} d_i \geq \sum_{i=z_{x+1}+t_{x+1}}^{m+s} g_i \geq \sum_{i=z_{x+1}+t_{x+1}-u}^{m} d_i + \sum_{i=u+1}^{s} a_i. \tag{5.24}$$

Let $(e_{z_{x+1}+t_{x+1}},\dots,e_{z_{x+1}+m_{x+1}})$ be the smallest $m_{x+1}-t_{x+1}+1$ elements among $(a_1,\dots,a_{m_{x+1}},d_1,\dots,d_{z_{x+1}})$.

Now we shall consider the possible values of u. We have three cases:

If $u < t_{x+1} \leq m_{x+1}$, then (5.24) gives

$$\sum_{i=x+1}^{h} c^i - \sum_{\substack{i > z_{x+1} \\ i \notin \Delta}} d_i - \sum_{i=m_{x+1}+1}^{s} a_i \geq - \sum_{i=z_{x+1}+1}^{z_{x+1}+t_{x+1}-u-1} d_i + \sum_{i=u+1}^{m_{x+1}} a_i. \tag{5.25}$$

In this case, we have that

$$a_{u+1} \geq \cdots \geq a_{m_{x+1}-t_{x+1}+u+1} \geq \cdots \geq a_{m_{x+1}} \geq c^{x+1}$$

$$> d_{z_{x+1}+1} \geq \cdots \geq d_{z_{x+1}+t_{x+1}-u-1}.$$

Hence, (5.25) implies

$$\sum_{i=x+1}^{h} c^i - \sum_{\substack{i > z_{x+1} \\ i \notin \Delta}} d_i - \sum_{i=m_{x+1}+1}^{s} a_i \geq \sum_{i=z_{x+1}+t_{x+1}}^{z_{x+1}+m_{x+1}} e_i. \tag{5.26}$$

If $u \geq m_{x+1} \geq t_{x+1}$, then (5.24) gives

$$\sum_{i=x+1}^{h} c^i - \sum_{\substack{i > z_{x+1} \\ i \notin \Delta}} d_i - \sum_{i=m_{x+1}+1}^{s} a_i \geq \sum_{i=z_{x+1}+t_{x+1}-u}^{z_{x+1}} d_i - \sum_{i=m_{x+1}+1}^{u} a_i. \tag{5.27}$$

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In this case, we have that

$$d_{z_{x+1}+t_{x+1}-u} \geq \cdots \geq d_{z_{x+1}} > c^{x+1} \geq a_{m_{x+1}+1} \geq \cdots \geq a_u.$$

Hence, (5.27) implies (5.26).

Finally, if $m_{x+1} > u \geq t_{x+1}$, then (5.24) becomes

$$\sum_{i=x+1}^{h} c^i - \sum_{\substack{i > z_{x+1} \\ i \notin \Delta}} d_i - \sum_{i=m_{x+1}+1}^{s} a_i \geq \sum_{i=z_{x+1}+t_{x+1}-u}^{z_{x+1}} d_i + \sum_{i=u+1}^{m_{x+1}} a_i.$$

Last, by the definition of $e_{z_{x+1}+t_{x+1}}, \ldots, e_{z_{x+1}+m_{x+1}}$ also implies (5.26).

Thus, we have proved that in all the cases (5.26) is valid.

From (5.26) and (5.15), by Lemma 4.7, we have that for all i such that $d_i \in (e_{z_{x+1}+t_{x+1}}, \ldots, \ldots, e_{z_{x+1}+m_{x+1}})$, we have that $i \notin \Delta$.

Denote by $E = (e_{z_{x+1}+t_{x+1}}, \ldots, e_{z_{x+1}+m_{x+1}})$. The rest of the proof is split into two cases.

Case 1. There are no d_i's in E. Then

$$E = (a_{t_{x+1}}, \ldots, a_{m_{x+1}}) = (a_{t_x+1}, \ldots, a_{m_{x+1}}).$$

Then (5.26) gives

$$\sum_{i=x+1}^{h} c^i - \sum_{\substack{i > z_{x+1} \\ i \notin \Delta}} d_i - \sum_{i=m_{x+1}+1}^{s} a_i \geq \sum_{i=t_x+1}^{m_{x+1}} a_i,$$

i.e., we have that (5.17) is valid, as wanted.

Case 2. There is $i \in \{1, \ldots, z_{x+1}\}$ such that $d_i \in E$. By the definition of z_{x+1} and by the definition of E, this implies that $d_{z_{x+1}} \in E$. Thus, by Lemma 4.7, $z_{x+1} \notin \Delta$. Since $w_x = 0$, we have $d_{z_{x+1}} > c^x$. Let $v \geq 0$ be such that $c^{x-v-1} > d_{z_{x+1}} > c^{x-v}$.

Then

$$c^{x-v-1} > d_{z_{x+1}} > c^{x-v} \geq \cdots \geq c^x \geq c^{x+1}.$$

Last, together with the fact that $z_{x+1} \notin \Delta$, by (3.2), gives

$$c^{x-v} + \cdots + c^x + \sum_{i=x+1}^{h} c^i \geq d_{z_{x+1}} + \sum_{\substack{i > z_{x+1} \\ i \notin \Delta}} d_i + \sum_{i=t_x-v+1}^{s} a_i. \tag{5.28}$$

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Moreover, since

$$\sum_{\substack{i > z_{x+1} \\ i \notin \Delta}} d_i = \sum_{\substack{i > z_x \\ i \notin \Delta}} d_i,$$

and since $d_{z_{x+1}} > c^{x-v}$, equation (5.28) implies

$$\sum_{i=x+1}^{h} c^i - \sum_{\substack{i > z_x \\ i \notin \Delta}} d_i - \sum_{i=t_x+1}^{s} a_i \geq \sum_{i=t_x-v+1}^{t_x} a_i - c^{x-v+1} - \cdots - c^x. \tag{5.29}$$

However, since $d_{z_{x+1}} \in E$, the number of a_i's in E is at most $m_{x+1} - t_{x+1}$, and so $a_{t_{x+1}} \notin E$. Hence, $a_{t_{x+1}}$ is bigger than or equal to all the elements from E, i.e., $a_{t_x+1} = a_{t_{x+1}} \geq d_{z_{x+1}} > c^{x-v}$. Hence, (5.29) implies (5.17), as wanted. ∎

Analogously, we obtain the dual result:

LEMMA 5.5. *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *and* $\mathbf{d}$ *be partitions as given in (2.29)–(2.32). Let* $\mathbf{g}$ *be a partition such that*

$$\mathbf{g} \prec' (\mathbf{c}, \mathbf{b}).$$

Let $x \in \{0, \ldots, h' - 1\}$.

Suppose that

$$d^i \geq g_{z'_i + t'_i}, \quad i \geq x+1,$$

$$\sum_{i=y+1}^{h'} d^i \geq \sum_{\substack{c_i < d^y \\ i \notin S}} c_i + \sum_{i=t'_y+1}^{k} b_i, \quad \textit{for} \quad y \geq x+1,$$

and that for all $c_i < d^{x+1}$, $i \in S$, *the following holds*

$$c_i \geq g_{i+t'_w}, \quad \textit{where } w \textit{ is such that} \quad d^w > c_i > d^{w+1}.$$

Then

$$\sum_{i=x+1}^{h'} d^i \geq \sum_{\substack{c_i < d^x \\ i \notin S}} c_i + \sum_{i=t'_x+1}^{k} b_i.$$

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6. Proof of the main result. In this section, we give a proof of Theorem 5.1.

6.1. Necessity. Suppose that there exists a partition $\mathbf{g}$ such that (5.1) and (5.2) are valid. Then, by the definition of the generalized majorization, we have that (i) is valid. So we are left with proving the necessity of the conditions (ii) and (iii).

In fact, we shall prove even more, that apart from (ii) and (iii), the following also hold

$$c^x \geq g_{z_x+t_x}, \quad x = 1, \dots, h, \tag{6.1}$$

$$d^y \geq g_{z'_y+t'_y}, \quad y = 1, \dots, h'. \tag{6.2}$$

The proof goes by induction on $x = 0, \dots, h$ and $y = 0, \dots, h'$. We prove that the conditions (ii), (iii), (6.1) and (6.2) are satisfied for all the elements from the set $X = \{c_i | i \in S\} \cup \{d_j | j \in \Delta\}$.

More precisely, denote and order the elements from the set X in the following way: $f_1 \geq \cdots \geq f_{h+h'}$. Then we shall prove that for every $\alpha \in \{1, \dots, h+h'\}$ the following is valid: if $f_\alpha = c^i$ for some $i = 1, \dots, h$, then (ii) and (6.1) are satisfied for $x = i$, and if $f_\alpha = d^j$ for some $j = 1, \dots, h'$, then (iii) and (6.2) are satisfied for $y = j$.

Before proceeding we note that by Lemma 4.5, the condition (6.1) is equivalent to the following:

For all $y = 1, \dots, h'$ and for all $c_i < d^y$, $i \in S$ the following is valid

$$c_i \geq g_{i+t'_w}, \quad \text{where } d^w > c_i > d^{w+1}.$$

Also, by Lemma 4.6, the condition (6.2) is equivalent to the following:

For all $x = 1, \dots, h$ and for all $d_i < c^x$, $i \in \Delta$ the following is valid

$$d_i \geq g_{i+t_w}, \quad \text{where } c^w > d_i > c^{w+1}.$$

The proof goes by the induction on $w \in \{1, \dots, h+h'\}$, starting from $h+h'$.

As the base of induction, consider $f_{h+h'}$. Suppose that $f_{h+h'} = c^h$. Then we need to prove the necessity of (ii) and (6.1) for $x = h$. However, by the definition of t_h and by Lemma 4.1, condition (ii) is trivially satisfied. Moreover, (6.1) for $x = h$ becomes

$$c^h \geq g_{z_h+s}. \tag{6.3}$$

Let $j \in \{0, \dots, n\}$ be such that $c^h = c_j$. Hence, by Lemma 4.5 (since $t_h = s$ and $d^{h'} > c_j$), (6.3) is equal to $c^h \geq g_{j+t'_{h'}}$, which follows by Lemma 5.3, as wanted.

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If $f_{h+h'} = d^{h'}$, then we are left with proving the necessity of (iii) and (6.2) for $y = h'$. By the definition of $t'_{h'}$ and by Lemma 4.3, condition (iii) is trivially satisfied. Also, completely analogously as in the proof of (6.1) for $x = h$, by Lemma 4.6, we have that (6.2) for $y = h'$ becomes $d^{h'} \geq g_{j+t_h}$. Last follows by Lemma 5.2, as wanted.

Now, we pass to the induction step. Let $w \in \{1, \ldots, h + h' - 1\}$. We are left with proving that if the conditions are satisfied for all f_j with $j \in \{w+1, \ldots, h+h'\}$, then they will be satisfied for f_w, as well. This is equivalent to the following:

If $f_w = c^i$, for some $i = 1, \ldots, h$, and if the conditions (ii) and (6.1) are valid for all $x = i+1, \ldots, h$, and the conditions (iii) and (6.2) are valid for all y such that $d^y < c^i$, then we are left with proving that the conditions (ii) and (6.1) are also valid for $x = i$.

In order to prove (ii), it is enough to apply the result of Lemma 5.4. As for (6.1), let $u \in \{0, \ldots, h'\}$ be such that

$$d^u > c^i > d^{u+1}.$$

Then by the induction hypothesis, we have that (6.2) and (iii) are satisfied for all $y = u+1, \ldots, h'$, and that (6.1) is satisfied for all $j \in S$ such that $c_j < d^{u+1}$. So, by applying Lemma 5.5, we have that the condition (iii) is satisfied for $y = u$ as well. Hence, we can now apply Lemma 5.3, and obtain (6.1) for $x = i$, as wanted.

Analogously as above, by the symmetry of the sets S and Δ and by applying Lemmas 5.2, 5.4 and 5.5, we obtain that if $f_w = d^i$, for some $i = 1, \ldots, h'$, and if the conditions (iii) and (6.2) are valid for all $y = i+1, \ldots, h$, and the conditions (ii) and (6.1) are valid for all x such that $c^x < d^i$, then the conditions (iii) and (6.2) are valid for $y = i$, as wanted.

Finally we are left with proving that the condition (ii) is satisfied for $x = 0$, and that (iii) is satisfied for $y = 0$. The first follows by Lemma 5.4 for $x = 0$, while the second follows by Lemma 5.5 for $x = 0$.

This ends the proof of the necessity of the conditions (i), (ii) and (iii). $\square$

6.2. Sufficiency. Let us suppose that the conditions (i), (ii) and (iii) are satisfied.

We need to define a certain partition $\mathbf{g}$ such that (5.1) and (5.2) are valid.

We shall do this in two steps. First we are going to define a partition $\bar{\mathbf{g}} = (\bar{g}_1, \ldots, \bar{g}_{m+s})$ that satisfies

$$d_i \geq \bar{g}_{i+s}, \quad i = 1, \ldots, m, \tag{6.4}$$

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$$c_i \geq \bar{g}_{i+k}, \quad i = 1, \ldots, n, \tag{6.5}$$

$$\sum_{i=\bar{h}_j+1}^{m+s} \bar{g}_i \geq \sum_{i=\bar{h}_j-j+1}^{m} d_i + \sum_{i=j+1}^{s} a_i, \quad j = 1, \ldots, s, \tag{6.6}$$

where $\bar{h}_j = \min\{i | d_{i-j+1} < \bar{g}_i\}$,

$$\sum_{i=\bar{h}'_j+1}^{n+k} \bar{g}_i \geq \sum_{i=\bar{h}'_j-j+1}^{n} c_i + \sum_{i=j+1}^{k} b_i, \quad j = 1, \ldots, k, \tag{6.7}$$

where $\bar{h}'_j = \min\{i | c_{i-j+1} < \bar{g}_i\}$, and

$$\sum_{i=1}^{m+s} \bar{g}_i \geq \sum_{i=1}^{m} d_i + \sum_{i=1}^{s} a_i. \tag{6.8}$$

Then we shall define the wanted partition $\mathbf{g}$, by decreasing the elements of $\bar{\mathbf{g}}$ so that we obtain the correct sum, while preserving the remaining properties of the generalized majorization.

From the definition of t_0 and t'_0, we have $t_0 = t'_0 = m + s - h - h'$. By Lemma 4.13, $t_0 \geq 0$, and we define

$$\bar{g}_j := \max(d_1, a_1, c_1) + 1, \quad j = 1, \ldots, t_0.$$

The remaining $h+h'$ of $\bar{g}_i$'s, i.e., $\bar{g}_{t_0+1}, \ldots, \bar{g}_{m+s}$, we define as a nonincreasing ordering of all c_i's with $i \in S$ and of all d_j's with $j \in \Delta$.

We can write this explicitly in the following way, by using the definitions of z_i, t_i and the property (4.2) on the placement of d_j's with $j \in \Delta$:

$$\bar{g}_j := \max(d_1, a_1, c_1) + 1, \quad j = 1, \ldots, t_0, \tag{6.9}$$

$$\bar{g}_j := d_{j-t_i}, \quad z_i + t_i < j < z_{i+1} + t_{i+1}, \text{ for some } i = 0, \ldots, h, \tag{6.10}$$

$$\bar{g}_{z_i+t_i} := c^i, \quad i = 1, \ldots, h. \tag{6.11}$$

Recall that by Lemma 4.9 the sequence $z_i + t_i$, $i = 0, \ldots, h$, is strictly increasing, and thus (6.10) and (6.11) are well-defined.

Dually, we can also write the explicit formula for $\bar{g}_i$'s by exchanging the roles of c_i's and d_j's:

$$\bar{g}_j := \max(d_1, a_1, c_1) + 1, \quad j = 1, \ldots, t'_0,$$

$$\bar{g}_j := c_{j-t'_i}, \quad z'_i + t'_i < j < z'_{i+1} + t'_{i+1}, \text{ for some } i = 0, \ldots, h', \tag{6.12}$$

$$\bar{g}_{z'_i+t'_i} := d^i, \quad i = 1, \ldots, h'. \tag{6.13}$$

Recall that by Lemma 4.10 the sequence $z'_i + t'_i$, $i = 0, \ldots, h'$, is strictly increasing, and thus (6.12) and (6.13) are well-defined.

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Proof of (6.4). By Lemma 4.18, we have that $t_0 \leq s$, and so $\bar{g}_j$'s involved in (6.4) are the ones defined by (6.10) and (6.11). From the definition, all such $\bar{g}_j$'s satisfy

$$\bar{g}_j \leq d_{j-t_i}, \quad \text{where } i \text{ is such that } z_i + t_i \leq j < z_{i+1} + t_{i+1}. \tag{6.14}$$

Finally, since $t_i \leq s$ (Lemma 4.18), for all $i = 0, \ldots, h$, (6.14) implies (6.4), as wanted.

Proof of (6.5). Equation (6.5) follows analogously as (6.4), by duality and by Lemma 4.19.

Proof of (6.6). From the definitions of $\bar{g}_1 \geq \cdots \geq \bar{g}_{m+k}$, we have that

$$\bar{h}_j = j, \qquad j = 1, \ldots, t_0, \tag{6.15}$$

$$\bar{h}_j = z_u + t_u, \quad j = t_0 + 1, \ldots, s, \tag{6.16}$$

where $u := \min\{i \in \{1, \ldots, h\} | t_i = j\}$ (note that u is well-defined by Lemma 4.15).

Indeed, (6.15) follows from the definition (6.9). As for (6.16), first note that $\bar{h}_j$, for some $j \in \{t_0 + 1, \ldots, s\}$ is always equal to $z_i + t_i$, for some $i \in \{1, \ldots, h\}$. Otherwise, let $i \in \{0, \ldots, h\}$ be such that $z_i + t_i < \bar{h}_j < z_{i+1} + t_{i+1}$. Then by (6.10), we have $\bar{g}_{\bar{h}_j} = d_{\bar{h}_j - t_i}$, and from the definition of $\bar{h}_j$, we have $d_{\bar{h}_j - j+1} < \bar{g}_{\bar{h}_j} = d_{\bar{h}_j - t_i}$, which implies $j \leq t_i$, and so $i \geq 1$. But then, from (6.11), $\bar{g}_{z_i+t_i} = c^i > d_{z_i+1} \geq d_{z_i+t_i-j+1}$, and so $\bar{h}_j \leq z_i + t_i$.

Thus, for every $j \in \{t_0+1, \ldots, s\}$, $\bar{h}_j$ is of the form z_i+t_i, for some $i \in \{1, \ldots, h\}$. Finally, we claim that $\bar{h}_j = z_u + t_u$, where u is the minimal index $i \in \{1, \ldots, h\}$ such that $t_i = j$. First of all, we have $\bar{g}_{z_u+t_u} = c^u > d_{z_u+1} = d_{z_u+t_u-j+1}$. Moreover, for all $\alpha < z_u + t_u$, we have that $\bar{g}_\alpha \leq d_{\alpha-j+1}$. Indeed, as shown in the previous paragraph it is enough to prove this fact for α of the form $z_i + t_i$, and since $\alpha < z_u + t_u$, we are left with proving $\bar{g}_{z_\beta+t_\beta} \leq d_{z_\beta+t_\beta-j+1}$, for $\beta < u$. From the definition of u and since $t_{i+1} \leq t_i + 1$, for all $i \in \{0, \ldots, h-1\}$, we have that $t_\beta < j$, for all $\beta < u$. Therefore $\bar{g}_{z_\beta+t_\beta} = c^\beta \leq d_{z_\beta} \leq d_{z_\beta+t_\beta-j+1}$, for all $\beta < u$, as wanted.

Hence, we have $\bar{h}_j = z_u + t_u$, where $u = \min\{i \in \{1, \ldots, h\} | t_i = j\}$, as wanted.

Now, the proof of (6.6) is split into two cases, depending on the value of $\bar{h}_j$, $j = 1, \ldots, s$.

Case $j = 1, \ldots, t_0$: Then $\bar{h}_j = j$, and the condition (6.6) becomes

$$\sum_{i=j+1}^{m+s} \bar{g}_i \geq \sum_{i=1}^{m} d_i + \sum_{i=j+1}^{s} a_i, \quad j = 1, \ldots, t_0. \tag{6.17}$$

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The condition (ii) for $x = 0$ gives

$$\sum_{i=1}^{h} c^i \geq \sum_{i \notin \Delta} d_i + \sum_{i=t_0+1}^{s} a_i,$$

i.e.,

$$\sum_{i=1}^{h} c^i + \sum_{j=1}^{h'} d^j \geq \sum_{i=1}^{m} d_i + \sum_{i=t_0+1}^{s} a_i.$$

Hence, we obtain

$$\sum_{i=t_0+1}^{m+s} \bar{g}_i \geq \sum_{i=1}^{m} d_i + \sum_{i=t_0+1}^{s} a_i.$$

Since $\bar{g}_{t_0} \geq a_1$, last gives (6.17), as wanted.

Case $j = t_0 + 1, \ldots, s$: Then $\bar{h}_j = z_u + t_u = z_u + j$, where $u \in \{1, \ldots, h\}$ is the minimal index such that $t_u = j$. To prove (6.6), we are going to prove

$$\sum_{i=z_x+t_x+1}^{m+s} \bar{g}_i \geq \sum_{i=z_x+1}^{m} d_i + \sum_{i=t_x+1}^{s} a_i, \quad \text{for all} \quad x = 1, \ldots, h.$$

By the definition of $\bar{g}_i$'s, we have that

$$\sum_{i=z_x+t_x+1}^{m+s} \bar{g}_i = \sum_{i=z_x+1}^{m} d_i - \sum_{\substack{i > z_x \\ i \notin \Delta}} d_i + \sum_{i=x+1}^{h} c^i.$$

Thus, we are left with proving

$$\sum_{i=x+1}^{h} c^i \geq \sum_{\substack{d_i < c^x \\ i \notin \Delta}} d_i + \sum_{i=t_x+1}^{s} a_i, \quad x = 1, \ldots, h.$$

Last is exactly the condition (ii), which finishes the proof.

Proof of (6.7). Equation (6.7) follows analogously as (6.6) by the symmetry of the sets S and Δ, i.e., by duality between c_i's and d_i's, and by the condition (iii).

Proof of (6.8). Again, from the condition (ii) for $x = 0$ and the definition of $\bar{g}_i$'s, we have:

$$\sum_{i=t_0+1}^{m+s} \bar{g}_i \geq \sum_{i=1}^{m} d_i + \sum_{i=t_0+1}^{s} a_i. \tag{6.18}$$

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Equation (6.18), together with $\bar{g}_{t_0} \geq a_1$, gives

$$\sum_{i=1}^{m+s} \bar{g}_i \geq \sum_{i=1}^{m} d_i + \sum_{i=1}^{s} a_i,$$

as wanted.

Now, our aim is to decrease some of the $\bar{g}_i$'s in order to define g_i's which will satisfy

$$d_i \geq g_{i+s}, \quad i = 1, \ldots, m, \tag{6.19}$$

$$c_i \geq g_{i+k}, \quad i = 1, \ldots, n, \tag{6.20}$$

$$\sum_{i=h_j+1}^{m+s} g_i \geq \sum_{i=h_j-j+1}^{m} d_i + \sum_{i=j+1}^{s} a_i, \quad j = 1, \ldots, s, \tag{6.21}$$

where $h_j = \min\{i | d_{i-j+1} < g_i\}$,

$$\sum_{i=h'_j+1}^{n+k} g_i \geq \sum_{i=h'_j-j+1}^{n} c_i + \sum_{i=j+1}^{k} b_i, \quad j = 1, \ldots, k, \tag{6.22}$$

where $h'_j = \min\{i | c_{i-j+1} < g_i\}$, and

$$\sum_{i=1}^{m+s} g_i = \sum_{i=1}^{m} d_i + \sum_{i=1}^{s} a_i. \tag{6.23}$$

We shall do this in the following way: start from $\bar{g}_1$ and decrease it till $\bar{g}_2$. If the sum is OK, stop. If not proceed by decreasing $\bar{g}_1$ and $\bar{g}_2$ till $\bar{g}_3$. And so on until we have decreased $\bar{g}_i$'s such that (6.23) is valid.

More precisely, let $\Omega := \sum_{i=1}^{m+s} \bar{g}_i - (\sum_{i=1}^{s} a_i + \sum_{i=1}^{m} d_i) \geq 0$ and let $f := \min\{i | \sum_{j=1}^{i} \bar{g}_j - i\bar{g}_i \geq \Omega\}$. Then we are going to define g_i, $i = 1, \ldots, m+s$, such that

$$\sum_{i=1}^{m+s} g_i = \sum_{i=1}^{m} d_i + \sum_{i=1}^{s} a_i,$$

$$g_i = \bar{g}_i, \quad \text{for all} \quad i \geq f,$$

$$\bar{g}_{f-1} \geq g_i \geq \bar{g}_f \quad \text{for all} \quad i = 1, \ldots, f-1,$$

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and

$$g_1 \geq g_{f-1} \geq g_1 - 1.$$

In other words, we decrease the smallest possible number of $\bar{g}_i$'s, such that the sum is correct, and such that $g_1 \geq g_2 \geq \ldots \geq g_{f-1}$ becomes the most homogeneous partition of $\bar{g}_1 + \bar{g}_2 + \cdots + \bar{g}_{f-1} - \Omega$. Such defined $g_1 \geq \cdots \geq g_{m+s}$ satisfy (6.23).

Since $\bar{g}_i \geq g_i$, $i = 1, \ldots, m+s$, from (6.4) and (6.5), we have that (6.19) and (6.20) are valid. So we are left with proving (6.21) and (6.22).

Proof of (6.21). Follows directly by applying Lemma 2.4 for $\bar{\mathbf{g}}$, $\mathbf{d}$, $\mathbf{a}$, f and $\mathbf{g}$.

Proof of (6.22). Follows by Lemma 2.4 for $\bar{\mathbf{g}}$, $\mathbf{c}$, $\mathbf{b}$, f and $\mathbf{g}$.

Now, conditions (6.19), (6.21) and (6.23) give (5.1), while conditions (6.20), (6.22) and (6.23) give (5.2). This finishes our proof. ☐

7. Some corollaries. In course of the proof of Theorem 5.1, we have in fact obtained a solution for the analogous slightly relaxed problem. Namely, we can define a weak generalized majorization:

Definition 7.1. Let $\mathbf{a} = (a_1, \ldots, a_s)$, $\mathbf{d} = (d_1, \ldots, d_m)$ and $\mathbf{g} = (g_1, \ldots, g_{m+s})$ be nonincreasing partitions. We write

$$\mathbf{g} \prec'' (\mathbf{d}, \mathbf{a})$$

if the following three conditions are satisfied:

$$d_i \geq g_{i+s}, \qquad i = 1, \ldots, m,$$
$$\sum_{i=h_j+1}^{m+s} g_i - \sum_{i=h_j-j+1}^{m} d_i \geq \sum_{i=j+1}^{s} a_i, \qquad j = 1, \ldots, s$$
$$\sum_{i=1}^{m+s} g_i \geq \sum_{i=1}^{m} d_i + \sum_{i=1}^{s} a_i,$$

where $h_j := \min\{i | d_{i-j+1} < g_i\}$, $j = 1, \ldots, s$.

Then by repeating the same proof as for Theorem 5.1, we have:

Theorem 7.2. *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *and* $\mathbf{d}$ *be partitions as given in (2.29)–(2.32).*

There exists a partition $\mathbf{g} = (g_1, \ldots, g_{m+s})$, *such that*

$$\mathbf{g} \prec'' (\mathbf{d}, \mathbf{a})$$
$$\mathbf{g} \prec'' (\mathbf{c}, \mathbf{b})$$

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if and only if the following conditions are valid:

$$(i) \sum_{i=x+1}^{h} c^i \geq \sum_{\substack{d_i \, < \, c^x \\ i \notin \Delta}} d_i + \sum_{i=t_x+1}^{s} a_i, \quad x = 0, \dots, h,$$

$$(ii) \sum_{i=y+1}^{h'} d^i \geq \sum_{\substack{c_i \, < \, d^y \\ i \notin S}} c_i + \sum_{i=t'_y+1}^{k} b_i, \quad y = 0, \dots, h'.$$

Recall that by the result obtained in Subsection 2.2, in Theorem 7.2 we are only considering partitions $\mathbf{c}$ and $\mathbf{d}$ such that for all $i = 1, \dots, n$, and for all $j = 1, \dots, m$, we have $c_i \neq d_j$.

7.1. Nonnegative partitions. In this subsection, we are considering Problem 1, in a case when the involved partitions consist only of nonnegative integers. In this case, some improvements can be made in Theorem 5.1. Also, this restriction is particularly important in the applications in the matrix and matrix pencils completion problems.

Let $\mathbf{c} = (c_1, \dots, c_n)$ and $\mathbf{d} = (d_1, \dots, d_m)$ be partitions such that $c_1 \geq \cdots \geq c_c > c_{c+1} = \cdots = c_n = 0$ and $d_1 \geq \cdots \geq d_d > d_{d+1} = \cdots = d_m = 0$. Then we can determine the number of nonzero elements of partitions $\bar{\mathbf{g}}$ and $\mathbf{g}$ obtained as in the sufficiency part of the proof of Theorem 5.1 (Section 6.2).

Namely, let $\bar{\mathbf{g}} = (\bar{g}_1, \dots, \bar{g}_{m+s})$, where $\bar{g}_1 \geq \cdots \geq \bar{g}_{\bar{g}} > \bar{g}_{\bar{g}+1} = \cdots = \bar{g}_{m+s} = 0$ be defined as in (6.9)–(6.11). Then,

$$m + s - \bar{g} = \max(n - c, m - d). \tag{7.1}$$

Indeed, if both c_i's and d_j's contain zeros (i.e., if $c < n$ and $d < m$), then by Proposition 2.6 we can "erase" the same number of zeros from both of them. So, without loss of generality, we can assume that only one of the partitions $\mathbf{c}$ or $\mathbf{d}$ contains zeros, say $\mathbf{d}$. Then from the definition of the set Δ, all $j = d+1, \dots, m$ satisfy $j \in \Delta$ (since for all of them $q_j = s+1$). Since $\{\bar{g}_i\}_{i=t_0+1}^{m+s} = \{c_i | i \in S\} \cup \{d_j | j \in \Delta\}$, we have that $m + s - \bar{g} = m - d$, as wanted. We get the analogous result in the case when only partition $\mathbf{c}$ contains zeros. Altogether, we have proved (7.1), as wanted.

Equation (7.1) gives

$$\bar{g} = \min(c + k, d + s).$$

Now, from the definition of $\mathbf{g} = (g_1, \dots, g_{m+s})$ with $g_1 \geq \cdots \geq g_g > g_{g+1} = \cdots = g_{m+s} = 0$, we have that either $g = \bar{g}$, or we have that $g_1 = \cdots = g_g = 1$,

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in which case $g \leq \bar{g}$. In the latter case we have $g = \sum_{i=1}^{m+s} g_i = \sum_{i=1}^{m} d_i + \sum_{i=1}^{s} a_i = \sum_{i=1}^{n} c_i + \sum_{i=1}^{k} b_i$, and thus we obtain:

Proposition 7.3. *Let* $\mathbf{c} = (c_1, \ldots, c_n)$ *and* $\mathbf{d} = (d_1, \ldots, d_m)$ *be nonnegative partitions with* $c_1 \geq \cdots \geq c_c > c_{c+1} = \cdots = c_n = 0$ *and* $d_1 \geq \cdots \geq d_d > d_{d+1} = \cdots = d_m = 0$. *Let* $\mathbf{g} = (g_1, \ldots, g_{m+s})$ *be partition defined in the sufficiency part of Theorem 5.1 in Section 6.2. Then* $g_1 \geq \cdots \geq g_g > g_{g+1} = \cdots = g_{m+s} = 0$ *with*

$$g = \min(c + k, d + s, \sum_{i=1}^{m} d_i + \sum_{i=1}^{s} a_i).$$

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