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# ON $K$-POTENT MATRICES*

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**Abstract.** The paper provides extensive and systematic investigations of $k$-potent complex matrices, with a particular attention paid to tripotent matrices. Several new properties of $k$-potent matrices are identified. Furthermore, some results known in the literature are reestablished with simpler proofs than in the original sources and often in a generalized form.

**Key words.** Projector, Idempotent matrix, Tripotent matrix, Powers of matrices, Sylvester's inequality, Frobenius' inequality, Rank, Column space, Null space, EP matrix, Group matrix, Partial isometry.

**AMS subject classifications.** 15A57, 14A24, 15A09.

**1. Introduction.** Let $\mathbb{C}_{m,n}$ denote the set of $m \times n$ complex matrices. The symbols $\mathbf{A}^*$, $\mathcal{R}(\mathbf{A})$, $\mathcal{N}(\mathbf{A})$, and $\mathrm{rk}(\mathbf{A})$ will stand for the conjugate transpose, column space (range), null space, and rank of $\mathbf{A} \in \mathbb{C}_{m,n}$, respectively. Moreover, $\mathbf{I}_n$ will be the identity matrix of order $n$, and for given $\mathbf{A} \in \mathbb{C}_{n,n}$ we define $\overline{\mathbf{A}} = \mathbf{I}_n - \mathbf{A}$ and $\widehat{\mathbf{A}} = \mathbf{I}_n + \mathbf{A}$. It will be also assumed that $\mathbf{A}^0 = \mathbf{I}_n$.

Customarily, with $\mathbf{A}^\dagger$ we will denote the Moore–Penrose inverse of $\mathbf{A} \in \mathbb{C}_{m,n}$, i.e., the unique matrix satisfying the following four Penrose conditions:

$$\mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{A}, \quad \mathbf{A}^\dagger\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger, \quad \mathbf{A}\mathbf{A}^\dagger = (\mathbf{A}\mathbf{A}^\dagger)^*, \quad \mathbf{A}^\dagger\mathbf{A} = (\mathbf{A}^\dagger\mathbf{A})^*. \tag{1.1}$$

Recall that $\mathbf{A}\mathbf{A}^\dagger$ and $\mathbf{I}_n - \mathbf{A}^\dagger\mathbf{A}$ are the orthogonal projectors onto $\mathcal{R}(\mathbf{A})$ and $\mathcal{N}(\mathbf{A})$, respectively. Correspondingly, $\mathbf{A}^\dagger\mathbf{A}$ and $\mathbf{I}_n - \mathbf{A}\mathbf{A}^\dagger$ are the orthogonal projectors onto $\mathcal{R}(\mathbf{A}^*)$ and $\mathcal{N}(\mathbf{A}^*)$.

Another generalized inverse exploited in the present paper is the group inverse of $\mathbf{A} \in \mathbb{C}_{n,n}$, which is understood as the unique matrix $\mathbf{A}^\#$ satisfying the equations:

$$\mathbf{A}\mathbf{A}^\#\mathbf{A} = \mathbf{A}, \quad \mathbf{A}^\#\mathbf{A}\mathbf{A}^\# = \mathbf{A}^\#, \quad \mathbf{A}\mathbf{A}^\# = \mathbf{A}^\#\mathbf{A}.$$

It is known that not every square matrix has a group inverse, and that its existence is restricted to so called group matrices, shortly called GP matrices.

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Several subsequent derivations will be based on the following result established in [22, Corollary 6] as a particular version of the singular value decomposition.

LEMMA 1.1. *Let $\mathbf{A} \in \mathbb{C}_{n,n}$ be of rank $r$. Then there exists unitary $\mathbf{U} \in \mathbb{C}_{n,n}$ such that*

$$\mathbf{A} = \mathbf{U} \begin{bmatrix} \mathbf{\Sigma K} & \mathbf{\Sigma L} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{U}^*, \tag{1.2}$$

*where $\mathbf{\Sigma} = \mathrm{diag}(\sigma_1 \mathbf{I}_{r_1}, \ldots, \sigma_t \mathbf{I}_{r_t})$ is the diagonal matrix of singular values of $\mathbf{A}$, $\sigma_1 > \sigma_2 > \cdots > \sigma_t > 0$, $r_1 + r_2 + \cdots + r_t = r$, and $\mathbf{K} \in \mathbb{C}_{r,r}$, $\mathbf{L} \in \mathbb{C}_{r,n-r}$ satisfy*

$$\mathbf{KK}^* + \mathbf{LL}^* = \mathbf{I}_r. \tag{1.3}$$

From Lemma 1.1 it follows that

$$\mathbf{A}^* = \mathbf{U} \begin{bmatrix} \mathbf{K}^*\mathbf{\Sigma} & \mathbf{0} \\ \mathbf{L}^*\mathbf{\Sigma} & \mathbf{0} \end{bmatrix} \mathbf{U}^* \quad \text{and} \quad \mathbf{A}^\dagger = \mathbf{U} \begin{bmatrix} \mathbf{K}^*\mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{L}^*\mathbf{\Sigma}^{-1} & \mathbf{0} \end{bmatrix} \mathbf{U}^*. \tag{1.4}$$

Another essential fact is that matrices $\mathbf{K}$ and $\mathbf{L}$ involved in representation (1.2) satisfy

$$\mathbf{L} = \mathbf{0} \;\Leftrightarrow\; \mathbf{K}^* = \mathbf{K}^{-1}. \tag{1.5}$$

Formulae (1.2)–(1.4) as well as equivalence (1.5) can be exploited to confirm the following characterizations of known classes of matrices; see e.g., [8, pp. 2799, 2800] or [10, p. 1224 and Corollary 1].

LEMMA 1.2. *Let $\mathbf{A} \in \mathbb{C}_{n,n}$ be of the form* (1.2). *Then:*

(i) $\mathbf{A}$ *Hermitian (i.e., $\mathbf{A}^* = \mathbf{A}$) if and only if $\mathbf{L} = \mathbf{0}$ and $\mathbf{\Sigma K} = \mathbf{K}^*\mathbf{\Sigma}$,*

(ii) $\mathbf{A}$ *is normal (i.e., $\mathbf{AA}^* = \mathbf{A}^*\mathbf{A}$) if and only if $\mathbf{L} = \mathbf{0}$ and $\mathbf{\Sigma K} = \mathbf{K\Sigma}$,*

(iii) $\mathbf{A}$ *is EP (i.e., $\mathbf{AA}^\dagger = \mathbf{A}^\dagger\mathbf{A}$) if and only if $\mathbf{L} = \mathbf{0}$,*

(iv) $\mathbf{A}$ *is GP (i.e., $\mathrm{rk}(\mathbf{A}^2) = \mathrm{rk}(\mathbf{A})$) if and only if $\mathrm{rk}(\mathbf{K}) = r$,*

(v) $\mathbf{A}$ *is a partial isometry (i.e., $\mathbf{A} = \mathbf{AA}^*\mathbf{A}$) if and only if $\mathbf{\Sigma} = \mathbf{I}_r$,*

(vi) $\mathbf{A}$ *is an orthogonal projector (i.e., $\mathbf{A}^2 = \mathbf{A} = \mathbf{A}^*$) if and only if $\mathbf{\Sigma} = \mathbf{I}_r$, $\mathbf{K} = \mathbf{I}_r$,*

(vii) $\mathbf{A}$ *is $k$-potent, $k \in \mathbb{N}$, (i.e., $\mathbf{A}^k = \mathbf{A}$) if and only if $(\mathbf{\Sigma K})^{k-1} = \mathbf{I}_r$,*

(viii) $\mathbf{A}$ *is nilpotent of index 2 (i.e., $\mathbf{A}^2 = \mathbf{0}$) if and only if $\mathbf{K} = \mathbf{0}$,*

(ix) $\mathbf{A}$ *is SR (i.e., $\mathcal{R}(\mathbf{A}) + \mathcal{R}(\mathbf{A}^*) = \mathbb{C}_{n,1}$) if and only if $\mathrm{rk}(\mathbf{L}) = n - r$,*

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(x) $\mathbf{A}$ *is DR (i.e.,* $\mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\mathbf{A}^*) = \{\mathbf{0}\}$*) if and only if* $\mathrm{rk}(\mathbf{L}) = r$*,*

(xi) $\mathbf{A}$ *is a generalized projector (i.e.,* $\mathbf{A}^2 = \mathbf{A}^*$*) if and only if* $\mathbf{L} = \mathbf{0}$*,* $\mathbf{\Sigma} = \mathbf{I}_r$*,* $\mathbf{K}^3 = \mathbf{I}_r$*,*

(xii) $\mathbf{A}$ *is a hypergeneralized projector (i.e.,* $\mathbf{A}^2 = \mathbf{A}^\dagger$*) if and only if* $\mathbf{L} = \mathbf{0}$*,* $(\mathbf{\Sigma}\mathbf{K})^3 = \mathbf{I}_r$*,*

(xiii) $\mathbf{A}$ *is a contraction (i.e., the length of* $\mathbf{A}\mathbf{x}$ *does not exceed the length of* $\mathbf{x}$ *for all* $\mathbf{x} \in \mathbb{C}_{n,1}$*) if and only if* $\mathbf{I}_r - \mathbf{\Sigma}^2 = \mathbf{B}\mathbf{B}^*$ *for some* $\mathbf{B} \in \mathbb{C}_{r,r}$*.*

The literature devoted to $k$-potent matrices, in particular to idempotents and tripotents, is quite extensive. The fact that such matrices attract lots of attention results mostly from their possible applications. Collections of results dealing with idempotent and tripotent matrices are available in several monographs emphasizing their usefulness in statistics, for instance [18, Section 12.4], [34, Chapter 5], [35, Chapter 7], and [36, Sections 8.6, 8.7, and 20.5.3]. Furthermore, a number of papers shed light on the role which those matrices play in the problems occurring in the distribution of quadratic forms, e.g., [1], [2], [4], [6], [11], [12], [29], [37], and [38] to mention just a few. Quadripotent matrices recently focused also some special interest, which originates mostly from the fact that they occur naturally in considerations dealing with generalized and hypergeneralized projectors introduced in [20]. In addition to the papers [25], [27], [30], [31], and [43], each of which contains a systematical study over a selected topic concerning $k$-potent matrices, a collection of related isolated results was published in recent years in a number of independent articles. Many of these results are recalled in the present paper, in which $k$-potent matrices are revisited and extensively investigated. Besides several original characteristics of such matrices, the paper reestablishes also selected results scattered in the literature, often in a generalized form and by means of new proofs. Apart from the Introduction, the paper consists of 3 sections, of which the next one is concerned with rank formulae, Section 3 is devoted to tripotent matrices, and the last one provides some results referring to known classes of matrices.

**2. Rank formulae.** An important role in the considerations of the present section is played by the well-known Frobenius inequality, which for $\mathbf{A} \in \mathbb{C}_{m,n}$, $\mathbf{B} \in \mathbb{C}_{n,p}$, and $\mathbf{C} \in \mathbb{C}_{p,q}$ has the form

$$\mathrm{rk}(\mathbf{ABC}) \geqslant \mathrm{rk}(\mathbf{AB}) + \mathrm{rk}(\mathbf{BC}) - \mathrm{rk}(\mathbf{B}). \tag{2.1}$$

As was shown in [39, Theorem 1], equality in (2.1) holds if and only if there exist matrices $\mathbf{X} \in \mathbb{C}_{q,p}$ and $\mathbf{Y} \in \mathbb{C}_{n,m}$ such that $\mathbf{BCX} + \mathbf{YAB} = \mathbf{B}$. From (2.1) it straightforwardly follows that $\mathbf{A} \in \mathbb{C}_{n,n}$ satisfies

$$\mathrm{rk}(\mathbf{A}^3) \geqslant 2\mathrm{rk}(\mathbf{A}^2) - \mathrm{rk}(\mathbf{A}), \tag{2.2}$$

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with equality holding if and only if there exist matrices $\mathbf{X}, \mathbf{Y} \in \mathbb{C}_{n,n}$ such that $\mathbf{A}^2\mathbf{X} + \mathbf{Y}\mathbf{A}^2 = \mathbf{A}$. It is seen that equality in (2.2) is satisfied whenever $\mathbf{A}^k = \mathbf{A}$ for some $k \in \mathbb{N}$, $k > 1$.

Replacing $\mathbf{B}$ in (2.1) with $\mathbf{I}_n$ leads to the so called Sylvester's inequality, which for $\mathbf{A} \in \mathbb{C}_{m,n}$ and $\mathbf{C} \in \mathbb{C}_{n,q}$ reads

$$\mathrm{rk}(\mathbf{AC}) \geqslant \mathrm{rk}(\mathbf{A}) + \mathrm{rk}(\mathbf{C}) - n. \tag{2.3}$$

It is known that equality in (2.3) holds if and only if $\mathcal{N}(\mathbf{A}) \subseteq \mathcal{R}(\mathbf{C})$; see [33, Theorem 3.4.14]. If $\mathbf{A} \in \mathbb{C}_{n,n}$, then an easy consequence of (2.3) is $\mathrm{rk}(\mathbf{A}^2) \geqslant 2\mathrm{rk}(\mathbf{A}) - n$, with equality holding if and only if $\mathcal{N}(\mathbf{A}) \subseteq \mathcal{R}(\mathbf{A})$. Actually, a lower bound for the rank of any power $k \in \mathbb{N}$ of $\mathbf{A}$ reads $\mathrm{rk}(\mathbf{A}^k) \geqslant k\mathrm{rk}(\mathbf{A}) - (k-1)n$.

The first theorem provides an expression for the rank of the difference $\mathbf{A} - \mathbf{ABA}$, which was mentioned in [26, p. 269].

Theorem 2.1. *Let* $\mathbf{A} \in \mathbb{C}_{m,n}$ *and* $\mathbf{B} \in \mathbb{C}_{n,m}$. *Then* $\mathrm{rk}(\mathbf{A} - \mathbf{ABA}) = \mathrm{rk}(\mathbf{A}) + \mathrm{rk}(\mathbf{I}_n - \mathbf{BA}) - n$.

*Proof.* Clearly, $\mathbf{A} - \mathbf{ABA} = \mathbf{A}(\mathbf{I}_n - \mathbf{BA})$. By (2.3), we have $\mathrm{rk}[\mathbf{A}(\mathbf{I}_n - \mathbf{BA})] \geqslant \mathrm{rk}(\mathbf{A}) + \mathrm{rk}(\mathbf{I}_n - \mathbf{BA}) - n$, with equality holding if and only if $\mathcal{N}(\mathbf{A}) \subseteq \mathcal{R}(\mathbf{I}_n - \mathbf{BA})$. To show that this inclusion is always satisfied, let a vector $\mathbf{x} \in \mathbb{C}_{n,1}$ be such that $\mathbf{x} \in \mathcal{N}(\mathbf{A})$. Then $\mathbf{Ax} = \mathbf{0}$, from where $(\mathbf{I}_n - \mathbf{BA})\mathbf{x} = \mathbf{x}$, whence $\mathbf{x} \in \mathcal{R}(\mathbf{I}_n - \mathbf{BA})$. ∎

Alternatively, the identity given in Theorem 2.1 could read $\mathrm{rk}(\mathbf{A} - \mathbf{ABA}) = \mathrm{rk}(\mathbf{A}) + \mathrm{rk}(\mathbf{I}_m - \mathbf{AB}) - m$. Moreover, by exploiting the corresponding expressions for the rank of $\mathbf{B} - \mathbf{BAB}$, we arrive at

$$\mathrm{rk}(\mathbf{A} - \mathbf{ABA}) - \mathrm{rk}(\mathbf{B} - \mathbf{BAB}) = \mathrm{rk}(\mathbf{A}) - \mathrm{rk}(\mathbf{B}). \tag{2.4}$$

An obvious consequence of (2.4) is that when $\mathbf{B}$ is a reflexive inverse of $\mathbf{A}$ (satisfying the first two of the four Penrose conditions given in (1.1)), then $\mathrm{rk}(\mathbf{A}) = \mathrm{rk}(\mathbf{B})$. Moreover, if $\mathbf{B}$ is an inner (outer) inverse of $\mathbf{A}$ (satisfying the first (second) condition in (1.1)) such that $\mathrm{rk}(\mathbf{A}) = \mathrm{rk}(\mathbf{B})$, then it is also an outer (inner) inverse. Another observation is that by replacing $\mathbf{B}$ with $\mathbf{A}^2$, from (2.4) we obtain $\mathrm{rk}(\mathbf{A} - \mathbf{A}^4) - \mathrm{rk}(\mathbf{A}^2 - \mathbf{A}^5) = \mathrm{rk}(\mathbf{A}) - \mathrm{rk}(\mathbf{A}^2)$, whence it follows that $\mathbf{A}$ is GP if and only if $\mathrm{rk}(\mathbf{A} - \mathbf{A}^4) = \mathrm{rk}(\mathbf{A}^2 - \mathbf{A}^5)$. Further consequences of Theorem 2.1 are given in what follows.

Corollary 2.2. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$. *Then:*

(i) $\mathrm{rk}(\mathbf{A}\overline{\mathbf{A}}) = \mathrm{rk}(\mathbf{A}) + \mathrm{rk}(\overline{\mathbf{A}}) - n$,

(ii) $\mathrm{rk}(\mathbf{A} - \mathbf{A}^3) = \mathrm{rk}(\mathbf{A}) + \mathrm{rk}(\mathbf{I}_n - \mathbf{A}^2) - n$.

*Proof.* The identities in points (i) and (ii) of the corollary are obtained from Theorem 2.1 by taking $\mathbf{B} = \mathbf{I}_n$ and $\mathbf{B} = \mathbf{A}$, respectively. ∎

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From Corollary 2.2 it follows that $\mathbf{A}$ is idempotent if and only if $\text{rk}(\mathbf{A}) + \text{rk}(\overline{\mathbf{A}}) = n$, whereas $\mathbf{A}$ is tripotent if and only if $\text{rk}(\mathbf{A}) + \text{rk}(\mathbf{I}_n - \mathbf{A}^2) = n$. The former of these facts is well known in the literature (see e.g., [33, Theorem 3.6.3]), but according to our knowledge the latter one was so far never mentioned. Combining the two points of Corollary 2.2 leads to

$$\text{rk}(\mathbf{A} - \mathbf{A}^3) + \text{rk}(\overline{\mathbf{A}}) = \text{rk}(\mathbf{A}\overline{\mathbf{A}}) + \text{rk}(\mathbf{I}_n - \mathbf{A}^2). \tag{2.5}$$

Another identity of interest is obtained by combining Theorem 2.1 and Corollary 2.2. Namely, by taking $\mathbf{B} = -\mathbf{I}_n$ in Theorem 2.1 we get

$$\text{rk}(\mathbf{A}\widehat{\mathbf{A}}) = \text{rk}(\mathbf{A}) + \text{rk}(\widehat{\mathbf{A}}) - n,$$

from where, by Corollary 2.2(ii),

$$\text{rk}(\mathbf{A} - \mathbf{A}^3) + \text{rk}(\widehat{\mathbf{A}}) = \text{rk}(\mathbf{A}\widehat{\mathbf{A}}) + \text{rk}(\mathbf{I}_n - \mathbf{A}^2). \tag{2.6}$$

From identities (2.5) and (2.6) we directly obtain original characteristics of idempotents, tripotents, involutions, and skew-idempotent matrices. Recall that involutions and skew-idempotent matrices are defined by conditions $\mathbf{A}^2 = \mathbf{I}_n$ and $\mathbf{A}^2 = -\mathbf{A}$, respectively. Surprisingly, none of the relationships (2.5) and (2.6) was mentioned in [1], where the following result was established.

Proposition 2.3. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$. *Then* $\text{rk}(\mathbf{A} - \mathbf{A}^3) + \text{rk}(\mathbf{A}) = \text{rk}(\mathbf{A}\overline{\mathbf{A}}) + \text{rk}(\mathbf{A}\widehat{\mathbf{A}})$.

*Proof.* Note that $\mathbf{A} - \mathbf{A}^3 = \overline{\mathbf{A}}\mathbf{A}\widehat{\mathbf{A}}$. Hence, $\text{rk}(\mathbf{A} - \mathbf{A}^3) \geqslant \text{rk}(\mathbf{A}\overline{\mathbf{A}}) + \text{rk}(\mathbf{A}\widehat{\mathbf{A}}) - \text{rk}(\mathbf{A})$, with equality holding if and only if there exist matrices $\mathbf{X}, \mathbf{Y} \in \mathbb{C}_{n,n}$ such that $\mathbf{A}\widehat{\mathbf{A}}\mathbf{X} + \mathbf{Y}\overline{\mathbf{A}}\mathbf{A} = \mathbf{A}$. Choosing $\mathbf{X} = \mathbf{Y} = \frac{1}{2}\mathbf{I}_n$ leads to the assertion. □

From Proposition 2.3 it follows that $\mathbf{A}^3 = \mathbf{A} \Leftrightarrow \text{rk}(\mathbf{A}) = \text{rk}(\mathbf{A}\overline{\mathbf{A}}) + \text{rk}(\mathbf{A}\widehat{\mathbf{A}})$, which was observed in [1, Lemma 3.1]. It should be mentioned that the proof of Proposition 2.3 provided above seems to be shorter and simpler than the one given in [1, p. 13].

Corollary 2.2 can be derived also as a particular case of the next theorem.

Theorem 2.4. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$ *and let* $k, l \in \mathbb{N} \cup \{0\}$, $l \geqslant k$. *Then* $\text{rk}(\mathbf{A}^k - \mathbf{A}^{k+l}) = \text{rk}(\mathbf{A}^k) + \text{rk}(\mathbf{I}_n - \mathbf{A}^l) - n$.

*Proof.* By $\mathbf{A}^k - \mathbf{A}^{k+l} = \mathbf{A}^k(\mathbf{I}_n - \mathbf{A}^l)$, the rank identity asserted in the theorem is satisfied if and only if $\mathcal{N}(\mathbf{A}^k) \subseteq \mathcal{R}(\mathbf{I}_n - \mathbf{A}^l)$. To show that this inclusion always holds, let a vector $\mathbf{x} \in \mathbb{C}_{n,1}$ be such that $\mathbf{x} \in \mathcal{N}(\mathbf{A}^k)$. Then $\mathbf{A}^k\mathbf{x} = \mathbf{0}$, from where $(\mathbf{I}_n - \mathbf{A}^l)\mathbf{x} = \mathbf{x}$, whence $\mathbf{x} \in \mathcal{R}(\mathbf{I}_n - \mathbf{A}^l)$. □

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A general formula for the rank of $\mathbf{I}_n - \mathbf{A}^{k+1}$ is established in what follows.

THEOREM 2.5. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$ *and let* $k \in \mathbb{N} \cup \{0\}$. *Then* $\mathrm{rk}(\mathbf{I}_n - \mathbf{A}^{k+1}) = \mathrm{rk}(\overline{\mathbf{A}}) + \mathrm{rk}(\mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^k) - n$.

*Proof.* It can be straightforwardly checked that $\mathbf{I}_n - \mathbf{A}^{k+1} = \overline{\mathbf{A}}(\mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^k)$. In consequence, the identity given in the theorem holds if and only if $\mathcal{N}(\overline{\mathbf{A}}) \subseteq \mathcal{R}(\mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^k)$. To verify that this inclusion indeed holds, let a vector $\mathbf{x} \in \mathbb{C}_{n,1}$ be such that $\mathbf{x} \in \mathcal{N}(\overline{\mathbf{A}})$, i.e., $\mathbf{Ax} = \mathbf{x}$. Then $(\mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^k)\mathbf{x} = (k+1)\mathbf{x}$, whence $\mathbf{x} = (\mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^k)\mathbf{x}/(k+1)$, which means that $\mathbf{x} \in \mathcal{R}(\mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^k)$. ◻

It is easy to show that the idempotency of $\mathbf{A} \in \mathbb{C}_{n,n}$ implies nonsingularity of $\widehat{\mathbf{A}}$ (in which case $\widehat{\mathbf{A}}^{-1} = \mathbf{I}_n - \frac{1}{2}\mathbf{A}$), but not the other way round. Thus, a question occurs by what condition should the requirement that $\mathrm{rk}(\widehat{\mathbf{A}}) = n$ be supplemented in order to make this implication reversible. The answer is given in what follows.

THEOREM 2.6. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$. *Then* $\mathbf{A}$ *is idempotent if and only if* $\widehat{\mathbf{A}}$ *is nonsingular and* $\mathrm{rk}(\mathbf{I}_n - \mathbf{A}^2) = n - \mathrm{rk}(\mathbf{A})$.

*Proof.* The necessity part is clearly satisfied. To establish the reverse implication, observe that Theorem 2.5 yields

$$\mathrm{rk}(\mathbf{I}_n - \mathbf{A}^2) = \mathrm{rk}(\overline{\mathbf{A}}) + \mathrm{rk}(\widehat{\mathbf{A}}) - n. \tag{2.7}$$

Hence, the nonsingularity of $\widehat{\mathbf{A}}$ entails $\mathrm{rk}(\mathbf{I}_n - \mathbf{A}^2) = \mathrm{rk}(\overline{\mathbf{A}})$, which combined with $\mathrm{rk}(\mathbf{I}_n - \mathbf{A}^2) = n - \mathrm{rk}(\mathbf{A})$ ensures that $\mathbf{A}$ is idempotent; see [33, Theorem 3.6.3]. ◻

Theorems 2.4 and 2.5 entail

$$\mathrm{rk}(\mathbf{A} - \mathbf{A}^{k+1}) = \mathrm{rk}(\mathbf{A}) + \mathrm{rk}(\overline{\mathbf{A}}) + \mathrm{rk}(\mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{k-1}) - 2n. \tag{2.8}$$

On the other hand, replacing $\mathbf{A}$ with $\mathbf{A}^2$ in Corollary 2.2(i) gives

$$\mathrm{rk}(\mathbf{A}^2 - \mathbf{A}^4) = \mathrm{rk}(\mathbf{A}^2) + \mathrm{rk}(\mathbf{I}_n - \mathbf{A}^2) - n.$$

Combining this identity with (2.7) leads to

$$\mathrm{rk}(\mathbf{A}^2 - \mathbf{A}^4) = \mathrm{rk}(\mathbf{A}^2) + \mathrm{rk}(\overline{\mathbf{A}}) + \mathrm{rk}(\widehat{\mathbf{A}}) - 2n. \tag{2.9}$$

Since (2.8) yields

$$\mathrm{rk}(\overline{\mathbf{A}}) + \mathrm{rk}(\widehat{\mathbf{A}}) - 2n = \mathrm{rk}(\mathbf{A} - \mathbf{A}^3) - \mathrm{rk}(\mathbf{A}), \tag{2.10}$$

equality (2.9) takes the form

$$\mathrm{rk}(\mathbf{A}^2 - \mathbf{A}^4) = \mathrm{rk}(\mathbf{A}^2) + \mathrm{rk}(\mathbf{A} - \mathbf{A}^3) - \mathrm{rk}(\mathbf{A}). \tag{2.11}$$

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Note that from (2.10) it follows that $\mathbf{A}^3 = \mathbf{A} \Leftrightarrow \mathrm{rk}(\mathbf{A}) + \mathrm{rk}(\overline{\mathbf{A}}) + \mathrm{rk}(\widehat{\mathbf{A}}) = 2n$.

An additional rank formula is provided below.

Theorem 2.7. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$ *and let* $k, l \in \mathbb{N} \cup \{0\}$, $l > 1$. *Then* $\mathrm{rk}(\mathbf{A}^k - \mathbf{A}^{kl}) = \mathrm{rk}(\mathbf{A}^k) + \mathrm{rk}(\mathbf{I}_n - \mathbf{A}^{k(l-1)}) - n$.

*Proof.* In view of $\mathbf{A}^k - \mathbf{A}^{kl} = \mathbf{A}^k(\mathbf{I}_n - \mathbf{A}^{k(l-1)})$, it follows that the equality claimed in the theorem is satisfied if and only if $\mathcal{N}(\mathbf{A}^k) \subseteq \mathcal{R}(\mathbf{I}_n - \mathbf{A}^{k(l-1)})$. Let a vector $\mathbf{x} \in \mathbb{C}_{n,1}$ be such that $\mathbf{x} \in \mathcal{N}(\mathbf{A}^k)$, which entails $\mathbf{A}^k\mathbf{x} = \mathbf{0}$. In consequence, $(\mathbf{I}_n - \mathbf{A}^{k(l-1)})\mathbf{x} = \mathbf{x}$, which means that $\mathbf{x} \in \mathcal{R}(\mathbf{I}_n - \mathbf{A}^{k(l-1)})$. □

Observe that Theorems 2.4 and 2.7 coincide when $k + l = kl$.

Yet another direct consequence of the Frobenius inequality is that $\mathbf{A} \in \mathbb{C}_{n,n}$ satisfies

$$\mathrm{rk}(\mathbf{A}^2\overline{\mathbf{A}}) \geqslant 2\mathrm{rk}(\mathbf{A}\overline{\mathbf{A}}) - \mathrm{rk}(\overline{\mathbf{A}}). \tag{2.12}$$

Hence, it is seen that when $\mathbf{A}$ is nilpotent of index 2, then $2\mathrm{rk}(\mathbf{A}) \leqslant \mathrm{rk}(\overline{\mathbf{A}})$. Similarly, $\mathbf{A}^2 = \mathbf{A}^3$ implies $2\mathrm{rk}(\mathbf{A}\overline{\mathbf{A}}) \leqslant \mathrm{rk}(\overline{\mathbf{A}})$. Note that equality happens in (2.12) if and only if there exist matrices $\mathbf{X}, \mathbf{Y} \in \mathbb{C}_{n,n}$ such that $\mathbf{A}\overline{\mathbf{A}}\mathbf{X} + \mathbf{Y}\mathbf{A}\overline{\mathbf{A}} = \overline{\mathbf{A}}$. A further observation is that if instead of the rank of $\mathbf{A}^2\overline{\mathbf{A}}$ we consider the rank of $\mathbf{A}\overline{\mathbf{A}}^2$, then we arrive at the conclusion that $\mathrm{rk}(\mathbf{A}\overline{\mathbf{A}}^2) = \mathrm{rk}(\mathbf{A}\overline{\mathbf{A}}) + \mathrm{rk}(\overline{\mathbf{A}}\,\overline{\mathbf{A}}) - \mathrm{rk}(\overline{\mathbf{A}})$ is always satisfied. The last claim follows from the fact that there exist $\mathbf{X}, \mathbf{Y} \in \mathbb{C}_{n,n}$ such that $\overline{\mathbf{A}}\,\overline{\mathbf{A}}\mathbf{X} + \mathbf{Y}\mathbf{A}\overline{\mathbf{A}} = \overline{\mathbf{A}}$ is fulfilled, namely $\mathbf{X} = \mathbf{I}_n$ and $\mathbf{Y} = \mathbf{I}_n$.

Inspired by the rank identities provided in Theorem 2.1 and (2.7), subsequently we identify two properties of column spaces.

Theorem 2.8. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{n,n}$. *Then:*

(i) $\mathcal{R}(\mathbf{A} - \mathbf{A}\mathbf{B}\mathbf{A}) = \mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\mathbf{I}_n - \mathbf{A}\mathbf{B})$,

(ii) $\mathcal{R}(\mathbf{I}_n - \mathbf{A}^2) = \mathcal{R}(\overline{\mathbf{A}}) \cap \mathcal{R}(\widehat{\mathbf{A}})$.

*Proof.* By replacing $\mathbf{A}$ with $\mathbf{A}^*$ and $\mathbf{B}$ with $\mathbf{B}^*$ in the relationships given in points (i) and (ii) of the theorem and taking orthogonal complements of the resulting equalities, we obtain:

(i′) $\mathcal{N}(\mathbf{A} - \mathbf{A}\mathbf{B}\mathbf{A}) = \mathcal{N}(\mathbf{A}) + \mathcal{N}(\mathbf{I}_n - \mathbf{B}\mathbf{A})$,

(ii′) $\mathcal{N}(\mathbf{I}_n - \mathbf{A}^2) = \mathcal{N}(\overline{\mathbf{A}}) + \mathcal{N}(\widehat{\mathbf{A}})$,

respectively. To prove relationship (i′), and simultaneously (i), first note that a vector $\mathbf{x} \in \mathbb{C}_{n,1}$ can be expressed as $\mathbf{x} = (\mathbf{I}_n - \mathbf{B}\mathbf{A})\mathbf{x} + \mathbf{B}\mathbf{A}\mathbf{x}$. Under the assumption that $\mathbf{x} \in \mathcal{N}(\mathbf{A} - \mathbf{A}\mathbf{B}\mathbf{A})$, i.e., $\mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{B}\mathbf{A}\mathbf{x}$, we have $\mathbf{A}(\mathbf{I}_n - \mathbf{B}\mathbf{A})\mathbf{x} = \mathbf{0}$ and $(\mathbf{I}_n - \mathbf{B}\mathbf{A})\mathbf{B}\mathbf{A}\mathbf{x} = \mathbf{0}$, from where we conclude that $(\mathbf{I}_n - \mathbf{B}\mathbf{A})\mathbf{x} \in \mathcal{N}(\mathbf{A})$ and $\mathbf{B}\mathbf{A}\mathbf{x} \in \mathcal{N}(\mathbf{I}_n - \mathbf{B}\mathbf{A})$. To show

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that $\mathcal{N}(\mathbf{A})+\mathcal{N}(\mathbf{I}_n-\mathbf{BA}) \subseteq \mathcal{N}(\mathbf{A}-\mathbf{ABA})$, let a vector $\mathbf{y} \in \mathbb{C}_{n,1}$ be given by $\mathbf{y} = \mathbf{u}+\mathbf{v}$, where $\mathbf{u} \in \mathcal{N}(\mathbf{A})$ and $\mathbf{v} \in \mathcal{N}(\mathbf{I}_n - \mathbf{BA})$, i.e., $\mathbf{Au} = \mathbf{0}$ and $\mathbf{BAv} = \mathbf{v}$. In consequence, $(\mathbf{A} - \mathbf{ABA})\mathbf{y} = (\mathbf{A} - \mathbf{ABA})(\mathbf{u} + \mathbf{v}) = \mathbf{0}$, which ensures that $\mathbf{y} \in \mathcal{N}(\mathbf{A} - \mathbf{ABA})$.

The proof of identity (ii′) (and, thus, also of (ii)) is obtained in a similar fashion. Observe that any $\mathbf{x} \in \mathbb{C}_{n,1}$ can be written as $\mathbf{x} = \frac{1}{2}\overline{\mathbf{A}}\mathbf{x} + \frac{1}{2}\widehat{\mathbf{A}}\mathbf{x}$. If now $\mathbf{x} \in \mathcal{N}(\mathbf{I}_n - \mathbf{A}^2)$, i.e., $\mathbf{A}^2\mathbf{x} = \mathbf{x}$, then $\widehat{\mathbf{A}}\mathbf{x} \in \mathcal{N}(\overline{\mathbf{A}})$ and $\overline{\mathbf{A}}\mathbf{x} \in \mathcal{N}(\widehat{\mathbf{A}})$. To derive the inclusion $\mathcal{N}(\overline{\mathbf{A}}) + \mathcal{N}(\widehat{\mathbf{A}}) \subseteq \mathcal{N}(\mathbf{I}_n - \mathbf{A}^2)$, take a vector $\mathbf{y}$ such that $\mathbf{y} = \mathbf{u} + \mathbf{v}$, where $\mathbf{u} \in \mathcal{N}(\overline{\mathbf{A}})$ and $\mathbf{v} \in \mathcal{N}(\widehat{\mathbf{A}})$. Then $(\mathbf{I}_n - \mathbf{A}^2)\mathbf{y} = \overline{\mathbf{A}}\widehat{\mathbf{A}}(\mathbf{u} + \mathbf{v}) = \widehat{\mathbf{A}}\overline{\mathbf{A}}(\mathbf{u} + \mathbf{v}) = \mathbf{0}$, which entails $\mathbf{y} \in \mathcal{N}(\mathbf{I}_n - \mathbf{A}^2)$. ∎

By taking $\mathbf{B} = \mathbf{I}_n$ and $\mathbf{B} = -\mathbf{I}_n$, from Theorem 2.8(i) we obtain the equivalences $\mathbf{A}^2 = \mathbf{A} \Leftrightarrow \mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\overline{\mathbf{A}}) = \{\mathbf{0}\}$ and $\mathbf{A}^2 = -\mathbf{A} \Leftrightarrow \mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\widehat{\mathbf{A}}) = \{\mathbf{0}\}$, respectively. The former of them is a known characteristic of idempotent matrices, whereas the latter property of skew-idempotent matrices seems to be not available in the literature. Another observation is that from Theorem 2.8(ii) it directly follows that $\mathbf{A}$ is an involution if and only if $\mathcal{R}(\overline{\mathbf{A}}) \cap \mathcal{R}(\widehat{\mathbf{A}}) = \{\mathbf{0}\}$.

Interestingly, several of the rank identities given above can be extended to the corresponding relationships between column and null spaces. The identities (i)–(iv) listed in Proposition 2.9 below were inspired by (2.5), (2.6), (2.11), and Proposition 2.3, respectively.

PROPOSITION 2.9. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$. *Then:*

(i) $\mathcal{R}(\overline{\mathbf{A}}) = \mathcal{R}(\mathbf{A}\overline{\mathbf{A}}) + \mathcal{R}(\mathbf{I}_n - \mathbf{A}^2)$,

(ii) $\mathcal{R}(\widehat{\mathbf{A}}) = \mathcal{R}(\mathbf{A}\widehat{\mathbf{A}}) + \mathcal{R}(\mathbf{I}_n - \mathbf{A}^2)$,

(iii) $\mathcal{R}(\mathbf{A}) = \mathcal{R}(\mathbf{A}^2) + \mathcal{R}(\mathbf{A} - \mathbf{A}^3)$,

(iv) $\mathcal{R}(\mathbf{A}) = \mathcal{R}(\mathbf{A}\overline{\mathbf{A}}) + \mathcal{R}(\mathbf{A}\widehat{\mathbf{A}})$.

*Proof.* The proof is established straightforwardly by considering the identities:

(i′) $\mathcal{N}(\overline{\mathbf{A}}) = \mathcal{N}(\mathbf{A}\overline{\mathbf{A}}) \cap \mathcal{N}(\mathbf{I}_n - \mathbf{A}^2)$,

(ii′) $\mathcal{N}(\widehat{\mathbf{A}}) = \mathcal{N}(\mathbf{A}\widehat{\mathbf{A}}) \cap \mathcal{N}(\mathbf{I}_n - \mathbf{A}^2)$,

(iii′) $\mathcal{N}(\mathbf{A}) = \mathcal{N}(\mathbf{A}^2) \cap \mathcal{N}(\mathbf{A} - \mathbf{A}^3)$,

(iv′) $\mathcal{N}(\mathbf{A}) = \mathcal{N}(\mathbf{A}\overline{\mathbf{A}}) \cap \mathcal{N}(\mathbf{A}\widehat{\mathbf{A}})$,

obtained from the characteristics given in points (i)–(iv) of the proposition by replacing $\mathbf{A}$ with $\mathbf{A}^*$ and taking orthogonal complement. ∎

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**3. Characteristics of tripotent matrices.** The present section provides an extensive study of tripotent matrices, with a particular attention paid to those matrices which are simultaneously Hermitian. It is clear that the set of Hermitian idempotent matrices (i.e., orthogonal projectors) is a proper subset of Hermitian tripotent matrices. For this reason, the latter set hereafter will be called the set of *extended orthogonal projectors*. Conditions necessary and sufficient for $\mathbf{A}$ of the form (1.2) to be an extended orthogonal projector are given in what follows.

LEMMA 3.1. *Let $\mathbf{A} \in \mathbb{C}_{n,n}$ be of the form* (1.2). *Then $\mathbf{A}$ is an extended orthogonal projector if and only if $\mathbf{L} = \mathbf{0}$, $\mathbf{\Sigma} = \mathbf{I}_r$, and $\mathbf{K}^* = \mathbf{K}$.*

*Proof.* From Lemma 1.2 we conclude that $\mathbf{A}$ is Hermitian and tripotent (i.e., is an extended orthogonal projector) if and only if

$$\mathbf{L} = \mathbf{0}, \quad \mathbf{\Sigma}\mathbf{K} = \mathbf{K}^*\mathbf{\Sigma}, \quad \text{and} \quad (\mathbf{\Sigma}\mathbf{K})^2 = \mathbf{I}_r. \tag{3.1}$$

However, in the light of (1.5), from the middle condition in (3.1) we get $\mathbf{\Sigma} = \mathbf{K}\mathbf{\Sigma}\mathbf{K}$, whence $\mathbf{\Sigma}^2 = \mathbf{\Sigma}\mathbf{K}\mathbf{\Sigma}\mathbf{K}$. Combining this identity with the last equality in (3.1) leads to $\mathbf{\Sigma}^2 = \mathbf{I}_r$, from where we arrive at $\mathbf{\Sigma} = \mathbf{I}_r$. In consequence, the middle condition in (3.1) yields $\mathbf{K}^* = \mathbf{K}$. The reverse implication is clearly satisfied. □

The theorem below sheds light on the relationship between idempotency and tripotency of a matrix. Condition (iii) therein was inspired by problem [24], which asserts that $\mathbf{A}^2 = \mathbf{A} \Leftrightarrow \mathrm{rk}(\mathbf{A}) = \mathrm{tr}(\mathbf{A})$ and $\mathrm{rk}(\overline{\mathbf{A}}) = \mathrm{tr}(\overline{\mathbf{A}})$, where tr(.) denotes the trace of a matrix argument.

THEOREM 3.2. *Let $\mathbf{A} \in \mathbb{C}_{n,n}$. Then $\mathbf{A}$ is idempotent if and only if $\mathbf{A}$ is tripotent and any of the following conditions is satisfied:*

(i) $\overline{\mathbf{A}}$ *is tripotent,*

(ii) $\widehat{\mathbf{A}}$ *is nonsingular,*

(iii) $\mathrm{rk}(\mathbf{A}) = \mathrm{tr}(\mathbf{A})$.

*Proof.* Clearly, $\mathbf{A}^2 = \mathbf{A}$ implies both $\mathbf{A}^3 = \mathbf{A}$ and $\overline{\mathbf{A}}^3 = \overline{\mathbf{A}}$. To prove the reverse implication, note that $\overline{\mathbf{A}}^3 = \overline{\mathbf{A}}$ is equivalent to $\mathbf{A}^3 - 3\mathbf{A}^2 + 2\mathbf{A} = \mathbf{0}$, which, by $\mathbf{A}^3 = \mathbf{A}$, reduces to $\mathbf{A}^2 = \mathbf{A}$.

In the light of Corollary 2.2(i), from (2.10) we get $\mathrm{rk}(\mathbf{A} - \mathbf{A}^3) = \mathrm{rk}(\mathbf{A}\overline{\mathbf{A}}) + \mathrm{rk}(\widehat{\mathbf{A}}) - n$. Hence, when $\mathbf{A}^3 = \mathbf{A}$ and $\mathrm{rk}(\widehat{\mathbf{A}}) = n$, then clearly $\mathbf{A}^2 = \mathbf{A}$. On the other hand, when $\mathbf{A}$ is idempotent, which yields tripotency of $\mathbf{A}$, then $\mathrm{rk}(\widehat{\mathbf{A}}) = n$, and the part of the theorem referring to its point (ii) follows.

To complete the proof, we only need to show that every tripotent matrix $\mathbf{A}$ such that $\mathrm{rk}(\mathbf{A}) = \mathrm{tr}(\mathbf{A})$ is necessarily idempotent. For this purpose, observe that when

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$\mathbf{A}^3 = \mathbf{A}$, then by Corollary 2.2(ii),

$$\mathrm{rk}(\mathbf{A}) + \mathrm{rk}(\mathbf{I}_n - \mathbf{A}^2) = n. \tag{3.2}$$

It easy to verify that tripotency of $\mathbf{A}$ implies idempotency of $\mathbf{I}_n - \mathbf{A}^2$, whence $\mathrm{rk}(\mathbf{I}_n - \mathbf{A}^2) = \mathrm{tr}(\mathbf{I}_n - \mathbf{A}^2)$. In consequence, taking into account condition (iii) of the theorem, identity (3.2) reduces to $\mathrm{tr}(\mathbf{A}\overline{\mathbf{A}}) = 0$. Since $\mathbf{A}^3 = \mathbf{A}$ implies also idempotency of $-\frac{1}{2}\mathbf{A}\overline{\mathbf{A}}$, we have $\mathrm{tr}(\mathbf{A}\overline{\mathbf{A}}) = 0 \Rightarrow \mathrm{rk}(\mathbf{A}\overline{\mathbf{A}}) = 0$, from where $\mathbf{A}^2 = \mathbf{A}$ follows. ◻

Characterization corresponding to Theorem 3.2(iii) is related to [27, Theorem 2], which asserts that when $\mathbf{A}$ is such that $\mathrm{rk}(\mathbf{A}) = \mathrm{tr}(\mathbf{A})$ and $\mathbf{A}^k = \mathbf{A}^l$ for some $k, l \in \mathbb{N}$, $k \neq l$, then $\mathbf{A}$ is idempotent. Clearly, Theorem 3.2 generalizes this implication to an equivalence when $k = 1$ and $l = 3$. It should be emphasized that the proof of Theorem 3.2 is much simpler than the proof of [27, Theorem 2]. Another fact is that the identity $\mathbf{A} - \mathbf{A}^3 = \overline{\mathbf{A}}\mathbf{A}\widehat{\mathbf{A}}$, already mentioned in the proof of Proposition 2.3, may be used to establish an alternative proof of Theorem 3.2(ii).

If we would impose in Theorem 3.2 a general assumption that $\mathbf{A}$ is Hermitian, then the theorem would demonstrate relationships between orthogonal projectors and extended orthogonal projectors. As is shown in what follows, the corresponding characteristics hold true also when hermitianness is replaced with the essentially weaker assumption that $\mathbf{A}$ is EP.

Corollary 3.3. *Let $\mathbf{A} \in \mathbb{C}_{n,n}$. Then $\mathbf{A}$ is an orthogonal projector if and only if $\mathbf{A}$ is tripotent, EP, and any of the following conditions is satisfied:*

(i) $\overline{\mathbf{A}}$ *is tripotent,*

(ii) $\widehat{\mathbf{A}}$ *is nonsingular,*

(iii) $\mathrm{rk}(\mathbf{A}) = \mathrm{tr}(\mathbf{A})$.

*Proof.* From Theorem 3.2 we know that combining $\mathbf{A}^3 = \mathbf{A}$ with any of the conditions (i)–(iii) listed in the corollary yields a conjunction equivalent to $\mathbf{A}^2 = \mathbf{A}$. Every Hermitian matrix is trivially EP. To show that every idempotent and EP matrix is an orthogonal projector, we utilize the condition $\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger\mathbf{A}$. Clearly, pre- or postmultiplying this identity by $\mathbf{A}$ leads to $\mathbf{A} = \mathbf{A}\mathbf{A}^\dagger$, which means that $\mathbf{A}$ is an orthogonal projector. ◻

When a tripotent matrix is Hermitian nonnegative definite, then its eigenvalues belong to the set $\{0, 1\}$, in which case, the matrix is idempotent. These facts lead to the following result.

Corollary 3.4. *Let $\mathbf{A} \in \mathbb{C}_{n,n}$. Then $\mathbf{A}$ is an orthogonal projector if and only if $\mathbf{A}$ is a nonnegative definite extended orthogonal projector.*

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It is known that $\mathbf{A} \in \mathbb{C}_{n,n}$ is an orthogonal projector if and only if $\mathbf{A}$ is idempotent and either generalized or hypergeneralized projector; see [3, Theorem 1]. It turns out that the idempotency condition can be replaced with tripotency.

Theorem 3.5. *Let $\mathbf{A} \in \mathbb{C}_{n,n}$. Then $\mathbf{A}$ is an orthogonal projector if and only if $\mathbf{A}$ is tripotent and any of the following conditions is satisfied:*

(i) $\mathbf{A}$ *is a generalized projector,*

(ii) $\mathbf{A}$ *is a hypergeneralized projector.*

*Proof.* The necessity parts are trivially satisfied. Since the set of generalized projectors is included in the set of hypergeneralized projectors, we need to prove the sufficiency only of point (ii) of the theorem. From Lemmas 1.2 and 3.1 we know that when $\mathbf{A}$ is tripotent and a hypergeneralized projector, then $(\mathbf{\Sigma K})^2 = \mathbf{I}_r$ and $(\mathbf{\Sigma K})^3 = \mathbf{I}_r$, whence $\mathbf{\Sigma K} = \mathbf{I}_r$. This condition can be rewritten as $\mathbf{\Sigma} = \mathbf{K}^{-1}$ or as $\mathbf{\Sigma} = (\mathbf{K}^*)^{-1}$, with the latter equality obtained from the former one by taking conjugate transpose. In consequence, by (1.5), we arrive at $\mathbf{\Sigma}^2 = (\mathbf{KK}^*)^{-1} = \mathbf{I}_r$. However, $\mathbf{\Sigma}^2 = \mathbf{I}_r \Rightarrow \mathbf{\Sigma} = \mathbf{I}_r$, which ensures that $\mathbf{K} = \mathbf{I}_r$. Concluding we have shown that when when $\mathbf{A}$ is tripotent and a hypergeneralized projector, then $\mathbf{\Sigma} = \mathbf{I}_r$ and $\mathbf{K} = \mathbf{I}_r$, i.e., $\mathbf{A}$ is an orthogonal projector. □

Next we derive three conditions which are mutually equivalent in the set of tripotent matrices.

Theorem 3.6. *Let $\mathbf{A} \in \mathbb{C}_{n,n}$ be tripotent. Then the following conditions are equivalent:*

(i) $\mathbf{A}$ *is Hermitian,*

(ii) $\mathbf{A}$ *is normal,*

(iii) $\mathbf{A}$ *is EP and a partial isometry.*

*Proof.* It is clear that when $\mathbf{A}$ is Hermitian, then it is normal, and that normality entails EP-ness. In consequence, the chain of implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) will be established when we show that within the set of tripotent matrices every normal matrix is necessarily a partial isometry. To do this, note that $\mathbf{A}$ of the form (1.2) is normal and tripotent if and only if

$$\mathbf{L} = \mathbf{0}, \quad \mathbf{K\Sigma} = \mathbf{\Sigma K}, \quad \text{and} \quad (\mathbf{\Sigma K})^2 = \mathbf{I}_r. \tag{3.3}$$

The last two conditions in (3.3) yield

$$\mathbf{K\Sigma}^2\mathbf{K} = \mathbf{I}_r. \tag{3.4}$$

In view of (1.5), by taking conjugate transpose of (3.4) and pre- and postmultiplying the resulting equality by $\mathbf{K}$ gives $\mathbf{\Sigma}^2 = \mathbf{K}^2$. Substituting this identity to (3.4) shows

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that $\mathbf{K}^4 = \mathbf{I}_r$, which means that $\mathbf{\Sigma}^4 = \mathbf{I}_r$. In consequence, we arrive at the conclusion that $\mathbf{\Sigma} = \mathbf{I}_r$, i.e., that $\mathbf{A}$ is a partial isometry.

In the last step of the proof, observe that when $\mathbf{A}$ is simultaneously tripotent and a partial isometry, then $\mathbf{K}^2 = \mathbf{I}_r$, which combined with $\mathbf{K}^{-1} = \mathbf{K}^*$ shows that $\mathbf{K}^* = \mathbf{K}$. Thus, the implication (iii) $\Rightarrow$ (i) follows. ∎

Note that alternative expressions for the conjunction in Theorem 3.6(iii) were provided in [8, Theorem 2].

It was shown in [16, Theorem 8] that $\mathbf{A}^3 = \mathbf{A}$ if and only if $\mathbf{A}$ is GP and group involutory, i.e., $\mathbf{A}^\# = \mathbf{A}$. In what follows we provide a modified version of this result.

THEOREM 3.7. *Let $\mathbf{A} \in \mathbb{C}_{n,n}$. Then the following conditions are equivalent:*

(i) $\mathbf{A}$ *is tripotent,*

(ii) $\mathbf{A}$ *is GP and* $\mathbf{A}^2$ *is idempotent,*

(iii) $\mathbf{A}$ *is GP and* $\mathbf{A}\mathbf{A}^\# = \mathbf{A}^2$*,*

(iv) $\mathbf{A}$ *is GP and* $\frac{1}{2}(\mathbf{A}\mathbf{A}^\# - \mathbf{A})$ *is idempotent.*

*Proof.* It is clear that $\mathrm{rk}(\mathbf{A}^2 - \mathbf{A}^4) \geqslant 0$ and $\mathrm{rk}(\mathbf{A}^2) - \mathrm{rk}(\mathbf{A}) \leqslant 0$. In view of these inequalities, the equivalence (i) $\Leftrightarrow$ (ii) follows straightforwardly from (2.11). The part (i) $\Rightarrow$ (iii) is obtained on account of [16, Theorem 8], and the reverse implication follows by multiplying $\mathbf{A}\mathbf{A}^\# = \mathbf{A}^2$ by $\mathbf{A}$. To conclude the proof note that direct calculations show that the condition given in point (iv) of the theorem is equivalent to the identity in point (iii). ∎

Observe that the requirement that $\frac{1}{2}(\mathbf{A}\mathbf{A}^\# - \mathbf{A})$ is idempotent present in Theorem 3.7(iv) can be replaced with the requirement that $\frac{1}{2}(\mathbf{A}\mathbf{A}^\# + \mathbf{A})$ is idempotent.

It is known that the following conditions are equivalent:

(a) $\mathbf{A}$ is idempotent,

(b) $\mathcal{R}(\mathbf{A}) \oplus \mathcal{R}(\overline{\mathbf{A}}) = \mathbb{C}_{n,1}$,

(c) $\mathcal{R}(\mathbf{A}) \subseteq \mathcal{N}(\overline{\mathbf{A}})$,

(d) $\mathbf{A}\mathbf{z} = \mathbf{z}$ for all $\mathbf{z} \in \mathcal{R}(\mathbf{A})$;

see e.g., [40, Theorem 1], where the real setup was considered. The next theorem provides counterparts of these equivalences when $\mathbf{A}$ is tripotent.

THEOREM 3.8. *Let $\mathbf{A} \in \mathbb{C}_{n,n}$. Then the following conditions are equivalent:*

(i) $\mathbf{A}$ *is tripotent,*

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(ii) $\mathcal{R}(\mathbf{A}) \oplus \mathcal{R}(\mathbf{I}_n - \mathbf{A}^2) = \mathbb{C}_{n,1}$,

(iii) $\mathcal{R}(\mathbf{A}) \subseteq \mathcal{N}(\mathbf{I}_n - \mathbf{A}^2)$,

(iv) $\mathbf{A}^2\mathbf{z} = \mathbf{z}$ *for all* $\mathbf{z} \in \mathcal{R}(\mathbf{A})$.

*Proof.* To prove that (i) $\Rightarrow$ (ii), first note that a vector $\mathbf{x} \in \mathbb{C}_{n,1}$ can be expressed as

$$\mathbf{x} = \mathbf{A}^2\mathbf{x} + (\mathbf{I}_n - \mathbf{A}^2)\mathbf{x},$$

where $\mathbf{A}^2\mathbf{x} \in \mathcal{R}(\mathbf{A})$ and $(\mathbf{I}_n - \mathbf{A}^2)\mathbf{x} \in \mathcal{R}(\mathbf{I}_n - \mathbf{A}^2)$. To show that the subspaces $\mathcal{R}(\mathbf{A})$ and $\mathcal{R}(\mathbf{I}_n - \mathbf{A}^2)$ are disjoint, take $\mathbf{y} \in \mathbb{C}_{n,1}$ such that $\mathbf{y} \in \mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\mathbf{I}_n - \mathbf{A}^2)$. Then $\mathbf{y} = \mathbf{A}\mathbf{u} = (\mathbf{I}_n - \mathbf{A}^2)\mathbf{v}$ for some $\mathbf{u}, \mathbf{v} \in \mathbb{C}_{n,1}$. Hence, $\mathbf{A}\mathbf{y} = \mathbf{A}(\mathbf{I}_n - \mathbf{A}^2)\mathbf{v} = \mathbf{0}$, which means that $\mathbf{A}^2\mathbf{u} = \mathbf{0}$. In consequence, $\mathbf{A}^3\mathbf{u} = \mathbf{A}\mathbf{u} = \mathbf{y} = \mathbf{0}$. To establish the part (ii) $\Rightarrow$ (iii), let $\mathcal{R}(\mathbf{A})$ and $\mathcal{R}(\mathbf{I}_n - \mathbf{A}^2)$ be disjoint. Then $\mathbf{x} \in \mathbb{C}_{n,1}$ such that $\mathbf{x} \in \mathcal{R}(\mathbf{A})$ satisfies $(\mathbf{I}_n - \mathbf{A}^2)\mathbf{x} = \mathbf{0}$, which means that $\mathbf{x} \in \mathcal{N}(\mathbf{I}_n - \mathbf{A}^2)$. The proof is thus complete, for the implications (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i) hold trivially. $\square$

It is clear that Theorem 3.8 could be extended by further two equivalent conditions, namely $\mathcal{N}(\mathbf{A}) \cap \mathcal{N}(\mathbf{I}_n - \mathbf{A}^2) = \{\mathbf{0}\}$ and $\mathcal{R}(\mathbf{I}_n - \mathbf{A}^2) \subseteq \mathcal{N}(\mathbf{A})$, obtained from points (ii) and (iii) of the theorem by replacing $\mathbf{A}$ with $\mathbf{A}^*$ and taking orthogonal complements of the resulting conditions.

As mentioned above Theorem 3.8, $\mathbf{A}^2 = \mathbf{A} \Leftrightarrow \mathcal{R}(\mathbf{A}) \subseteq \mathcal{N}(\overline{\mathbf{A}})$. It can be verified that $\mathbf{A}$ is an orthogonal projector if and only if $\mathcal{R}(\mathbf{A}^*) \subseteq \mathcal{N}(\overline{\mathbf{A}})$. In the light of Theorem 3.8, this observation leads to a question whether the requirement that $\mathbf{A}$ is an extended orthogonal projector can be equivalently expressed as

$$\mathcal{R}(\mathbf{A}^*) \subseteq \mathcal{N}(\mathbf{I}_n - \mathbf{A}^2). \tag{3.5}$$

The answer to this question is negative, what can be confirmed by considering the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix},$$

which is tripotent and satisfies (3.5), but is clearly not Hermitian (for an illustrative collection of matrices fulfilling various conditions involving powers of matrices see [42]). It can be shown that inclusion (3.5) ensures that $\mathbf{A}$ is tripotent and $\mathbf{A}^2$ is Hermitian, which is a conjunction of conditions involved also in the next theorem.

It is known that $\mathbf{A} \in \mathbb{C}_{n,n}$ is an orthogonal projector if and only if $\mathbf{A}$ is idempotent and $\mathbf{A}^\dagger = \mathbf{A}$; for real case see [40, Theorem 12]. Another relevant fact is given in Exercise 105 in [18, Chapter 12], which claims that within the class of real symmetric matrices we have $\mathbf{A}^3 = \mathbf{A} \Leftrightarrow \mathbf{A}^\dagger = \mathbf{A}$. In what follows we generalize the latter

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equivalence to the complex setup and extend it by several further conditions. One of them refers to so called Core inverse of a matrix $\mathbf{A} \in \mathbb{C}_{n,n}$, introduced in [9], and understood as the unique matrix $\mathbf{A}^{\oplus}$ such that

$$\mathbf{A}\mathbf{A}^{\oplus} = \mathbf{A}\mathbf{A}^{\dagger} \quad \text{and} \quad \mathcal{R}(\mathbf{A}^{\oplus}) \subseteq \mathcal{R}(\mathbf{A}).$$

Similarly as in the case of the group inverse, also an existence of the Core inverse is restricted to GP matrices only.

THEOREM 3.9. *Let $\mathbf{A} \in \mathbb{C}_{n,n}$. Then the following conditions are equivalent:*

(i) $\mathbf{A}$ *is tripotent and EP,*

(ii) $\mathbf{A}^{\dagger} = \mathbf{A}$,

(iii) $\mathbf{A}$ *is tripotent and* $\mathbf{A}^2$ *is Hermitian,*

(iv) $\mathbf{A}^2 = \mathbf{A}\mathbf{A}^{\dagger}$,

(v) $\mathbf{A}^{\dagger} = \mathbf{A}^2\mathbf{A}^{\dagger}$,

(vi) $\mathbf{A}(\mathbf{A}^*)^2 = \mathbf{A}$,

(vii) $(\mathbf{A}^*)^2\mathbf{A} = \mathbf{A}$,

(viii) $\mathbf{A}$ *is GP and* $\mathbf{A}^2$ *is an orthogonal projector,*

(ix) $\mathbf{A}$ *is GP and* $\mathbf{A}^{\oplus} = \mathbf{A}$,

(x) $\mathbf{A}$ *is GP and* $\mathbf{A}^{\#} = \mathbf{A}^2\mathbf{A}^{\dagger}$.

*Proof.* From Lemma 1.2 we know that $\mathbf{A}$ is tripotent and EP if an only if $(\mathbf{\Sigma}\mathbf{K})^2 = \mathbf{I}_r$ and $\mathbf{L} = \mathbf{0}$. Direct calculations with representation (1.2) show that also the conditions in points (ii)–(viii) are satisfied if and only if this conjunction holds. The proof of the remaining two points is established similarly, but here we need formulae for the group and Core inverses of $\mathbf{A}$. Both of them were provided in [9] and read

$$\mathbf{A}^{\#} = \mathbf{U}\begin{bmatrix}\mathbf{K}^{-1}\mathbf{\Sigma}^{-1} & \mathbf{K}^{-1}\mathbf{\Sigma}^{-1}\mathbf{K}^{-1}\mathbf{L} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{U}^* \quad \text{and} \quad \mathbf{A}^{\oplus} = \mathbf{U}\begin{bmatrix}(\mathbf{\Sigma}\mathbf{K})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{U}^*. \tag{3.6}$$

The proof is complete. ☐

Note that the part (i) ⇔ (ii) of Theorem 3.9 is a particular case of [7, Theorem 4], according to which $\mathbf{A}^{\dagger} = \mathbf{A}^k$ if and only if $\mathbf{A}$ is EP and $\mathbf{A}^{k+2} = \mathbf{A}$. Furthermore, the equivalence (ii) ⇔ (viii) of Theorem 3.9 was posted as Exercise 21 in [13, Section 1.6]. An additional comment is that a modified version of the equivalence (i) ⇔ (viii) of the theorem (without requirements that $\mathbf{A}$ is EP and that $\mathbf{A}^2$ is Hermitian in points (i) and (viii), respectively) was derived with a different proof in [34, Lemma 5.6.3]. It

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is also worth mentioning that the conditions $\mathbf{A} = \mathbf{A}^2\mathbf{A}^\dagger$ and $\mathbf{A}^* = \mathbf{A}^2\mathbf{A}^\dagger$, obtained by modifying the identities in points (v) and/or (x) of Theorem 3.9, are satisfied if and only if $\mathbf{A}$ is EP and Hermitian, respectively.

From Theorem 3.9, we obtain what follows.

COROLLARY 3.10. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$. *Then* $\mathbf{A}$ *is an extended orthogonal projector if and only if:*

(i) $\mathbf{A}$ *is Hermitian and any of the conditions in Theorem 3.9 is satisfied,*

(ii) $\mathbf{A}$ *is a partial isometry and any of the conditions in Theorem 3.9 is satisfied,*

(iii) $\mathbf{A}$ *is GP and* $\mathbf{A} = \mathbf{A}^\dagger = \mathbf{A}^* = \mathbf{A}^\# = \mathbf{A}^{\oplus}$.

*Proof.* Points (i) and (ii) of the corollary are established in a similar fashion as Theorem 3.9 by taking into account that $\mathbf{A}$ is Hermitian if and only if $\mathbf{L} = \mathbf{0}$ and $\mathbf{\Sigma K} = \mathbf{K}^*\mathbf{\Sigma}$, whereas is a partial isometry if and only if $\mathbf{\Sigma} = \mathbf{I}_r$. The point (iii) is obtained straightforwardly from Theorem 3.9 and point (i) of the corollary; see also [16, Theorem 8]. □

Another characterization of the class of extended orthogonal projectors was given in [27, Theorem 3] and asserts that $\mathbf{A}$ belongs to this class if and only if $\mathbf{A}(\mathbf{A}^*\mathbf{A})^k = \mathbf{A}^*(\mathbf{A}\mathbf{A}^*)^l$ for some $k, l \in \mathbb{N}$, $k \neq l$.

Theorem 3.6 asserts that within the set of tripotent matrices, a matrix is Hermitian if and only if it is EP and a partial isometry. The next theorem is related to this characterization, for it identifies conditions which combined with the requirement that $\mathbf{A}$ is EP and a partial isometry are necessary and sufficient for $\mathbf{A}$ to be an extended orthogonal projector.

THEOREM 3.11. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$. *Then* $\mathbf{A}$ *is an extended orthogonal projector if and only if* $\mathbf{A}$ *is a partial isometry, EP, and either* $\frac{1}{2}(\mathbf{A}\mathbf{A}^\dagger - \mathbf{A})$ *or* $\frac{1}{2}(\mathbf{A}\mathbf{A}^\dagger + \mathbf{A})$ *is idempotent.*

*Proof.* When $\mathbf{A}$ is EP, then idempotency of both $\frac{1}{2}(\mathbf{A}\mathbf{A}^\dagger - \mathbf{A})$ and $\frac{1}{2}(\mathbf{A}\mathbf{A}^\dagger + \mathbf{A})$ is equivalent to $\mathbf{A}^2 = \mathbf{A}\mathbf{A}^\dagger$. Since this identity occurs in Theorem 3.9(iv) , the assertion follows by Corollary 3.10(ii). □

A different characterization of the set of extended orthogonal projectors is given below. It refers to the notion of a contraction. Recall that a matrix $\mathbf{A} \in \mathbb{C}_{n,n}$ is a contraction if and only if $\mathbf{I}_n - \mathbf{A}\mathbf{A}^* \geqslant_{\mathsf{L}} \mathbf{0}$, where $\geqslant_{\mathsf{L}}$ stands for the Löwner partial ordering introduced in [28], and for $\mathbf{B}, \mathbf{C} \in \mathbb{C}_{n,n}$ specified as $\mathbf{B} \geqslant_{\mathsf{L}} \mathbf{C} \Leftrightarrow \mathbf{B} - \mathbf{C} = \mathbf{D}\mathbf{D}^*$ for some $\mathbf{D} \in \mathbb{C}_{n,n}$.

THEOREM 3.12. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$. *Then* $\mathbf{A}$ *is an extended orthogonal projector if and only if* $\mathbf{A}$ *is tripotent, EP, and a contraction.*

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*Proof.* The necessity part is satisfied trivially. For the proof of sufficiency, let $\mathbf{A}$ be of the form (1.2). As we know $\mathbf{A}^3 = \mathbf{A}$ yields $(\mathbf{\Sigma K})^2 = \mathbf{I}_r$, which can be alternatively expressed either as $\mathbf{K\Sigma K} = \mathbf{\Sigma}^{-1}$ or as $\mathbf{K}^*\mathbf{\Sigma K}^* = \mathbf{\Sigma}^{-1}$. Hence, $\mathbf{K\Sigma KK}^*\mathbf{\Sigma K}^* = \mathbf{\Sigma}^{-2}$ from where, on account of the fact that the EP-ness of $\mathbf{A}$ entails $\mathbf{KK}^* = \mathbf{I}_r$, we obtain $\mathbf{K\Sigma}^2\mathbf{K}^* = \mathbf{\Sigma}^{-2}$. On the other hand, when $\mathbf{A}$ is a contraction, then

$$\mathbf{I}_r - \mathbf{\Sigma}^2 \geqslant_{\mathsf{L}} \mathbf{0}, \tag{3.7}$$

which implies $\mathbf{KK}^* - \mathbf{K\Sigma}^2\mathbf{K}^* \geqslant_{\mathsf{L}} \mathbf{0}$. In consequence, $\mathbf{I}_r - \mathbf{\Sigma}^{-2} \geqslant_{\mathsf{L}} \mathbf{0}$, from where it follows that $\mathbf{I}_r - \mathbf{\Sigma}^2 \leqslant_{\mathsf{L}} \mathbf{0}$. Combining this condition with (3.7) gives $\mathbf{\Sigma}^2 = \mathbf{I}_r$, which can be rewritten as $\mathbf{\Sigma} = \mathbf{I}_r$. So, we have shown that when $\mathbf{A}$ is tripotent, EP, and a contraction, then it is necessarily also a partial isometry. By Theorem 3.6 we see that in such a situation $\mathbf{A}$ is Hermitian, and the proof is complete. ◻

It is known that the commutativity of two idempotent matrices is a sufficient condition for a product of the matrices to be idempotent. Moreover, the commutativity becomes necessary and sufficient condition when the two matrices are Hermitian (i.e., are orthogonal projectors); see e.g., [5, Section 1]. In the next theorem we show that the commutativity plays a similar role when idempotent matrices are replaced with tripotent ones.

THEOREM 3.13. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{n,n}$ *be tripotent. Then* $\mathbf{AB} = \mathbf{BA}$ *is a sufficient condition for* $\mathbf{AB}$ *to be tripotent. Moreover, if* $\mathbf{A}$ *and* $\mathbf{B}$ *are Hermitian (i.e., are extended orthogonal projectors), then* $\mathbf{AB}$ *is an extended orthogonal projector if and only if* $\mathbf{AB} = \mathbf{BA}$.

*Proof.* The first claim of the theorem follows from $(\mathbf{AB})^3 = \mathbf{ABABAB} = \mathbf{A}^3\mathbf{B}^3 = \mathbf{AB}$. To prove the remaining part, first observe that when $\mathbf{A}$ and $\mathbf{B}$ are Hermitian, then their commutativity yields $(\mathbf{AB})^* = \mathbf{B}^*\mathbf{A}^* = \mathbf{BA} = \mathbf{AB}$. On the other hand, if $\mathbf{AB}$ is an extended orthogonal projector, then $(\mathbf{AB})^* = \mathbf{AB}$ and $(\mathbf{AB})^* = \mathbf{B}^*\mathbf{A}^* = \mathbf{BA}$, from where the commutativity follows. ◻

It is known that in general $(\mathbf{AB})^\dagger \neq \mathbf{B}^\dagger\mathbf{A}^\dagger$; see e.g., [13, Section 4.4]. Below we provide two conditions necessary and sufficient for $(\mathbf{AB})^\dagger = \mathbf{B}^\dagger\mathbf{A}^\dagger$ when $\mathbf{A}$ and $\mathbf{B}$ are extended orthogonal projectors.

THEOREM 3.14. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{n,n}$ *be extended orthogonal projectors. Then the following conditions are equivalent:*

(i) $(\mathbf{AB})^\dagger = \mathbf{B}^\dagger\mathbf{A}^\dagger$,

(ii) $\mathbf{A}^2\mathbf{B}^2$ *is Hermitian,*

(iii) $(\mathbf{A}^2\mathbf{B}^2)^2 = \mathbf{B}^2\mathbf{A}^2$.

*Proof.* The theorem follows directly from [34, Complement 3.7]. ◻

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Observe that condition (ii) of Theorem 3.14 is of particular interest. Since when $\mathbf{A}$ and $\mathbf{B}$ are extended orthogonal projectors, then $\mathbf{A}^2$ and $\mathbf{B}^2$ are orthogonal projectors, it follows that this condition expresses the requirement that $\mathbf{A}^2\mathbf{B}^2$ is an orthogonal projector.

Subsequently, we consider so called star matrix partial ordering introduced in [17], which for $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{m,n}$ can be defined as

$$\mathbf{A} \overset{*}{\leqslant} \mathbf{B} \;\Leftrightarrow\; \mathbf{A}^*\mathbf{A} = \mathbf{A}^*\mathbf{B} \quad \text{and} \quad \mathbf{A}\mathbf{A}^* = \mathbf{B}\mathbf{A}^*. \tag{3.8}$$

Original characterizations of the conditions $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$ and $\mathbf{B} \overset{*}{\leqslant} \mathbf{A}$, when $\mathbf{A}$ and $\mathbf{B}$ are extended orthogonal projectors, are provided in what follows.

LEMMA 3.15. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{n,n}$ *be extended orthogonal projectors, of which* $\mathbf{A}$ *is of the form* (1.2) *and* $\mathbf{B}$ *is partitioned as*

$$\mathbf{B} = \mathbf{U} \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_2^* & \mathbf{B}_4 \end{bmatrix} \mathbf{U}^*, \tag{3.9}$$

*where* $\mathbf{B}_1 \in \mathbb{C}_{r,r}$. *Then:*

(i) $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$ *if and only if* $\mathbf{B}_1 = \mathbf{K}$ *and* $\mathbf{B}_2 = \mathbf{0}$,

(ii) $\mathbf{B} \overset{*}{\leqslant} \mathbf{A}$ *if and only if* $\mathbf{B}_1\mathbf{K} = \mathbf{B}_1^2$, $\mathbf{B}_2 = \mathbf{0}$, *and* $\mathbf{B}_4 = \mathbf{0}$.

*Proof.* Let us first prove point (i) of the lemma. Since $\mathbf{A}$ and $\mathbf{B}$ are Hermitian, from (3.8) we get $\mathbf{A} \overset{*}{\leqslant} \mathbf{B} \Leftrightarrow \mathbf{A}^2 = \mathbf{A}\mathbf{B} = \mathbf{B}\mathbf{A}$. Taking into account that $\mathbf{A}^2$ is Hermitian as well, the conditions on the right-hand side of this equivalence can be simplified to $\mathbf{A}^2 = \mathbf{A}\mathbf{B}$. From (1.2) and (3.9) we get

$$\mathbf{A}^2 = \mathbf{U} \begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{U}^* \quad \text{and} \quad \mathbf{A}\mathbf{B} = \mathbf{U} \begin{bmatrix} \mathbf{K}\mathbf{B}_1 & \mathbf{K}\mathbf{B}_2 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{U}^*,$$

whence it is seen that $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$ if and only if $\mathbf{K}\mathbf{B}_1 = \mathbf{I}_r$ and $\mathbf{K}\mathbf{B}_2 = \mathbf{0}$. By $\mathbf{K}^* = \mathbf{K}$, this conjunction can clearly be simplified to $\mathbf{B}_1 = \mathbf{K}$, $\mathbf{B}_2 = \mathbf{0}$.

From the proof of point (i) we conclude that $\mathbf{B} \overset{*}{\leqslant} \mathbf{A}$ if and only if $\mathbf{B}^2 = \mathbf{B}\mathbf{A}$, and Lemma 3.1 and (3.9) yield

$$\mathbf{B}^2 = \mathbf{U} \begin{bmatrix} \mathbf{B}_1^2 + \mathbf{B}_2\mathbf{B}_2^* & \mathbf{B}_1\mathbf{B}_2 + \mathbf{B}_2\mathbf{B}_4 \\ \mathbf{B}_2^*\mathbf{B}_1 + \mathbf{B}_4\mathbf{B}_2^* & \mathbf{B}_2^*\mathbf{B}_2 + \mathbf{B}_4^2 \end{bmatrix} \mathbf{U}^* \quad \text{and} \quad \mathbf{B}\mathbf{A} = \mathbf{U} \begin{bmatrix} \mathbf{B}_1\mathbf{K} & \mathbf{0} \\ \mathbf{B}_2^*\mathbf{K} & \mathbf{0} \end{bmatrix} \mathbf{U}^*.$$

Since $\mathbf{B}_2^*\mathbf{B}_2 + \mathbf{B}_4^2 = \mathbf{0}$ is equivalent to $\mathbf{B}_2 = \mathbf{0}$ and $\mathbf{B}_4 = \mathbf{0}$, we obtain the asserted equivalence. ∎

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Point (i) of Lemma 3.15 proves to be useful to establish the following results.

THEOREM 3.16. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{n,n}$ *be extended orthogonal projectors. Then the following conditions are equivalent:*

(i) $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$,

(ii) $\mathbf{ABA} = \mathbf{A}$ *and* $\mathbf{BAB} = \mathbf{A}$,

(iii) $\mathbf{ABA} = \mathbf{A}$ *and* $\mathbf{A}^2 \overset{*}{\leqslant} \mathbf{B}^2$.

*Proof.* To show that (i) $\Rightarrow$ (ii) note that $\mathbf{A}^2 = \mathbf{AB}$ implies $\mathbf{ABA} = \mathbf{A}^3 = \mathbf{A}$. Furthermore, since $\mathbf{AB} = \mathbf{BA}$, it also follows that $\mathbf{A}^2 = \mathbf{AB}$ yields $\mathbf{BAB} = \mathbf{BA}^2 = \mathbf{ABA} = \mathbf{A}$. For the reverse implication, observe that $\mathbf{A}^2 = (\mathbf{ABA})(\mathbf{BAB}) = \mathbf{A}(\mathbf{BAB})\mathbf{AB} = \mathbf{A}^3\mathbf{B} = \mathbf{AB}$, from where we conclude that $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$.

Let us now consider condition (iii). From (1.2) and (3.9), we obtain $\mathbf{A}^2 \overset{*}{\leqslant} \mathbf{B}^2$ if and only if $\mathbf{B}_1\mathbf{B}_2 + \mathbf{B}_2\mathbf{B}_4 = \mathbf{0}$ and $\mathbf{B}_1^2 + \mathbf{B}_2\mathbf{B}_2^* = \mathbf{I}_r$. On account of $\mathbf{B}_1 = \mathbf{K}$, which proves to be equivalent to $\mathbf{ABA} = \mathbf{A}$, the latter of these conditions reduces to $\mathbf{B}_2 = \mathbf{0}$. Thus, the part (i) $\Leftrightarrow$ (iii) follows by Lemma 3.15(i). □

It is known that within the class of orthogonal projectors, the Löwner and star partial orderings are equivalent; see [23, Theorem 5.8]. However, in the class of extended orthogonal projectors the two orderings prove to be independent. This fact can be confirmed with the matrices:

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Direct calculations show that $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$, but not $\mathbf{A} \leqslant_{\mathsf{L}} \mathbf{B}$ and $\mathbf{A} \leqslant_{\mathsf{L}} \mathbf{C}$, but not $\mathbf{A} \overset{*}{\leqslant} \mathbf{C}$. On the other hand, when $\mathbf{A}$ and $\mathbf{B}$ are extended orthogonal projectors, then $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$ if and only if $\mathbf{A} \overset{-}{\leqslant} \mathbf{B}$, where $\mathbf{A} \overset{-}{\leqslant} \mathbf{B}$ denotes the minus (also called rank-subtractivity) partial ordering introduced in [21], and understood as $\mathbf{A} \overset{-}{\leqslant} \mathbf{B} \Leftrightarrow \mathrm{rk}(\mathbf{B} - \mathbf{A}) = \mathrm{rk}(\mathbf{B}) - \mathrm{rk}(\mathbf{A})$. This result is a consequence of [32, Theorem 5.4.16], which asserts that the star and minus orderings coincide within the class of partial isometries.

**4. Results referring to known classes of matrices.** The present section provides several results dealing with the classes of matrices referred to in Section 3. One of the questions explored below is how to characterize matrices which belong to a given class and are $k$-potent. The section begins with a lemma which will be used to establish a subsequent theorem.

LEMMA 4.1. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$ *be of the form* (1.2). *Then* $\mathbf{K}^k = \mathbf{I}_r$ *for some* $k \in \mathbb{N}$ *is a sufficient condition for* $\mathbf{A}$ *to be EP.*

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*Proof.* It is known that the set of EP matrices is included in the set of GP matrices. Taking this into account, the assertion follows by the obvious fact that $\mathbf{K}^k = \mathbf{I}_r$ for some $k \in \mathbb{N}$ ensures that $\mathbf{K}$ is nonsingular, i.e., that $\mathbf{A}$ is GP. ◻

The theorem below provides a number of characterizations of normal and $(k+1)$-potent matrices. Condition (iii) given therein was inspired by Theorems 2.1 and 2.2 in [31]. From the former of these results it follows that when $\mathbf{A} \in \mathbb{C}_{n,n}$ is normal and $k$-potent, then it is a partial isometry. On the other hand, Theorem 2.2 in [31] asserts that when $\mathbf{A}$ is normal and $k \in \mathbb{N}$, $k \neq 3$, then $\mathbf{A}^k = \mathbf{A}$ if and only if $\mathbf{A}^{k-1} = \mathbf{A}\mathbf{A}^*$. Even though the theorem below generalizes these characteristics, its proof seems to be simpler than the proofs of [31, Theorems 2.1 and 2.2]. Conditions (v) and (vi) of the theorem below are recalled after [19, Theorem 5] and the remaining two conditions given therein are new.

THEOREM 4.2. *Let $\mathbf{A} \in \mathbb{C}_{n,n}$ and let $k \in \mathbb{N}$, $k > 2$. Then the following conditions are equivalent:*

(i) $\mathbf{A}$ *is normal and $(k+1)$-potent,*

(ii) $\mathbf{A}$ *is a partial isometry and $(k+1)$-potent,*

(iii) $\mathbf{A}$ *is a partial isometry and $\mathbf{A}^k = \mathbf{A}\mathbf{A}^*$,*

(iv) $\mathbf{A}$ *is GP, a partial isometry, and $\mathbf{A}^{k-1} = \mathbf{A}^{\#}$,*

(v) $\mathbf{A}$ *is GP, $\mathbf{A}^* = \mathbf{A}^{\#}$, and $\mathbf{A}^{k-1} = \mathbf{A}^{\#}$,*

(vi) $\mathbf{A}^{k-1} = \mathbf{A}^*$.

*Proof.* The part (i) $\Rightarrow$ (ii) follows from [31, Theorem 2.1]. Alternatively, the implication can be derived by exploiting representation (1.2) and observing that when $\mathbf{A}$ is normal and $(k+1)$-potent, then $\mathbf{\Sigma}^k = \mathbf{K}^{-k}$. Hence, similar arguments to those utilized in the proof of Theorem 3.5 leads to $\mathbf{\Sigma} = \mathbf{I}_r$, i.e., $\mathbf{A}$ is a partial isometry.

The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are based on the observation that when $\mathbf{A}$ is a partial isometry, then each of the conditions $\mathbf{A}^{k+1} = \mathbf{A}$, $\mathbf{A}^k = \mathbf{A}\mathbf{A}^*$, and $\mathbf{A}^{k-1} = \mathbf{A}^{\#}$ is equivalent to $\mathbf{K}^k = \mathbf{I}_r$.

The fact that when $\mathbf{\Sigma} = \mathbf{I}_r$ and $\mathbf{K}^k = \mathbf{I}_r$, then $\mathbf{A}^* = \mathbf{A}^{\#}$ follows by comparing the left-hand side representations in (1.4) with (3.6). In consequence, it is seen that (iv) implies (v).

Since trivially (v) $\Rightarrow$ (vi), to complete the proof it remains to show that (vi) $\Rightarrow$ (i). First observe that premultiplying and postmultiplying $\mathbf{A}^{k-1} = \mathbf{A}^*$ by $\mathbf{A}$ gives $\mathbf{A}^k = \mathbf{A}\mathbf{A}^*$ and $\mathbf{A}^k = \mathbf{A}^*\mathbf{A}$, respectively, which shows that the equality in point (vi) yields normality of $\mathbf{A}$. Substituting (1.2) and the left-hand side formula in (1.4) into

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$\mathbf{A}^{k-1} = \mathbf{A}^*$ leads to $\mathbf{L} = \mathbf{0}$ and $(\mathbf{\Sigma K})^{k-1} = \mathbf{K}^*\mathbf{\Sigma}$. Since normality of $\mathbf{A}$ means that $\mathbf{\Sigma}$ and $\mathbf{K}$ commute, we arrive at $\mathbf{\Sigma}^{-(k-2)} = \mathbf{K}^k$. Utilizing the same arguments as in the proof of Theorem 3.5, we conclude that $\mathbf{\Sigma} = \mathbf{I}_r$, which means that $\mathbf{K}^k = \mathbf{I}_r$. In consequence, it is seen that the equality in point (vi) entails $(\mathbf{\Sigma K})^k = \mathbf{I}_r$, which is necessary and sufficient for $\mathbf{A}^{k+1} = \mathbf{A}$ to hold. The proof is complete. ☐

In a comment to Theorem 4.2 it is worth mentioning that matrices $\mathbf{A}$ such that $\mathbf{A}^{k-1} = \mathbf{A}^{\#}$, $k \in \mathbb{N}$, $k > 1$, were in [15, p. 9] called $\{k\}$-group periodic matrices. It was pointed out in [15, p. 9] that $\mathbf{A}^{k-1} = \mathbf{A}^{\#} \Leftrightarrow \mathbf{A}^{k+1} = \mathbf{A}$, and several equivalent conditions for a matrix to be $\{k\}$-group periodic were given in [15, Theorem 2.1]. A similar observation is that matrices $\mathbf{A}$ such that $\mathbf{A}^{k-1} = \mathbf{A}^*$, $k \in \mathbb{N}$, $k > 2$, were in [14, p. 151] called $k$-generalized projectors, and the equivalence (i) $\Leftrightarrow$ (vi) of Theorem 4.2 was originally established in [14, Theorem 2.1]. Parenthetically note that for $k = 2$, point (vi) of Theorem 4.2 would express the requirement that $\mathbf{A}$ is Hermitian, which trivially implies normality of $\mathbf{A}$, but not its $(k+1)$-potency. Actually, from the proof of Theorem 4.2 it follows that the implication (vi) $\Rightarrow$ (i) is the only one which requires that $k > 2$, for the remaining implications established therein hold also for $k = 2$.

The identity given in Theorem 4.2(iii) can be replaced with $\mathbf{A}^k = \mathbf{A}\mathbf{A}^\dagger$. This condition is characterized in the next theorem.

Theorem 4.3. *Let $\mathbf{A} \in \mathbb{C}_{n,n}$ and let $k \in \mathbb{N}$. Then the following conditions are equivalent:*

(i) $\mathbf{A}$ *is EP and* $(k+1)$*-potent,*

(ii) $\mathbf{A}^k = \mathbf{A}\mathbf{A}^\dagger$.

*Proof.* The result follows directly by exploiting representation (1.2). ☐

Two further conditions equivalent to the requirement that $\mathbf{A}$ is EP and $k$-potent were derived in [19, Theorem 4].

Matrices which are simultaneously EP and partial isometries are characterized in what follows.

Theorem 4.4. *Let $\mathbf{A} \in \mathbb{C}_{n,n}$. Then the following conditions are equivalent:*

(i) $\mathbf{A}$ *is EP and a partial isometry,*

(ii) $\mathbf{A} = \mathbf{A}^*\mathbf{A}^2$,

(iii) $\mathbf{A} = \mathbf{A}^2\mathbf{A}^*$.

*Proof.* The result follows directly by exploiting representation (1.2). ☐

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The next theorem provides a characterization of $k$-potency of a matrix. It was originally given in [19, Theorem 3]; see also [30, Theorem 5.2].

THEOREM 4.5. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$ *and let* $k \in \mathbb{N}$, $k > 1$. *Then the following conditions are equivalent:*

(i) $\mathbf{A}$ *is* $k$*-potent,*

(ii) $\mathbf{A}$ *is GP and* $\mathbf{A}^{k+1} = \mathbf{A}^2$.

*Proof.* It is clear that when $k > 1$, then $\mathbf{A}^k = \mathbf{A}$ implies both $\mathbf{A}^{k+1} = \mathbf{A}^2$ and $\mathrm{rk}(\mathbf{A}^2) = \mathrm{rk}(\mathbf{A})$, which proves that (i) $\Rightarrow$ (ii). For the reverse implication multiply $\mathbf{A}^{k+1} = \mathbf{A}^2$ by $\mathbf{A}^{\#}$ to get $\mathbf{A}^k = \mathbf{A}$. □

Theorem 4.5 shows that within the class of GP matrices, $\mathbf{A}^2 = \mathbf{A} \Leftrightarrow \mathbf{A}^3 = \mathbf{A}^2$, $\mathbf{A}^3 = \mathbf{A} \Leftrightarrow \mathbf{A}^4 = \mathbf{A}^2$, etc. A related characterization, namely that when $\mathbf{A}$ is GP, then $\mathbf{A}^2 = \mathbf{A} \Leftrightarrow \mathbf{A}^{k+1} = \mathbf{A}^k$ for some $k \in \mathbb{N}$, follows from [41, Theorem 2.6].

Without going into details of the necessary calculations, below we provide representations of the orthogonal projectors onto the column spaces of $\mathbf{A}^2$ and $(\mathbf{A}^*)^2$, when $\mathbf{A}$ is of the form (1.2).

LEMMA 4.6. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$ *be of the form* (1.2). *Then the orthogonal projectors onto* $\mathcal{R}(\mathbf{A}^2)$ *and* $\mathcal{R}[(\mathbf{A}^2)^*]$ *are of the forms*

$$\mathbf{P}_{\mathcal{R}(\mathbf{A}^2)} = \mathbf{U}\begin{bmatrix} \mathbf{\Sigma K}(\mathbf{\Sigma K})^\dagger & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{U}^* \tag{4.1}$$

*and*

$$\mathbf{P}_{\mathcal{R}[(\mathbf{A}^2)^*]} = \mathbf{U}\begin{bmatrix} \mathbf{K}^*(\mathbf{\Sigma K \Sigma})^\dagger \mathbf{\Sigma K \Sigma K} & \mathbf{K}^*(\mathbf{\Sigma K \Sigma})^\dagger \mathbf{\Sigma K \Sigma L} \\ \mathbf{L}^*(\mathbf{\Sigma K \Sigma})^\dagger \mathbf{\Sigma K \Sigma K} & \mathbf{L}^*(\mathbf{\Sigma K \Sigma})^\dagger \mathbf{\Sigma K \Sigma L} \end{bmatrix}\mathbf{U}^*, \tag{4.2}$$

*respectively.*

*Proof.* Direct verifications confirm that the Moore–Penrose inverse of $\mathbf{A}^2$ is given by

$$(\mathbf{A}^2)^\dagger = \mathbf{U}\begin{bmatrix} \mathbf{K}^*(\mathbf{\Sigma K \Sigma})^\dagger & \mathbf{0} \\ \mathbf{L}^*(\mathbf{\Sigma K \Sigma})^\dagger & \mathbf{0} \end{bmatrix}\mathbf{U}^*.$$

Hence, the projector $\mathbf{P}_{\mathcal{R}(\mathbf{A}^2)} = \mathbf{A}^2(\mathbf{A}^2)^\dagger$ takes the form

$$\mathbf{P}_{\mathcal{R}(\mathbf{A}^2)} = \mathbf{U}\begin{bmatrix} \mathbf{\Sigma K \Sigma}(\mathbf{\Sigma K \Sigma})^\dagger & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{U}^*. \tag{4.3}$$

Since $\mathcal{R}(\mathbf{\Sigma K \Sigma}) \subseteq \mathcal{R}(\mathbf{\Sigma K})$ and $\mathrm{rk}(\mathbf{\Sigma K \Sigma}) = \mathrm{rk}(\mathbf{\Sigma K})$, representation (4.3) can be rewritten as in (4.1).

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The formula in (4.2) is obtained in a similar way by utilizing the fact that $\mathbf{P}_{\mathcal{R}[(\mathbf{A}^2)^*]} = (\mathbf{A}^2)^\dagger \mathbf{A}^2$. $\square$

Lemma 4.6 is helpful to establish the following result, providing two new characterizations of the EP-ness.

THEOREM 4.7. *Let $\mathbf{A} \in \mathbb{C}_{n,n}$. Then the following conditions are equivalent:*

(i) $\mathbf{A}$ *is EP,*

(ii) $\mathcal{R}(\mathbf{A}^2) = \mathcal{R}(\mathbf{A}^*)$*,*

(iii) $\mathbf{A}$ *is GP and* $\mathbf{A}^2$ *is EP.*

*Proof.* The equality in point (ii) of the theorem holds if and only if the orthogonal projectors onto $\mathcal{R}(\mathbf{A}^2)$ and $\mathcal{R}(\mathbf{A}^*)$ coincide. The representation of the first of these projectors is given in Lemma 4.6, whereas the latter one is obtained from (1.2) and the right-hand formula in (1.4), and reads

$$\mathbf{A}^\dagger \mathbf{A} = \mathbf{U} \begin{bmatrix} \mathbf{K}^*\mathbf{K} & \mathbf{K}^*\mathbf{L} \\ \mathbf{L}^*\mathbf{K} & \mathbf{L}^*\mathbf{L} \end{bmatrix} \mathbf{U}^*.$$

In consequence, $\mathcal{R}(\mathbf{A}^2) = \mathcal{R}(\mathbf{A}^*)$ if and only if $\mathbf{L} = \mathbf{0}$ and $\mathbf{K}^*\mathbf{K} = \mathbf{\Sigma}\mathbf{K}(\mathbf{\Sigma}\mathbf{K})^\dagger$. However, in the light of (1.5), the latter of these conditions is redundant. Thus, we have shown that (i) $\Leftrightarrow$ (ii).

To prove that (iii) $\Rightarrow$ (i), note that from (4.1) and (4.2) it follows that for $\mathbf{A}^2$ to be EP, i.e., for the identity $\mathbf{P}_{\mathcal{R}(\mathbf{A}^2)} = \mathbf{P}_{\mathcal{R}[(\mathbf{A}^*)^2]}$ to be satisfied, it is necessary that $\mathbf{K}^*(\mathbf{\Sigma}\mathbf{K}\mathbf{\Sigma})^\dagger \mathbf{\Sigma}\mathbf{K}\mathbf{\Sigma}\mathbf{L} = \mathbf{0}$. If $\mathbf{A}$ is GP, then this condition reduces to $\mathbf{L} = \mathbf{0}$, which means that $\mathbf{A}$ is EP. The fact that the reverse implication holds as well is seen clearly by comparing the projectors given in (4.1) and (4.2) under the assumption that $\mathbf{L} = \mathbf{0}$. $\square$

As was observed at the beginning of Section 2,

$$\text{rk}(\mathbf{A}^2) = 2\text{rk}(\mathbf{A}) - n \ \Leftrightarrow\ \mathcal{N}(\mathbf{A}) \subseteq \mathcal{R}(\mathbf{A}). \tag{4.4}$$

By referring to the representation (1.2), we obtain yet another condition equivalent to $\mathcal{N}(\mathbf{A}) \subseteq \mathcal{R}(\mathbf{A})$.

LEMMA 4.8. *Let $\mathbf{A} \in \mathbb{C}_{n,n}$ be of the form* (1.2)*. Then:*

(i) $\mathcal{N}(\mathbf{A}) \subseteq \mathcal{R}(\mathbf{A})$ *if and only if* $\mathbf{L}^*\mathbf{L} = \mathbf{I}_{n-r}$*,*

(ii) $\mathcal{R}(\mathbf{A}) \subseteq \mathcal{N}(\mathbf{A})$ *if and only if* $\mathbf{L}\mathbf{L}^* = \mathbf{I}_r$*,*

(iii) $\mathcal{R}(\mathbf{A}) \cap \mathcal{N}(\mathbf{A}) = \{\mathbf{0}\}$ *if and only if* $\mathcal{R}(\mathbf{L}) \subseteq \mathcal{R}(\mathbf{K})$*.*

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*Proof.* It is known that any matrices $\mathbf{B} \in \mathbb{C}_{n,p}$, and $\mathbf{C} \in \mathbb{C}_{n,q}$ satisfy $\mathcal{R}(\mathbf{B}) \subseteq \mathcal{R}(\mathbf{C}) \Leftrightarrow \mathbf{C}\mathbf{C}^\dagger\mathbf{B} = \mathbf{B}$. Thus, since $\mathcal{N}(\mathbf{A}) = \mathcal{R}(\mathbf{I}_n - \mathbf{A}^\dagger\mathbf{A})$, the inclusion given in point (i) of the lemma can be equivalently expressed as $\mathbf{A}\mathbf{A}^\dagger(\mathbf{I}_n - \mathbf{A}^\dagger\mathbf{A}) = \mathbf{I}_n - \mathbf{A}^\dagger\mathbf{A}$. Hence, by involving representation (1.2), we have $\mathcal{N}(\mathbf{A}) \subseteq \mathcal{R}(\mathbf{A}) \Leftrightarrow \mathbf{L}^*\mathbf{L} = \mathbf{I}_{n-r},\ \mathbf{L}^*\mathbf{K} = \mathbf{0}$. Since $\mathbf{L}^*\mathbf{L} = \mathbf{L}^*\mathbf{K}(\mathbf{L}^*\mathbf{K})^* + (\mathbf{L}^*\mathbf{L})^2$, which is a direct consequence of the idempotency of $\mathbf{A}^\dagger\mathbf{A}$, it is clear that $\mathbf{L}^*\mathbf{L} = \mathbf{I}_{n-r} \Rightarrow \mathbf{L}^*\mathbf{K} = \mathbf{0}$, and the equivalence in point (i) is established.

The inclusion $\mathcal{R}(\mathbf{A}) \subseteq \mathcal{N}(\mathbf{A})$ is equivalent to $(\mathbf{I}_n - \mathbf{A}^\dagger\mathbf{A})\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}\mathbf{A}^\dagger$, or, in other words, $\mathbf{A}^\dagger\mathbf{A}^2\mathbf{A}^\dagger = \mathbf{0}$. Pre- and postmultiplying this condition by $\mathbf{A}$ gives $\mathbf{A}^2 = \mathbf{0}$. In consequence, we conclude that $\mathcal{R}(\mathbf{A}) \subseteq \mathcal{N}(\mathbf{A})$ is equivalent to the requirement that $\mathbf{A}$ is nilpotent of index 2. However, from Lemma 1.2 and (1.3) we know that $\mathbf{A}^2 = \mathbf{0}$ if and only if $\mathbf{L}\mathbf{L}^* = \mathbf{I}_r$, which establishes point (ii) of the lemma.

In the proof of the next point, first recall that $\mathcal{R}(\mathbf{A}) \cap \mathcal{N}(\mathbf{A}) = \{\mathbf{0}\}$ if and only if $\mathbf{A}$ is GP; see e.g., [13, Section 4.4]. In view of Lemma 1.2, this means that $\mathcal{R}(\mathbf{A}) \cap \mathcal{N}(\mathbf{A}) = \{\mathbf{0}\}$ is equivalent to the nonsingularity of $\mathbf{K}$. Hence, the necessity part of the equivalence asserted in point (iii) is clear. Since, trivially, $\mathcal{R}(\mathbf{A}^2) \subseteq \mathcal{R}(\mathbf{A})$, the reverse implication will be established when we show that $\mathcal{R}(\mathbf{L}) \subseteq \mathcal{R}(\mathbf{K})$ implies $\mathcal{R}(\mathbf{A}) \subseteq \mathcal{R}(\mathbf{A}^2)$, in which case $\mathbf{A}$ will be necessarily GP. The inclusion $\mathcal{R}(\mathbf{A}) \subseteq \mathcal{R}(\mathbf{A}^2)$ is equivalent to $\mathbf{P}_{\mathcal{R}(\mathbf{A}^2)}\mathbf{A} = \mathbf{A}$, which, by (1.2) and (4.1), holds if and only if $\mathbf{\Sigma}\mathbf{K}(\mathbf{\Sigma}\mathbf{K})^\dagger\mathbf{\Sigma}\mathbf{L} = \mathbf{\Sigma}\mathbf{L}$. This equality can be equivalently expressed as $\mathcal{R}(\mathbf{\Sigma}\mathbf{L}) \subseteq \mathcal{R}(\mathbf{\Sigma}\mathbf{K})$ or, in yet another form, as $\mathcal{R}(\mathbf{L}) \subseteq \mathcal{R}(\mathbf{K})$. ◻

The paper is concluded with some remarks concerning classes of SR and DR matrices. On account of Lemma 1.2, it is seen that the conditions involved in Lemma 4.8(i) ensure that $\mathbf{A}$ is necessarily SR, whereas the conditions occurring in Lemma 4.8(ii) imply that $\mathbf{A}$ is DR.

As was shown in [10, p. 1225], in general $\mathrm{rk}(\mathbf{K}^*\mathbf{L}) = \mathrm{rk}(\mathbf{K}) + \mathrm{rk}(\mathbf{L}) - r$ holds (which is equivalent to $\mathcal{N}(\mathbf{K}^*) \subseteq \mathcal{R}(\mathbf{L})$). In consequence, when $\mathbf{A}$ is such that $\mathcal{N}(\mathbf{A}) \subseteq \mathcal{R}(\mathbf{A})$, then $\mathrm{rk}(\mathbf{A}) = \frac{1}{2}[\mathrm{rk}(\mathbf{K}) + n]$. Another observation refers to the known fact that $\mathbf{A}$ is SR if and only if $\mathbf{A}^*$ is SR if and only if $\mathbf{A}^\dagger$ is SR; see [10, p. 1228]. It turns out that the same inheritance property holds within the considered subset of the SR class, for

$$\mathcal{N}(\mathbf{A}) \subseteq \mathcal{R}(\mathbf{A}) \ \Leftrightarrow\ \mathcal{N}(\mathbf{A}^*) \subseteq \mathcal{R}(\mathbf{A}^*) \ \Leftrightarrow\ \mathcal{N}(\mathbf{A}^\dagger) \subseteq \mathcal{R}(\mathbf{A}^\dagger). \tag{4.5}$$

The former of the equivalences in (4.5) is a direct consequence of (4.4), whereas the latter follows from $\mathcal{N}(\mathbf{A}^*) = \mathcal{N}(\mathbf{A}^\dagger)$ and $\mathcal{R}(\mathbf{A}^*) = \mathcal{R}(\mathbf{A}^\dagger)$. The last remark is that by combining Lemma 4.8(iii) with [10, Lemma 3] we conclude that $\mathbf{A}$ is simultaneously GP and DR if and only if $\mathcal{R}(\mathbf{L}) = \mathcal{R}(\mathbf{K})$.

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