

A NOTE ON THE POSITIVE SEMIDEFINITE MINIMUM RANK OF A SIGN PATTERN MATRIX*

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Abstract. The positive semidefinite minimum rank of a sign pattern matrix A , denoted by $\text{pmr}(A)$, is the smallest possible rank among all real positive semidefinite matrices M with $\text{sgn}(M) = A$. By counting the number of signs and using the inner structure of A and the underlying graph $G(A)$, various bounds for $\text{pmr}(A)$ are given in this paper.

Key words. Minimum rank, Minimum semidefinite rank, Sign pattern.

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1. Introduction. Let $A = (a_{ij})$ be any $m \times n$ real matrix. The $m \times n$ matrix $\text{sgn}(A) = (\text{sgn}(a_{ij}))$ is called the *sign pattern* of A , where $\text{sgn}(x) = -1, 0, 1$ when $x <, =, > 0$ respectively. A matrix with $-1, 0, 1$ entries is also called a *sign pattern matrix*. The concept of sign pattern matrix first appeared in Paul A. Samuelson's book [24], and it was related to the stability problem in economic models. Later on, additional applications for the sign pattern matrix have been found. Since the 1990s, the sign pattern matrix has been an important research topic in combinatorial matrix theory. The interested readers many refer to [12] and the bibliography therein.

In this paper, we concentrate on the following positive semidefinite minimum rank problem: Given an $n \times n$ symmetric sign pattern matrix A , let $\mathcal{Q}(A)$ be the set of all positive semidefinite real matrices M with $\text{sgn}(M) = A$. Determine the *positive semidefinite minimum rank* of A ,

$$\text{pmr}(A) = \min_{M \in \mathcal{Q}(A)} \text{rank}(M).$$

The positive semidefinite minimum rank problem is related to the minimum rank problem for a sign pattern which in general is still an open problem. The positive semidefinite minimum rank is also related to the dot product dimension of graphs

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[10, 22, 23]. Let $G = (V, E)$ be a simple graph. If there is a map $f : V \rightarrow \mathbb{R}^d$ such that for any two vertices $u \neq v$, $uv \in E$ if and only if $f(u) \cdot f(v) \geq 1$, then f is called a *dot product representation of G* , and the minimal possible integer d is called the *dot product dimension of G* .

Recently, the minimum rank problem and the minimum semidefinite rank problem for graphs have attracted the attention of many researchers. For example, Barrett et al. [3] characterize those graphs G with minimum rank $\text{mr}(G) \geq n - 2$; Hogben [14] surveys the results on minimum rank of all types of trees from a unified perspective and solves the minimum rank problem for simple directed trees; Hogben also [15] uses various graph parameters to bound the minimum rank of sign patterns and determines the minimum rank of small sign patterns; van der Holst [16] characterizes those graphs G with minimum semidefinite rank $\text{msr}(G) \geq n - 2$; Booth et al. [4] give the upper and lower bounds for $\text{msr}(G)$ and determine $\text{msr}(G)$ when G is a chordal graph; several authors [2, 7, 8, 19, 20, 25] explore the connection between $\text{msr}(G)$ and the graph properties of G , such as triangle-free graph, outer-planar graph, complement of a partial k -tree, chordal supergraph, etc.

The minimum rank problem and minimum semidefinite rank problem for graphs make use of the zero-nonzero pattern of the adjacency matrix. Let $G = (V, E)$ be a simple graph with $V = \{v_1, \dots, v_n\}$ and $\mathcal{P}(G)$ be the set of all $n \times n$ positive semidefinite real matrices $A = (a_{ij})$ such that for any $i \neq j$, $a_{ij} \neq 0$ if and only if $v_i v_j \in E$. The minimum semidefinite rank of G is defined to be $\text{msr}(G) = \min_{A \in \mathcal{P}(G)} \text{rank}(A)$. The minimum rank of G , denoted by $\text{mr}(G)$, is defined similarly. Any symmetric sign pattern matrix A is associated with its underlying graph $G = G(A)$ defined as below. Since $\mathcal{Q}(A) \subset \mathcal{P}(G)$, we have $\text{pmr}(A) \geq \text{msr}(G)$. However, the results for $\text{msr}(G)$ can not be extended to the $\text{pmr}(A)$.

In Section 2, we give some lower bounds for $\text{pmr}(A)$ in terms of the number of positive, negative, and zero entries of A . In Section 3, we give some upper bounds for $\text{pmr}(A)$ in terms of the pmr of its submatrix. In Section 4, we pay more attention to those sign pattern matrices A whose underlying graph $G(A)$ has a particular graph structure.

Here are some definitions and notations that are used throughout this paper.

- $J_{m \times n}$ stands for the $m \times n$ matrix of all ones, and is shortened to J when the size is clear.
- $n - \text{pmr}(A)$ is called the *maximum nullity* of an $n \times n$ symmetric sign pattern matrix A .
- $N_+(A)$, $N_-(A)$, $N_0(A)$ are the number of positive, negative and zero entries of a real matrix A , respectively.
- By changing some entries of a permutation matrix from 1 to -1 , the resulting

matrix is called a *signed permutation matrix*.

- For any symmetric matrix $A = (a_{ij})$, the *underlying graph* of A , denoted by $G(A)$, is the simple graph with adjacency matrix $B = (b_{ij})$, where $b_{ij} = 1$ if $i \neq j$ and $a_{ij} \neq 0$; otherwise, $b_{ij} = 0$.

2. Lower bound in terms of the number of signs.

LEMMA 2.1. *Let $\alpha_1, \dots, \alpha_m$ be nonzero vectors in an n -dimensional Euclidean space V with inner product $(\cdot, \cdot)$, then we have*

- (1) *if $(\alpha_i, \alpha_j) \leq 0$ for all $i \neq j$, then $m \leq 2n$;*
- (2) *if $(\alpha_i, \alpha_j) < 0$ for all $i \neq j$, then $m \leq n + 1$.*

Proof. We prove (1) by induction on n . When $n = 1$, since the vectors are nonzero, $(\alpha_1, \alpha_i) \leq 0$ implies $\alpha_i = \lambda_i \alpha_1$, where $\lambda_i < 0$. If $m \geq 3$, then $(\alpha_2, \alpha_3) > 0$ leads to a contradiction. Hence, $m \leq 2$. When $n \geq 2$, let

$$\beta_i = \alpha_i - \frac{(\alpha_i, \alpha_m)}{(\alpha_m, \alpha_m)} \alpha_m, \quad i = 1, \dots, m-1.$$

All β_i belong to the $(n-1)$ -dimensional Euclidean space

$$W = \{v \in V \mid (v, \alpha_m) = 0\}.$$

If $\beta_i = 0$, then $\alpha_i = \lambda \alpha_m$, where $\lambda = \frac{(\alpha_i, \alpha_m)}{(\alpha_m, \alpha_m)} < 0$. Thus, at most one of $\beta_1, \dots, \beta_{m-1}$ is zero. Since

$$(\beta_i, \beta_j) = (\alpha_i, \alpha_j) - \frac{(\alpha_i, \alpha_m)(\alpha_j, \alpha_m)}{(\alpha_m, \alpha_m)} \leq 0$$

for all $i \neq j$, applying the induction hypothesis to W , we have $m-2 \leq 2(n-1)$, $m \leq 2n$. The proof of (2) is similar. $\square$

LEMMA 2.2. *Let k and n be positive integers and $x_1, \dots, x_k$ be nonnegative integers such that $x_1 + \dots + x_k = n$. Then we have*

$$x_1^2 + \dots + x_k^2 \geq \frac{n^2 - t^2 + kt}{k},$$

where t is the remainder of n divided by k .

Proof. Without loss of generality, we may assume that $x_1 \leq \dots \leq x_k$. If $x_k - x_1 > 1$, then

$$(x_1 + 1)^2 + (x_k - 1)^2 - (x_1^2 + x_k^2) = 2(1 + x_1 - x_k) < 0.$$

So $x_1^2 + \dots + x_k^2$ reaches the minimum value when

$$x_1 = \dots = x_{k-t} = \frac{n-t}{k}, \quad x_{k-t+1} = \dots = x_k = \frac{n-t}{k} + 1.$$

Therefore,

$$x_1^2 + \cdots + x_k^2 \geq (k-t) \left(\frac{n-t}{k} \right)^2 + t \left(\frac{n-t}{k} + 1 \right)^2 = \frac{n^2 - t^2 + kt}{k}. \quad \square$$

LEMMA 2.3. Let A be an $n \times n$ positive semidefinite real matrix with positive diagonal entries. If $\text{rank}(A) = r$, then we have

$$(1) N_-(A) \leq n^2 - f_{r+1}(n); \quad (2) N_0(A) \leq n^2 - f_r(n); \quad (3) N_+(A) \geq f_{2r}(n),$$

where $f_k(n) = \frac{n^2 - t^2 + kt}{k}$ and t is the remainder of n divided by k .

Proof. We know that there exist nonzero column vectors $\alpha_1, \dots, \alpha_n \in \mathbb{R}^r$ so that $A = (\alpha_i^T \alpha_j)$. Next, we will prove the conclusion by adjusting these α_i 's.

(1) For any $i < j$ so that $\alpha_i^T \alpha_j \geq 0$, we replace α_i by α_j if the j -th row of A has more negative entries than the i -th row; otherwise we replace α_j by α_i . Repeating the above process, we finally get some nonzero column vectors $\beta_1, \dots, \beta_n \in \mathbb{R}^r$ so that for any i, j either $\beta_i = \beta_j$ or $\beta_i^T \beta_j < 0$, and the matrix $B = (\beta_i^T \beta_j)$ satisfies $N_-(B) \geq N_-(A)$. Without loss of generality, we may assume that $\{\beta_1, \dots, \beta_n\}$ consist of x_1 copies of $\beta_1, \dots, x_k$ copies of β_k , where $x_1 + \cdots + x_k = n$ and $\beta_1, \dots, \beta_k$ are distinct. By Lemma 2.1, $k \leq r + 1$. By Lemma 2.2,

$$N_-(B) = n^2 - (x_1^2 + \cdots + x_k^2) \leq n^2 - f_{r+1}(n).$$

(2) Similar to (1), there exist some nonzero column vectors $\beta_1, \dots, \beta_n \in \mathbb{R}^r$ so that for any i, j either $\beta_i = \beta_j$ or $\beta_i \perp \beta_j$, and the matrix $B = (\beta_i^T \beta_j)$ satisfies $N_0(B) \geq N_0(A)$. Assume that $\{\beta_1, \dots, \beta_n\}$ consist of x_1 copies of $\beta_1, \dots, x_k$ copies of β_k , where $x_1 + \cdots + x_k = n$ and $\beta_1, \dots, \beta_k$ are distinct, then $k \leq r$. By Lemma 2.2,

$$N_0(B) = n^2 - (x_1^2 + \cdots + x_k^2) \leq n^2 - f_r(n).$$

(3) Similar to (1), there exist some nonzero column vectors $\beta_1, \dots, \beta_n \in \mathbb{R}^r$ so that for any i, j either $\beta_i = \beta_j$ or $\beta_i^T \beta_j \leq 0$, and the matrix $B = (\beta_i^T \beta_j)$ satisfies $N_+(B) \leq N_+(A)$. Assume that $\{\beta_1, \dots, \beta_n\}$ consist of x_1 copies of $\beta_1, \dots, x_k$ copies of β_k , where $x_1 + \cdots + x_k = n$ and $\beta_1, \dots, \beta_k$ are distinct. By Lemma 2.1, $k \leq 2r$. By Lemma 2.2,

$$N_+(B) = x_1^2 + \cdots + x_k^2 \geq f_{2r}(n). \quad \square$$

We demonstrate the adjustment procedure in the proof of Lemma 2.3 (1) with

the following. Suppose $A = (\alpha_i^T \alpha_j)_{1 \leq i, j \leq 4}$ and

$$\operatorname{sgn}(A) = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & -1 \\ -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix}.$$

Since $a_{12} \geq 0$ and the first row of A has more negative entries than the second row, after replacing α_2 by α_1 , $B = (\alpha_1, \alpha_1, \alpha_3, \alpha_4)^T (\alpha_1, \alpha_1, \alpha_3, \alpha_4)$ satisfies

$$\operatorname{sgn}(B) = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix}$$

and $N_-(B) \geq N_-(A)$. Since $b_{34} \geq 0$ and the third row of B has the same number of negative entries as the fourth row, after replacing α_4 by α_3 , $C = (\alpha_1, \alpha_1, \alpha_3, \alpha_3)^T (\alpha_1, \alpha_1, \alpha_3, \alpha_3)$ satisfies

$$\operatorname{sgn}(C) = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

and $N_-(C) = N_-(B)$. So we have $N_-(C) \geq N_-(A)$ and $\operatorname{pmr}(A) \geq \operatorname{rank}(C)$.

By Lemma 2.3 and $f_k(n) = \frac{n^2 + (k-t)t}{k} \geq \frac{n^2}{k}$, we immediately get a rough estimate on the positive semidefinite minimum rank of a sign pattern matrix.

THEOREM 2.4. *For any $n \times n$ symmetric sign pattern matrix A with diagonal entries 1,*

$$\operatorname{pmr}(A) \geq \max \left(\frac{N_-(A)}{n^2 - N_-(A)}, \frac{n^2}{n^2 - N_0(A)}, \frac{n^2}{2N_+(A)} \right).$$

In particular, $\operatorname{pmr}(A) \geq n - 1$ when $N_-(A) = n^2 - n$.

REMARK. For any signed permutation matrix P , $\operatorname{pmr}(PAP^T) = \operatorname{pmr}(A)$ while $N_{\pm}(PAP^T)$ may be different from $N_{\pm}(A)$. It is possible to obtain better estimate of $\operatorname{pmr}(A)$ by applying Theorem 2.4 to PAP^T .

3. Upper bound in terms of the pmr of submatrix.

THEOREM 3.1. *Let $A = \begin{bmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{bmatrix}$ be an $n \times n$ symmetric sign pattern matrix with diagonal entries 1, where the size of A_1 is $n_1 \times n_1$. We have*

- (1) $\text{pmr}(A_1) \leq \text{pmr}(A)$.
- (2) $\text{pmr}(A_1) = \text{pmr}(A)$ if and only if there exist $M_1 \in \mathcal{Q}(A_1)$, $X \in \mathbb{R}^{n_1 \times (n-n_1)}$ with $\text{rank}(M_1) = \text{pmr}(A_1)$, $\text{sgn}(M_1 X) = A_2$ and $\text{sgn}(X^T M_1 X) = A_3$;
- (3) $\text{pmr}(A) \leq \text{pmr}(A_1) + n - n_1$ if A_1 has no zero entries;
- (4) $\text{pmr}(A) \leq n - 1$ if $A \neq I$;
- (5) If A has no zero entries and $\text{pmr}(A) = n - 1$, then $A = P(2I - J)P^T$ for some signed permutation matrix P .

Proof.

(1) $\text{pmr}(A_1) \leq \text{pmr}(A)$ follows from the definition.

(2) Suppose $\text{pmr}(A_1) = \text{pmr}(A)$. Let $M = \begin{bmatrix} M_1 & M_2 \\ M_2^T & M_3 \end{bmatrix} \in \mathcal{Q}(A)$ with $\text{rank}(M) = \text{pmr}(A)$. Since $\text{rank}(M_1) \geq \text{pmr}(A_1)$, we have

$$\text{rank}(M_1) = \text{rank}(M) = \text{rank}(M_1 \ M_2).$$

Hence, $M_2 = M_1 X$ for some $X \in \mathbb{R}^{n_1 \times (n-n_1)}$. By

$$\text{rank}(M) = \text{rank}(M_1) + \text{rank}(M_3 - X^T M_1 X),$$

we have $M_3 = X^T M_1 X$.

Next, suppose $M_1 \in \mathcal{Q}(A_1)$, $X \in \mathbb{R}^{n_1 \times (n-n_1)}$ with $\text{rank}(M_1) = \text{pmr}(A_1)$, $\text{sgn}(M_1 X) = A_2$ and $\text{sgn}(X^T M_1 X) = A_3$. Thus,

$$M = \begin{bmatrix} M_1 & M_1 X \\ X^T M_1 & X^T M_1 X \end{bmatrix} \in \mathcal{Q}(A),$$

and hence,

$$\text{pmr}(A) \leq \text{rank}(M) = \text{rank}(M_1) = \text{pmr}(A_1).$$

(3) Let $M_1 \in \mathcal{Q}(A_1)$ with $\text{rank}(M_1) = \text{pmr}(A_1)$ and ϵ be the smallest absolute value of entries of M_1 . Since M_1 has no zero entries, $\epsilon > 0$. When $\lambda > \|A_3\| + \|A_2\|^2/\epsilon$,

$$\begin{aligned} \|A_2(\lambda I + A_3)^{-1} A_2^T\| &= \left\| \sum_{k=0}^{\infty} \lambda^{-k-1} A_2 (-A_3)^k A_2^T \right\| \\ &\leq \sum_{k=0}^{\infty} \lambda^{-k-1} \|A_2\|^2 \|A_3\|^k = \frac{\|A_2\|^2}{\lambda - \|A_3\|} < \epsilon, \end{aligned}$$

where $\|\cdot\|$ is the matrix 2-norm (the largest singular value of a matrix). Therefore, the absolute value of each entry of $A_2(\lambda I + A_3)^{-1} A_2^T$ is less than ϵ , $M_1 + A_2(\lambda I + A_3)^{-1} A_2^T$ has the same sign pattern as M_1 , and

$$M = \begin{bmatrix} M_1 + A_2(\lambda I + A_3)^{-1} A_2^T & A_2 \\ A_2^T & \lambda I + A_3 \end{bmatrix} \in \mathcal{Q}(A).$$

Hence, using Schur complements we conclude that

$$\text{pmr}(A) \leq \text{rank}(M) = \text{rank}(M_1) + \text{rank}(\lambda I + A_3) \leq \text{pmr}(A_1) + n - n_1.$$

(4) If $A \neq I$, there is a permutation matrix P such that the last row of PAP^T has at least two nonzero entries. Without loss of generality, we may assume that $n_1 = n - 1$ and A_2 is a nonzero vector. Since $M_1 = A_1 + nI$ is positive definite,

$$M = \begin{bmatrix} M_1 & A_2 \\ A_2^T & A_2^T M_1^{-1} A_2 \end{bmatrix} \in \mathcal{Q}(A).$$

Hence,

$$\text{pmr}(A) \leq \text{rank}(M) = \text{rank}(M_1) = n - 1.$$

(5) We use induction on n . When $n = 2$,

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ or } A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

and the conclusion holds. When $n \geq 3$, let $n_1 = n - 1$, $\text{pmr}(A_1) = n - 2$ follows from (3) and (4). By the induction hypothesis, $A_1 = P_1(2I - J)P_1^T$ for some signed permutation matrix P_1 , so

$$A = P \begin{bmatrix} 2I_s - J & -J & -J \\ -J & 2I_t - J & J \\ -J & J & 1 \end{bmatrix} P^T,$$

where P is a signed permutation matrix, $s + t = n - 1$. If $st \neq 0$, then

$$M = P \begin{bmatrix} 2sI_s - J & -J & -J \\ -J & 2tI_t - J & J \\ -J & J & 1 \end{bmatrix} P^T \in \mathcal{Q}(A)$$

and $\text{rank}(M) = n - 2$. This is a contradiction. If $s = 0$, then

$$\begin{bmatrix} 2I - J & J \\ J & 1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2I - J & -J \\ -J & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix}.$$

By Theorem 2.4 and (4),

$$\text{pmr}(A) = \text{pmr}(2I - J) = n - 1. \quad \square$$

Theorem 3.1 (3) is a special case of the following theorem.

THEOREM 3.2. *Let $A = \begin{bmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{bmatrix}$ be an $n \times n$ symmetric sign pattern matrix with diagonal entries 1, where the size of A_1 is $n_1 \times n_1$. If*

$$A_1 = P \begin{bmatrix} B_{11} & \cdots & B_{1k} \\ \vdots & \ddots & \vdots \\ B_{k1} & \cdots & B_{kk} \end{bmatrix} P^T,$$

where P is a permutation matrix and each diagonal block B_{ii} is a square matrix without zero entries, then

$$\text{pmr}(A) \leq \text{pmr}(A_1) + (n - n_1)k.$$

Proof. Without loss of generality, we may assume that $P = I$. When $\lambda > n$, $M_3 = \lambda I + A_3$ is positive definite, $M_3 = Y^2$ for some positive definite real matrix Y . Let $M_1 \in \mathcal{Q}(A_1)$ with $\text{rank}(M_1) = \text{pmr}(A_1)$. Write

$$M_1 = \begin{bmatrix} X_{11} & \cdots & X_{1k} \\ \vdots & \ddots & \vdots \\ X_{k1} & \cdots & X_{kk} \end{bmatrix}, \quad A_2 = \begin{bmatrix} C_1 \\ \vdots \\ C_k \end{bmatrix},$$

where each X_{ii} and C_i has the same number of rows as B_{ii} . Let λ be large enough so that

$$\text{sgn}(X_{ii} + C_i(\lambda I + A_3)^{-1}C_i^T) = B_{ii}$$

for all i , then

$$M = \begin{bmatrix} M_1 & O \\ O & O \end{bmatrix} + \begin{bmatrix} C_1 Y^{-1} & & \\ & \ddots & \\ & & C_k Y^{-1} \\ Y & \cdots & Y \end{bmatrix} \begin{bmatrix} Y^{-1} C_1^T & & Y \\ & \ddots & \vdots \\ & & Y^{-1} C_k^T & Y \end{bmatrix} \in \mathcal{Q}(A),$$

and hence,

$$\text{pmr}(A) \leq \text{rank}(M) \leq \text{pmr}(A_1) + (n - n_1)k. \quad \square$$

The following example shows that $\text{pmr}(A) = \text{pmr}(A_1) + (n - n_1)k$ can hold in Theorem 3.2. Let

$$A = \begin{bmatrix} \text{diag}(B_1, \dots, B_k) & J \\ J & 1 \end{bmatrix} \text{ where } B_1 = \cdots = B_k = 2I_s - J.$$

Then $n_1 = ks$, $n = n_1 + 1$, $\text{pmr}(A_1) = k(s - 1)$, $\text{pmr}(A) = n - 1 = \text{pmr}(A_1) + k$.

REMARK. The minimal possible k in Theorem 3.2 is the chromatic number of the complement graph of $G(A_1)$.

Naturally we raise the following question: find all the $n \times n$ sign pattern matrices with maximum nullity 1. In Section 4, we will give several classes of such matrices.

4. pmr and the underlying graph.

THEOREM 4.1. *Let A be an $n \times n$ symmetric sign pattern matrix with diagonal entries 1. Suppose*

$$A = \begin{bmatrix} A_{11} & O & A_{13} \\ O & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{bmatrix}, \quad B_i = \begin{bmatrix} A_{ii} & A_{i3} \\ A_{i3}^T & A_{33} \end{bmatrix},$$

where $A_{i3} \neq O$, and the size of A_{ii} is $n_i \times n_i$, $i = 1, 2, 3$. Then

$$\text{pmr}(A) \leq \text{pmr}(B_1) + \text{pmr}(B_2) \leq \text{pmr}(A) + 2(n_3 - 1).$$

Proof. Let $X_i = \begin{bmatrix} M_{ii} & M_{i3} \\ M_{i3}^T & M_{33}^{(i)} \end{bmatrix} \in \mathcal{Q}(B_i)$ with $\text{rank}(X_i) = \text{pmr}(B_i)$, $i = 1, 2$.

$$M = \begin{bmatrix} M_{11} & O & M_{13} \\ O & M_{22} & M_{23} \\ M_{13}^T & M_{23}^T & M_{33}^{(1)} + M_{33}^{(2)} \end{bmatrix} \in \mathcal{Q}(A),$$

and hence,

$$\text{pmr}(A) \leq \text{rank}(M) \leq \text{pmr}(B_1) + \text{pmr}(B_2).$$

Next, let $M = \begin{bmatrix} M_{11} & O & M_{13} \\ O & M_{22} & M_{23} \\ M_{13}^T & M_{23}^T & M_{33} \end{bmatrix} \in \mathcal{Q}(A)$ with $\text{rank}(M) = \text{pmr}(A)$. Then $M_{i3} = M_{ii}Y_i$ for some $Y_i \in \mathbb{R}^{n_i \times n_3}$. For each i , there is a diagonal matrix D_i such that

$$M_{33} - D_i \in \mathcal{Q}(A_{33}) \text{ and } \det(M_{33} - Y_i^T M_{ii} Y_i - D_i) = 0.$$

Hence,

$$\begin{aligned} \text{pmr}(B_1) + \text{pmr}(B_2) &\leq \text{rank} \begin{bmatrix} M_{11} & M_{13} \\ M_{13}^T & M_{33} - D_1 \end{bmatrix} + \text{rank} \begin{bmatrix} M_{22} & M_{23} \\ M_{23}^T & M_{33} - D_2 \end{bmatrix} \\ &\leq \text{rank}(M_{11}) + \text{rank}(M_{22}) + 2(n_3 - 1) \leq \text{rank}(M) + 2(n_3 - 1). \quad \square \end{aligned}$$

Theorem 4.1 shows that if $G(A)$ is the union of two induced subgraph $G(B_1)$ and $G(B_2)$, and k is the number of vertices in $G(B_1) \cap G(B_2)$, then $\text{pmr}(A)$ and $\text{pmr}(B_1) + \text{pmr}(B_2)$ differ by at most $2k - 2$. In particular, if $k = 1$, then $\text{pmr}(A) = \text{pmr}(B_1) + \text{pmr}(B_2)$. Furthermore, if $G(A)$ has a vertex v of degree 1 and $G(B)$ is the induced subgraph of $G(A)$ by deleting v , then $\text{pmr}(A) = \text{pmr}(B) + 1$. As a consequence, we have the following.

THEOREM 4.2. *Let A be an $n \times n$ symmetric sign pattern matrix with diagonal entries 1. If $n \geq 2$ and $G(A)$ is a tree, then $\text{pmr}(A) = n - 1$.*

THEOREM 4.3. *Let $A = (a_{ij})$ be an $n \times n$ sign pattern matrix,*

$$a_{ij} = \begin{cases} 1, & \text{if } |i - j| \leq 1; \\ \delta, & \text{if } |i - j| = n - 1; \\ 0, & \text{otherwise,} \end{cases}$$

where $n \geq 3$ and $\delta = \pm 1$, then

$$\text{pmr}(A) = \begin{cases} n - 1, & \text{if } \delta = (-1)^n; \\ n - 2, & \text{if } \delta = (-1)^{n-1}. \end{cases}$$

Proof. We use induction on n . When $n = 3$,

$$\text{pmr} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 1, \quad \text{pmr} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \text{pmr} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} = 2$$

because of

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & -1 & \\ & & 1 \end{bmatrix},$$

the conclusion holds. When $n \geq 4$, write $A = \begin{bmatrix} A_1 & \alpha \\ \alpha^T & 1 \end{bmatrix}$. Let $M = \begin{bmatrix} M_1 & \beta \\ \beta^T & c \end{bmatrix} \in \mathcal{Q}(A)$ with $\text{rank}(M) = \text{pmr}(A)$. Since $M_2 = M_1 - \frac{1}{c}\beta\beta^T$ is positive semidefinite and

$$\text{sgn}(M_2) = (b_{ij}) \text{ where } b_{ij} = \begin{cases} 1, & \text{if } |i - j| \leq 1; \\ -\delta, & \text{if } |i - j| = n - 2; \\ 0, & \text{otherwise.} \end{cases}$$

By the induction hypothesis,

$$\text{pmr}(A) = \text{rank}(M_2) + 1 \geq \text{pmr}(\text{sgn}(M_2)) + 1 = \begin{cases} n - 1, & \text{if } \delta = (-1)^n; \\ n - 2, & \text{if } \delta = (-1)^{n-1}. \end{cases}$$

If $\delta = (-1)^n$, then by Theorem 3.1 (4), $\text{pmr}(A) = n - 1$. If $\delta = (-1)^{n-1}$, then let

$$P = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & 1 \\ 0 & \cdots & 0 & 1 \\ (-1)^{n-1} & \cdots & -1 & 1 \end{bmatrix} \in \mathbb{R}^{n \times (n-2)},$$

and note that $PP^T \in \mathcal{Q}(A)$ and $\text{pmr}(A) \leq \text{rank}(PP^T) = n - 2$. Hence, $\text{pmr}(A) = n - 2$. $\square$

The matrices A_δ ($\delta = \pm 1$) in Theorem 4.3 are two representatives of those sign pattern matrix A so that $G(A)$ is the n -cycle. In general, if $G(A)$ is the n -cycle, then there exists a signed permutation matrix P so that PAP^T is one of the A_δ , thus $\text{pmr}(A) = n - 2$ or $n - 1$. Similarly, if $G(A)$ is a connected unicyclic graph, by Theorem 4.1, we also have $\text{pmr}(A) = n - 2$ or $n - 1$.

By Theorem 3.1 (5), Theorem 4.2 and Theorem 4.3, we get three kinds of sign pattern matrices with maximum nullity 1. Let S_1 be the set of these three kinds of sign pattern matrices, and $S_2, S_3, \dots$ be defined in the following successive way.

Given S_k , S_{k+1} is the the set of all matrix $P \begin{bmatrix} A & O & \alpha \\ O & B & \beta \\ \alpha^T & \beta^T & 1 \end{bmatrix} P^T$, where $\begin{bmatrix} A & \alpha \\ \alpha^T & 1 \end{bmatrix} \in S_k$,

$\begin{bmatrix} B & \beta \\ \beta^T & 1 \end{bmatrix} \in S_k$, $\alpha \neq 0$, $\beta \neq 0$, P is a signed permutation matrix. By Theorem 4.1, every matrix in S_{k+1} has maximum nullity 1. In fact, the underlying graph of a sign pattern matrix with maximum nullity 1 could be any connected simple graph. The Laplacian matrix L of any connected simple G on n vertices is positive semidefinite and $\text{rank}(L) = n - 1$ (see [6, Theorem 2.3.2]).

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