

DIAGONAL ENTRY RESTRICTIONS IN MINIMUM RANK MATRICES*

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Abstract. Let F be a field, let G be a simple graph on n vertices, and let $\mathcal{S}^F(G)$ be the class of all F -valued symmetric $n \times n$ matrices whose nonzero off-diagonal entries occur in exactly the positions corresponding to the edges of G . For each graph G , there is an associated minimum rank class, $\mathcal{MR}^F(G)$ consisting of all matrices $A \in \mathcal{S}^F(G)$ with $\text{rank } A = \text{mr}^F(G)$, the minimum rank among all matrices in $\mathcal{S}^F(G)$. Although no restrictions are applied to the diagonal entries of matrices in $\mathcal{S}^F(G)$, this work explores when the diagonal entries corresponding to specific vertices of G must be zero or nonzero for all matrices $A \in \mathcal{MR}^F(G)$. These vertices are denoted as nil or nonzero, respectively. Vertices whose corresponding diagonal entries are not similarly restricted for all matrices in $\mathcal{MR}^F(G)$ are called neutral. The minimum rank of a graph following an edge-subdivision is determined by the existence of a nil vertex, and several relations between diagonal restrictions and the rank-spread parameter are found. This is followed by the rather different approach of using the graph parameter $\widehat{Z}$ to identify nil and nonzero vertices. The nil, nonzero and neutral vertices of trees are classified in terms of rank-spread. Finally, it is shown that except for K_3 , 2-connected graphs with maximum nullity 2 have all neutral vertices and, moreover, the graphs with maximum nullity 2 that have nil or nonzero vertices are completely classified.

Key words. Combinatorial matrix theory, Edge subdivision, Graph, Minimum positive semi-definite rank, Minimum rank, Nil vertex, Rank-spread, Symmetric.

AMS subject classifications. 05C50, 15A03, 15B57.

1. Introduction. One area of combinatorial matrix theory uses graphs to study rank and spectral properties of symmetric $n \times n$ matrices. Given a graph G on n vertices and a field F , we define $\mathcal{S}^F(G)$ to be the set of all symmetric $n \times n$ matrices $A = [a_{ij}]$ such that $a_{ij} \in F$ and $a_{ij} \neq 0, i \neq j$ if and only if ij is an edge of G . In this definition, no restrictions are placed on the diagonal entries. The minimum rank of a graph G is defined as the minimum rank among all matrices in $\mathcal{S}^F(G)$ and is denoted as $\text{mr}^F(G)$. Much work has been done on finding the minimum rank of specific graphs. In [7] and [8], we began studying the structure of minimum rank matrices. In this paper, we continue our investigation and present results that specify when diagonal entries of minimum rank matrices must be zero, nonzero, or neither.

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Even though matrices in $\mathcal{S}^F(G)$ have no restrictions placed on the diagonal entries, for some graphs a particular diagonal entry must be zero or nonzero in order for the matrix to achieve the minimum rank of the graph. Given a vertex v of G , if every minimum rank matrix corresponding to G has a zero diagonal entry corresponding to v , v is called a nil vertex. Similarly, if every minimum rank matrix has a nonzero diagonal entry corresponding to v , v is called a nonzero vertex. If minimum rank matrices exist where the diagonal entry for v is zero and nonzero, then we call v a neutral vertex. Nil, nonzero, and neutral vertices are completely classified for graphs whose minimum rank is two in [8]. We present further results about nil, nonzero, and neutral vertices and present several methods that can determine when a vertex is nil or nonzero. We note that classifying the nil, nonzero, and neutral vertices of a graph has applications to determining the possible inertias of the graph. For example, if G is a connected graph that has a nil vertex, the inertia of G is not trapezoidal (see [5] for definition) since in a positive semi-definite matrix, the entire row and column corresponding to the nil vertex would have to be zero, contradicting the fact that G is connected.

We also present results that relate nil, nonzero, and neutral vertices to rank-spreads. In particular, we completely classify the nil, nonzero, and neutral vertices of trees by means of rank-spreads. We also offer one solution to the following question which was posed in [6]: Suppose e is an edge in a graph G and G_e is the graph obtained from G by subdividing e . When does $\text{mr}^F(G_e) = \text{mr}^F(G)$? We prove that $\text{mr}^F(G_e) = \text{mr}^F(G)$ if and only if the new vertex created by the edge subdivision is a nil vertex.

2. Definitions.

DEFINITION 2.1. If G is a graph and $e = vw \in E(G)$, *subdividing* e is the action of creating a new graph G_e from G by adding a new vertex u , and adjusting the edge set as shown:

$$G_e = (V(G) \cup \{u\}, (E(G) \setminus \{vw\}) \cup \{uv, uw\}).$$

DEFINITION 2.2.

- We denote the *complete graph* on n vertices by K_n .
- The *diamond* is the graph obtained from K_4 by deleting one edge.
- The *paw* is the graph obtained from K_4 by deleting two incident edges.
- A *double cycle* is a graph obtained through successive edge subdivisions of exterior edges of the diamond graph.
- We denote the *complete bipartite graph* with independent vertex sets of size m and size n by $K_{m,n}$; i.e., $K_{m,n} = (K_m \cup K_n)^c$, where c denotes the graph

complement.

- We denote the *complete tripartite graph* with independent vertex sets of size m , size n , and size ℓ by $K_{m,n,\ell}$; i.e., $K_{m,n,\ell} = (K_m \cup K_n \cup K_\ell)^c$.
- The *star* on $n \geq 3$ vertices, $K_{1,n-1}$, is denoted S_n .

DEFINITION 2.3. Let G and H be graphs with at least two vertices, each with a vertex labeled v . The *vertex-sum at v of G and H* is the graph on $|G| + |H| - 1$ vertices obtained by identifying the vertex v in G with the vertex v in H .

DEFINITION 2.4. A *2-tree* is a graph that can be built up from a K_2 by adding one vertex at a time adjacent to exactly the vertices in an existing K_2 . A *2-path* is a 2-tree which is either K_3 or has exactly two vertices of degree 2. A *partial 2-path* is a subgraph of a 2-path.

DEFINITION 2.5. Given a proper subgraph H of a graph G , let $\tilde{H}$ be the graph with vertex set $V(G)$ and edge set $E(H)$.

DEFINITION 2.6. For a graph G the *path cover number*, denoted $P(G)$, is the minimum number of vertex-disjoint paths, occurring as induced subgraphs of G , that cover all the vertices of G .

DEFINITION 2.7. Given a graph G on n vertices and a field F , let $\mathcal{S}^F(G)$ be the set of all symmetric $n \times n$ matrices $A = [a_{ij}]$ such that $a_{ij} \in F$ and $a_{ij} \neq 0$, $i \neq j$, if and only if ij is an edge of G . Then the *minimum rank* of G over F is

$$\text{mr}^F(G) = \min_{A \in \mathcal{S}^F(G)} \{\text{rank } A\}.$$

The *maximum nullity* of G over F is

$$M^F(G) = \max_{A \in \mathcal{S}^F(G)} \{\text{nullity } A\}.$$

DEFINITION 2.8. Given a graph G on n vertices, let $\mathcal{S}_+(G)$ be the set of all real $n \times n$ positive semidefinite matrices $A = [a_{ij}]$ such that $a_{ij} \neq 0$, $i \neq j$, if and only if ij is an edge of G . Then the *positive semidefinite minimum rank* of G is

$$\text{mr}_+(G) = \min_{A \in \mathcal{S}_+(G)} \{\text{rank } A\}.$$

The *positive semidefinite maximum nullity* of G is

$$M_+(G) = \max_{A \in \mathcal{S}_+(G)} \{\text{nullity } A\}.$$

Note that for any graph G and field F , $\text{mr}^F(G) + M^F(G) = |G|$. Similarly, $\text{mr}_+(G) + M_+(G) = |G|$.

DEFINITION 2.9. Given a graph G and a field F , let

$$\mathcal{MR}^F(G) = \{A \in \mathcal{S}^F(G) \mid 4 \text{rank } A = \text{mr}^F(G)\},$$

$$\mathcal{MR}_+(G) = \{A \in \mathcal{S}_+(G) \mid \text{rank } A = \text{mr}_+(G)\}.$$

DEFINITION 2.10. Let F be a field. The *rank-spread* of a vertex v of a graph G , denoted $r_v^F(G)$, is the difference between the minimum rank over F of G and $G - v$ (the graph obtained from G by deleting the vertex v and all edges incident to v); i.e.,

$$r_v^F(G) = \text{mr}^F(G) - \text{mr}^F(G - v).$$

REMARK 2.11. Recall (see [13],[3]) that $0 \leq r_v^F(G) \leq 2$ for any vertex v of G . Also, by taking G_2 as defined in Theorem 2.6 of [3] to be an isolated vertex we have that if v is pendant, $0 \leq r_v^F(G) \leq 1$.

Most of our results and arguments do not depend on the field F , so we often suppress it in later use of these definitions. We adopt the convention of including the F in statements of theorems to emphasize field independence while excluding it from proofs unless the particular field is important.

DEFINITION 2.12. Given a field F and a graph G , a vertex v in G is a

- *nil vertex* if its corresponding diagonal entry d_v is zero in every matrix in $\mathcal{MR}^F(G)$.
- *nonzero vertex* if its corresponding diagonal entry d_v is nonzero in every matrix in $\mathcal{MR}^F(G)$.
- *neutral vertex* if it is neither a nil vertex nor a nonzero vertex.

It is somewhat unexpected that a particular diagonal entry must be zero in order to achieve minimum rank. The same can be said of nonzero vertices. Therefore, we provide the following examples.

EXAMPLE 2.13. Let F be a field and let S_4 be the star on 4 vertices with $V = \{1, 2, 3, 4\}$ and $E = \{12, 13, 14\}$. Every matrix in $\mathcal{MR}^F(S_4)$ is of the form

$$A = \begin{bmatrix} d_1 & a & b & c \\ a & d_2 & 0 & 0 \\ b & 0 & d_3 & 0 \\ c & 0 & 0 & d_4 \end{bmatrix},$$

where a, b and c are not zero and d_2, d_3, d_4 correspond to the pendant vertices of S_4 . Since $\text{mr}^F(S_4) = 2$, $\text{rank } A = 2$. If any of d_2, d_3 , or d_4 is not 0, then $\text{rank } A$ is greater than 2. Hence, every pendant vertex of S_4 is a nil vertex.

Further, both $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ are in $\mathcal{MR}^F(S_4)$, and thus,

vertex 1 is a neutral vertex.

Similarly, any star S_n with $n \geq 4$ has the property that all its pendant vertices are nil and its central vertex is neutral.

EXAMPLE 2.14. Let F be a field and consider $K_n, n \geq 2$. Let $A \in \mathcal{MR}^F(K_n)$. Since $\text{mr}^F(K_n) = 1$, $\text{rank } A = 1$. If any diagonal entry of A were zero, then the rank of A would be at least two. Therefore, every diagonal entry of A is nonzero, and thus, every vertex of K_n is a nonzero vertex. We note that this classifies the vertices of connected minimum rank one graphs since every such graph is a complete graph on two or more vertices.

DEFINITION 2.15. Let G be a graph. A *cover* of G is a set of subgraphs of G such that the union of the edge sets is equal to $E(G)$.

REMARK 2.16. In this paper we only use covers for which each pair of subgraphs in the cover is edge-disjoint.

DEFINITION 2.17. A *K_2 -star cover* of G is a cover of G consisting of only K_2 's and stars.

DEFINITION 2.18. Given a graph G and a cover $\mathcal{C}$ of G , we say a vertex (edge) of G is *covered by* an element of the cover $H \in \mathcal{C}$ if the vertex (edge) is in the vertex (edge) set of H .

DEFINITION 2.19. The *rank sum* of a cover $\mathcal{C}$ over a field F , denoted $\text{rs}^F(\mathcal{C})$, is the sum of the minimum ranks over F of the graphs in $\mathcal{C}$.

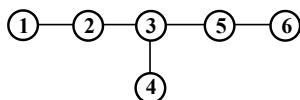
REMARK 2.20. Given a field F and a graph G , the rank sum of any edge-disjoint cover $\mathcal{C} = \{H_1, \dots, H_m\}$ of G is an upper bound on $\text{mr}^F(G)$. To see this, let $\widetilde{A}_i \in \mathcal{MR}^F(\widetilde{H}_i)$ for $i = 1, \dots, m$ and let $A = \widetilde{A}_1 + c \cdots + \widetilde{A}_m$. Then since $\mathcal{C}$ is edge-disjoint, $A \in \mathcal{S}^F(G)$ and

$$\text{mr}^F(G) \leq \text{rank } A \leq \sum_{i=1}^m \text{rank}(\widetilde{A}_i) = \sum_{i=1}^m \text{mr}^F(\widetilde{H}_i) = \sum_{i=1}^m \text{mr}^F(H_i) = \text{rs}^F(\mathcal{C}).$$

DEFINITION 2.21. A *minimum rank cover* of a graph G over a field F is a cover

$\mathcal{C}$ of G such that $\text{rs}^F(\mathcal{C}) = \text{mr}^F(G)$.

EXAMPLE 2.22. Let G be the following graph:



The minimum rank of G over any field F is 4. Consider the cover of G consisting of the two K_2 's formed by vertices 1 and 2 and vertices 5 and 6, which we denote as H_1 and H_2 respectively, and the star consisting of vertices 2, 3, 4 and 5, which we denote as S . We construct a matrix in $\mathcal{S}^F(G)$ by taking a matrix from each of $\mathcal{MR}^F(\widetilde{H}_1)$, $\mathcal{MR}^F(\widetilde{H}_2)$, and $\mathcal{MR}^F(\widetilde{S})$ and summing them together. For example, we can take the matrices

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathcal{MR}^F(\widetilde{H}_1), \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \in \mathcal{MR}^F(\widetilde{H}_2),$$

$$\text{and} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathcal{MR}^F(\widetilde{S}).$$

Then $\text{rs}^F(\{H_1, H_2, S\}) = \text{mr}^F(H_1) + \text{mr}^F(H_2) + \text{mr}^F(S) = 1 + 1 + 2 = \text{mr}^F(G)$, so $\{H_1, H_2, S\}$ is a minimum rank cover of G and the sum of these three matrices belongs to $\mathcal{MR}^F(G)$.

3. Previous results. The following theorem was published by Johnson and Duarte in [10]. Van der Holst proved a field independent version in Theorem 8 of [15].

THEOREM 3.1. *If a graph T is a tree then $P(T) = M^F(T)$.*

The following theorem was published by Hsieh in [9], and independently by Bar-ioli, Fallat, and Hogben (see Theorem 2.3 in [3]). Van der Holst in [15] and Barrett, Grout, and Loewy in [4] proved field independent versions.

THEOREM 3.2. *Let G_1 and G_2 be graphs on at least two vertices each with a vertex labeled v and let G be the vertex-sum at v of G_1 and G_2 . Let F be any field.*

Then

$$\text{mr}^F(G) = \min\{\text{mr}^F(G_1) + \text{mr}^F(G_2), \text{mr}^F(G_1 - v) + \text{mr}^F(G_2 - v) + 2\}.$$

Equivalently,

$$r_v^F(G) = \min\{r_v^F(G_1) + r_v^F(G_2), 2\}.$$

The following is Lemma 8 from [6], which is a field independent version of a lemma due to Johnson, Loewy, and Smith in [11].

LEMMA 3.3. *Let F be any field, let G be any graph, and let e be an edge of G . Then $\text{mr}^F(G) \leq \text{mr}^F(G_e) \leq \text{mr}^F(G) + 1$.*

The following is Theorem 4.1 of [16].

THEOREM 3.4. *Let G be a graph. Then $M_+(G) \leq 1$ if and only if G is a tree.*

The following result appears in [7].

THEOREM 3.5. *Let F be any field and let G be the vertex-sum at v of G_1 and G_2 , and let S_{k+1} be the star subgraph of G formed by the degree k vertex v and all of its neighbors.*

1. *If $r_v^F(G_1) + r_v^F(G_2) < 2$, then*

$$\mathcal{MR}^F(G) = \mathcal{MR}^F(\widetilde{G_1}) + \mathcal{MR}^F(\widetilde{G_2}).$$

2. *If $r_v^F(G_1) + r_v^F(G_2) > 2$, then*

$$\mathcal{MR}^F(G) = \mathcal{MR}^F(\widetilde{G_1 - v}) + \mathcal{MR}^F(\widetilde{G_2 - v}) + \mathcal{MR}^F(\widetilde{S_{k+1}}).$$

3. *If $r_v^F(G_1) + r_v^F(G_2) = 2$, then*

$$\begin{aligned} \mathcal{MR}^F(G) = & \left(\mathcal{MR}^F(\widetilde{G_1}) + \mathcal{MR}^F(\widetilde{G_2}) \right) \\ & \cup \left(\mathcal{MR}^F(\widetilde{G_1 - v}) + \mathcal{MR}^F(\widetilde{G_2 - v}) + \mathcal{MR}^F(\widetilde{S_{k+1}}) \right). \end{aligned}$$

The following two results are Theorems 5.2 and 5.6 in [8], respectively, and represent what has been previously known about nil, nonzero, and neutral vertices. The former shows that nil and nonzero vertices ascend to G from induced subgraphs with the same minimum rank. The latter completely classifies nil, nonzero and neutral vertices for graphs with minimum rank equal to 2 over the real field. In the sections following we classify the nil, nonzero and neutral vertices of some other families of

graphs and present various techniques for identifying these vertices in more general graphs.

THEOREM 3.6. *Let F be a field. If H is an induced subgraph of a graph G with $\text{mr}^F(H) = \text{mr}^F(G)$ and v is a nil (nonzero) vertex in H , then v is also a nil (nonzero) vertex in G .*

Consequently, if v is neutral in G , then v is neutral in H .

THEOREM 3.7. *Let G be a connected graph with $\text{mr}^{\mathbb{R}}(G) = 2$ and v be a vertex of G . Then*

- *v is a nonzero vertex if and only if v is either a non-dominating vertex of an induced paw of G or else is the dominating vertex of an induced $K_{3,3,1}$.*
- *v is a nil vertex if and only if v is in an independent set of size three or greater.*
- *v is a neutral vertex if and only if it does not meet either of the previous two conditions.*

REMARK 3.8. The minimum rank of the diamond over any field is 2. Over the real field, Theorem 3.7 implies that the degree 2 vertices of the diamond are neutral. However, over the field of two elements, the vertices of degree 2 in the diamond are nil vertices. This is seen by assuming to the contrary that an entry corresponding to a degree 2 vertex in a minimum rank matrix A is nonzero. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & a & 1 & 1 \\ 1 & 1 & b & 1 \\ 0 & 1 & 1 & c \end{bmatrix} \in \mathcal{S}^{F_2}(\text{diamond}).$$

Considering the 3×3 submatrix obtained by deleting the second row and third column

$$A(2,3) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & c \end{bmatrix},$$

we note $\det A(2,3) = 1$, so that A cannot be a minimum rank matrix for the diamond.

We now show the diamond's degree 3 vertices are neutral over F_2 . Let

$$B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \in \mathcal{S}^{F_2}(\text{diamond}).$$

If $a = 0$ or 1 , $\text{rank } B = 2 = \text{mr}^{F_2}(\text{diamond})$.

4. Edge subdivision. One of the questions posed in [6] asked if it were possible to characterize when the minimum rank increases after subdividing an edge. Here we show that the minimum rank stays the same after subdividing an edge if and only if the newly created vertex is a nil vertex.

THEOREM 4.1. *Let F be a field, G be a connected graph, and G_e be the graph obtained from G by subdividing an edge $e \in E(G)$. Let v be the vertex created by subdividing the edge e . Then $\text{mr}^F(G_e) = \text{mr}^F(G)$ if and only if v is a nil vertex.*

Proof. We prove the contrapositive of the reverse implication. Let

$$A = \begin{bmatrix} d_1 & a & x^T \\ a & d_2 & y^T \\ x & y & B \end{bmatrix} \in \mathcal{S}(G)$$

and $\text{mr}(G) = \text{rank } A$. The vertices corresponding to e have been labeled one and two, and thus, $a \neq 0$. In the following matrix, A_e , the labeling of the vertices of G has been increased by one, and the new vertex v created by the subdivision of e has been labeled one.

$$\text{Then, } A_e = \begin{bmatrix} 0 & 0 & 0 & 0^T \\ 0 & d_1 & a & x^T \\ 0 & a & d_2 & y^T \\ 0 & x & y & B \end{bmatrix} - \begin{bmatrix} a & a & a & 0^T \\ a & a & a & 0^T \\ a & a & a & 0^T \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathcal{S}(G_e). \quad \text{Now } \text{mr}(G_e) \leq$$

$\text{rank } A_e \leq \text{rank } A + 1 = \text{mr}(G) + 1$. By hypothesis, $\text{mr}(G_e) \neq \text{mr}(G)$ so by Lemma 3.3, $\text{mr}(G_e) = \text{mr}(G) + 1$. Thus, $\text{rank } A_e = \text{mr}(G_e)$. Since $a \neq 0$, v is not a nil vertex.

We now prove the contrapositive of the forward implication. Suppose v is not a nil vertex so that there exists a matrix $C \in \mathcal{S}(G_e)$ such that the diagonal entry of C corresponding to v is nonzero and $\text{rank } C = \text{mr}(G_e)$. Without loss of generality, label v one and the neighbors of v two and three. Then we have

$$C = \begin{bmatrix} d_1 & a & b & 0^T \\ a & d_2 & 0 & x^T \\ b & 0 & d_3 & y^T \\ 0 & x & y & D \end{bmatrix}$$

with $a, b, d_1 \neq 0$. Since $d_1 \neq 0$, we can consider the Schur complement $C/[d_1]$.

$$C/[d_1] = \begin{bmatrix} d_2 & 0 & x^T \\ 0 & d_3 & y^T \\ x & y & D \end{bmatrix} - \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \frac{1}{d_1} \begin{bmatrix} a & b & 0^T \end{bmatrix} = \begin{bmatrix} d_4 & c & x^T \\ c & d_5 & y^T \\ x & y & D \end{bmatrix},$$

where $d_4 = d_2 - \frac{a^2}{d_1}$, $d_5 = d_3 - \frac{b^2}{d_1}$, and $c = -\frac{ab}{d_1}$. Note that $C/[d_1] \in \mathcal{S}(G)$, since

$c \neq 0$. Since Schur complements are nullity preserving, $\text{rank } C/[d_1] + 1 = \text{rank } C$. So

$$\text{mr}(G) + 1 \leq \text{rank } C/[d_1] + 1 = \text{rank } C = \text{mr}(G_e).$$

Thus, $\text{mr}(G) < \text{mr}(G_e)$ which completes the proof. $\square$

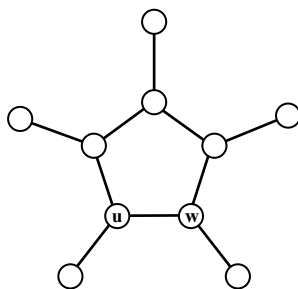
5. Rank-spread. We now investigate the relationship between rank-spread and nil, nonzero and neutral vertices. For a tree it will be shown that the vertices which are nil are exactly those that have rank-spread zero. We begin with some results that hold for graphs in general.

LEMMA 5.1. *Let F be a field and v be a vertex of a graph G . If $r_v^F(G) = 2$, then v is neutral in G .*

Proof. Assume the degree of v is k . By Remark 2.11, $k \geq 2$. Recall that the central vertex of a star is neutral. Choose $B \in \mathcal{MR}(\widehat{S_{k+1}})$ and $C \in \mathcal{MR}(\widehat{G-v})$ such that $B + C = A \in \mathcal{S}(G)$. Notice that $\text{mr}(G) \leq \text{rank } A \leq \text{rank } B + \text{rank } C = 2 + \text{mr}(G-v) = \text{mr}(G)$. Therefore, $A \in \mathcal{MR}(G)$. Since the central vertex of a star is neutral, we may choose B such that the diagonal entry corresponding to d_v in A is either zero or nonzero. We conclude that v is neutral in G . $\square$

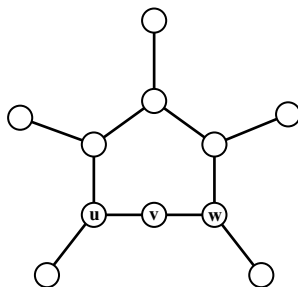
Thus, we see that a nil vertex or nonzero vertex must either be rank-spread 0 or 1. The vertices of the graph K_2 are examples of rank-spread 1 vertices which are nonzero, while the vertices of K_n for $n \geq 3$ are examples of rank-spread 0 vertices which are nonzero. The pendant vertices of S_n for $n \geq 4$ are all vertices which have rank-spread 0 and are nil. We also point out that although rank-spread 2 implies a vertex is neutral, the converse is not true. The pendant vertices of P_3 are neutral with rank-spread 1 and the vertices of degree 3 in $K_{2,3}$ are neutral with rank-spread 0.

Finding an example of a nil vertex with rank-spread 1 is not as simple. Let H_5 be the 5-sun:



It is well known that $\text{mr}^F(H_5) = 8$ for any field F . Let G be the graph obtained by

subdividing the edge uw :



To determine $\text{mr}^F(G)$, we consider the graph $G - u$. The graph $G - u$ is the union of a tree T on 9 vertices and an isolated vertex. The minimum rank of an isolated vertex is zero, and thus, $\text{mr}^F(G - u) = \text{mr}^F(T)$. Since T has 5 pendant vertices, $P(T) \geq 3$. By Theorem 3.1, $\text{mr}^F(T) = 9 - M^F(T) = 9 - P(T) \leq 6$. As noted after Definition 2.10 $\text{mr}^F(G) - \text{mr}^F(G - u) \leq 2$ for all u . Thus, $\text{mr}^F(G) \leq \text{mr}^F(G - u) + 2 = \text{mr}^F(T) + 2 \leq 8$. By Lemma 3.3, $8 = \text{mr}^F(H_5) \leq \text{mr}^F(G) \leq 8$. Thus, $\text{mr}^F(G) = 8$. By Theorem 4.1, v is a nil vertex of G . The graph $G - v$ is a tree on 10 vertices, 5 of which are pendant. Thus, $P(G - v) \geq 3$. Note that a path cover of size three exists and so $P(G - v) = 3$. By Theorem 3.1, $\text{mr}^F(G - v) = 10 - M^F(G - v) = 10 - P(G - v) = 7$ so $r_v^F(G) = 1$.

We now examine several situations where the existence of a nil vertex implies the rank-spread of said vertex is 0.

A vertex v is *simplicial* if its neighborhood $N(v)$ induces a complete subgraph.

LEMMA 5.2. *Let v be a simplicial vertex of a graph G and let F be an infinite field. If v is a nil vertex, then $r_v^F(G) = 0$.*

Proof. We prove the contrapositive, that if $r_v^F(G) \geq 1$, v is not a nil vertex. Let $A \in \mathcal{MR}^F(\widetilde{G - v})$. Since v is simplicial its neighborhood induces a complete graph. Let $H = K_m$ be the complete graph induced by v and its neighbors. Let B be the matrix formed by appropriately embedding the $m \times m$ all ones matrix into a $|G| \times |G|$ matrix so that $B \in \mathcal{MR}^F(\widetilde{H})$. Since F is infinite there exists a $k \in F^\times$ such that $kB + A \in \mathcal{S}^F(G)$. Using the fact that $r_v^F(G) \geq 1$, $\text{mr}^F(G) \leq \text{rank}(kB + A) \leq \text{rank } A + \text{rank}(kB) = \text{rank } A + 1 = \text{mr}^F(G - v) + 1 \leq \text{mr}^F(G)$. Thus, $kB + A$ is in $\mathcal{MR}^F(G)$. Since the diagonal entry of A corresponding to vertex v is zero and the diagonal entry of kB corresponding to v is k , v is not a nil vertex of G . $\square$

REMARK 5.3. Although in Lemma 5.2 we require F to be infinite, the field need only have a sufficient number of elements to ensure the existence of the k in the proof. The minimum size of the field is determined by the order of the complete graph. From Remark 3.8, a degree 2 vertex v in a diamond is simplicial and over F_2

is nil but $r_v^{F_2}(\text{diamond}) = 1$.

From the proof of Lemma 5.2, we have:

COROLLARY 5.4. *Let p be a pendant vertex of a graph G and F a field. If p is a nil vertex, then $r_p^F(G) = 0$*

COROLLARY 5.5. *Let v be a simplicial vertex of degree 2 and $F \neq F_2$ a field. If v is a nil vertex, then $r_v^F(G) = 0$.*

LEMMA 5.6. *Let p be a pendant vertex of a graph G and F a field. Then $r_p^F(G) = 0$ if and only if p is a nil vertex.*

Proof. By Corollary 5.4, we need only prove the contrapositive of the forward implication. Suppose that p is not a nil vertex. Label vertex p as 1 and its neighbor

as 2. Then there exists a matrix $A \in \mathcal{MR}(G)$ of the form $\begin{bmatrix} d_1 & a & 0 \\ a & d_2 & x^T \\ 0 & x & C \end{bmatrix}$, where $d_1, a \neq 0$. Since $d_1 \neq 0$ we may consider the Schur complement

$$A/[d_1] = \begin{bmatrix} d_2 & x^T \\ x & C \end{bmatrix} - \begin{bmatrix} a \\ 0 \end{bmatrix} \frac{1}{d_1} \begin{bmatrix} a & 0 \end{bmatrix} = \begin{bmatrix} d_3 & x^T \\ x & C \end{bmatrix} \text{ where } d_3 = d_2 - \frac{a^2}{d_1}.$$

Thus, $\text{rank } A = \text{rank } A/[d_1] + 1$. Since $A/[d_1] \in \mathcal{S}(G - p)$, $\text{mr}(G) = \text{rank } A = \text{rank } A/[d_1] + 1 \geq \text{mr}(G - p) + 1$ and $r_p(G) \geq 1$. $\square$

LEMMA 5.7. *Let F be a field and G be the vertex-sum at v of $G_1, G_2, \dots, G_k$. If v is a nil vertex in each G_i and $r_v^{F_i}(G_i) \leq 1$, then v is a nil vertex in G .*

Proof. We proceed by induction on k , the number of graphs in the vertex-sum.

Let $k = 2$. Since $r_v(G) \leq 1$, Theorems 3.2 and 3.5 imply that, $\mathcal{MR}(G) = \mathcal{MR}(\widetilde{G_1}) + \mathcal{MR}(\widetilde{G_2})$ so any $A \in \mathcal{MR}(G)$ can be written as a sum of matrices $A_i \in \mathcal{MR}(\widetilde{G_i})$. Since v is a nil vertex in each G_i , then the diagonal entry of A_i corresponding to v must be zero for both i . Thus, the diagonal entry of A corresponding to v must be zero as well. Therefore, v is nil in G .

Assume the claim is true for $k \leq m$ and let G be the vertex-sum at v of $m + 1$ graphs. Let H be the vertex-sum at v of $G_1, \dots, G_m$. By Theorem 3.2, $r_v(G) \geq r_v(H)$. Thus, $r_v(H) \leq 1$. By the inductive hypothesis v is nil in H . Now G is the vertex-sum at v of H and G_{m+1} . By the case $k = 2$, v is nil in G . $\square$

THEOREM 5.8. *Let G be a graph and F a field. Let v be a vertex of G which is not in any cycle of G . Then v is a nil vertex of G if and only if $r_v^F(G) = 0$.*

Proof. Since v does not belong to any cycle of G , v is either a pendant vertex or a cut-vertex. If v is pendant, then by Lemma 5.6, $r_v(G) = 0$ if and only if it is a nil

vertex.

If v is a cut-vertex not on a cycle then G is the vertex-sum at v of $G_1, \dots, G_k$ such that v is pendant in each G_i . By Theorem 3.2, if $r_v(G) = 0$ then $r_v(G_i) = 0$ for all i . Thus, by Lemma 5.6, v is a nil vertex in each G_i . Then by Lemma 5.7, v is nil in G . Hence, $r_v(G) = 0$ implies that v is nil in G .

So it remains to show that if v is nil in G , then $r_v(G) = 0$. By Lemma 5.1, $r_v(G) \neq 2$. Suppose by way of contradiction that $r_v(G) = 1$. By Theorem 3.2, there exists exactly one G_i such that $r_v(G_i) = 1$. Renaming if necessary let $r_v(G_1) = 1$. Let H be the vertex-sum at v of $G_2, \dots, G_k$. Thus, G is the vertex-sum at v of G_1 and H where $r_v(G_1) = 1$ and $r_v(H) = 0$. Since $r_v(H) = 0$, by Lemma 5.7 v is nil in H . Since v is pendant in G_1 and $r_v(G_1) = 1$, by Lemma 5.6, v is not nil. Thus, there exists $A \in \mathcal{MR}(\widetilde{G_1})$ such that $a_{vv} \neq 0$. Also there exists $B \in \mathcal{MR}(\widetilde{H})$ such that $b_{vv} = 0$. Then $A + B \in \mathcal{S}(G)$. Since $r_v(G) = 1$, $\text{mr}(G) = \text{mr}(G_1) + \text{mr}(H)$. Thus, $\text{mr}(G) \leq \text{rank}(A + B) \leq \text{rank } A + \text{rank } B = \text{mr}(G_1) + \text{mr}(H) = \text{mr}(G)$. Thus, $A + B \in \mathcal{MR}(G)$, and the diagonal entry corresponding to v is equal to $a_{vv} + b_{vv} \neq 0$. Thus, v is not nil, a contradiction. $\square$

COROLLARY 5.9. *Let F be a field, T be a tree, and v be a vertex of T . Then $r_v^F(T) = 0$ if and only if v is a nil vertex.*

Proof. Since T is a tree, it is acyclic. Thus, the result follows by Theorem 5.8. $\square$

Rank-spread zero vertices in graphs other than trees do not necessarily have to be nil vertices as we have seen with complete graphs with as least three vertices. The following theorem and corollary somewhat generalize the graph structure sufficient to determine whether a rank-spread zero vertex is nil. We note that the forward implication of Corollary 5.9 follows from Theorem 5.10.

THEOREM 5.10. *Let F be a field. Let G be a graph with $\text{mr}^F(G) = r$ and let H be a proper induced subgraph of G with $\text{mr}^F(H) = r$. Let $v \notin V(H)$ be a vertex adjacent to exactly one vertex of each component of H . Then v is a nil vertex.*

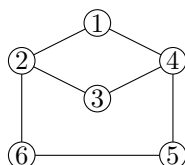
Proof. Let $H_1, H_2, \dots, H_k$ denote the disjoint components of H . Let H_v be the induced subgraph containing H and v . Similarly, let $H_{iv}, i = 1, 2, \dots, k$ denote the induced subgraph of H_v containing H_i and v . Since $\text{mr}(H) = \text{mr}(G)$ it follows that $\text{mr}(H_v) = \text{mr}(H)$, and thus, $r_v(H_v) = 0$. Since v is adjacent to exactly one vertex of each component of H , H_v is a vertex-sum of $H_{iv}, i = 1, 2, \dots, k$. It follows from Theorem 3.2 that $0 = r_v(H_v) = \min\{(r_v(H_{1v}) + r_v(H_{2v}) + \dots + r_v(H_{kv})), 2\}$. Therefore, $r_v(H_{iv}) = 0, i = 1, 2, \dots, k$. We note that v is pendant in each H_{iv} and thus by Lemma 5.6 v is a nil vertex in each H_{iv} . Now by Lemma 5.7 v is a nil vertex

in H_v , and thus, by Theorem 3.6, v is a nil vertex in G . $\square$

COROLLARY 5.11. *Let F be a field. Let G be a graph with $\text{mr}^F(G) = r$ and let H be a connected induced subgraph of G with $\text{mr}^F(H) = r$. Let $v \notin V(H)$ be a vertex adjacent to exactly one vertex of H . Then v is a nil vertex.*

Proof. This is a special case of Theorem 5.10. $\square$

EXAMPLE 5.12. Let G be the following graph:



We show that vertex 3 is a nil vertex using Corollary 5.11. Note G is a subdivision of $K_{2,3}$ which has minimum rank 2 over every field F . By Lemma 3.3, $\text{mr}^F(G) \leq 3$. Since P_4 is induced and $\text{mr}^F(P_4) = 3$, $\text{mr}^F(G) \geq 3$. Thus, $\text{mr}^F(G) = 3$. Let H be the path induced by vertex set $\{1, 4, 5, 6\}$. Then $\text{mr}^F(G) = 3 = \text{mr}^F(P_4) = \text{mr}^F(H)$. Vertex 3 is not in H , is adjacent to exactly one vertex of H , and thus is nil.

Another method which can be used to determine both nil and nonzero vertices uses the graph parameter $\hat{Z}$. This is the subject of the following section.

6. The $\hat{Z}$ method. We now introduce two graph parameters, the zero forcing number Z and the enhanced zero forcing number $\hat{Z}$, which we use to develop another method to determine nil and nonzero vertices. The parameter Z first appeared in [1] and was used to put an upper bound on the maximum nullity of a graph. The parameter $\hat{Z}$, which appears in [2] is a modification of Z and is also used to put an upper bound on the maximum nullity of a graph. The following definitions from [1] and [2] define Z and $\hat{Z}$.

DEFINITION 6.1.

- *Color-change rule for a simple graph:* If G is a graph with each vertex colored either white or black, u is a black vertex of G , and exactly one neighbor v of u is white, then change the color of v to black.
- Given a coloring of G , the *derived coloring* is the result of applying the color-change rule for a simple graph until no more changes are possible.
- A *zero forcing set* for a graph G is a subset of vertices Z such that if initially the vertices in Z are colored black and the remaining vertices are colored white, the derived coloring of G is all black.

• The *zero forcing number* of a graph G , $Z(G)$ is the minimum of $|Z|$ over all zero forcing sets $Z \subset V(G)$.

The following two results are Propositions 2.4 and 4.2 in [1].

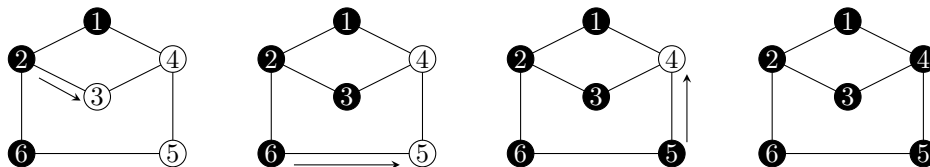
THEOREM 6.2. For any graph G and any field F , $M^F(G) \leq Z(G)$.

THEOREM 6.3. For any tree T and any field F , $M^F(T) = Z(T)$.

REMARK 6.4. It has been verified that $M^{\mathbb{R}}(G) = Z(G)$ for all graphs G on fewer than 7 vertices.

We briefly review how to determine Z for a graph.

EXAMPLE 6.5. Consider the graph G from Example 5.12. We show that one zero forcing set consists of vertices 1, 2 and 6. We first color 1, 2 and 6 black (see the illustration below). Since 2 has exactly one white neighbor, 3, it can force 3 black by the color-change rule for a simple graph. Since 6 has exactly one white neighbor, 5, it can force 5 black. Lastly since 5 has exactly one white neighbor, 4, it can force 4 black.



We note that the zero forcing set 1, 2 and 6 is not unique nor is the order in which we forced vertices black in the above example. Also, there are no two vertices in G that if colored black can force the rest of the graph black. Thus, $Z(G) = 3$.

DEFINITION 6.6. A *loop graph* is a graph that allows single loops at vertices, i.e., $\widehat{G} = (V_{\widehat{G}}, E_{\widehat{G}})$ where $V_{\widehat{G}}$ is the set of vertices of $\widehat{G}$ and the set of edges $E_{\widehat{G}}$ is a set of two-element multisets. Vertex u is a *neighbor* of vertex v in $\widehat{G}$ if $\{u, v\} \in E_{\widehat{G}}$; note that u is a neighbor of itself if and only if the loop $\{u, u\}$ is an edge. The *underlying simple graph* of a loop graph $\widehat{G}$ is the graph G obtained from $\widehat{G}$ by deleting all loops.

Note that if we ever write $\widehat{G}$ we think of the graph as coming with extra information, namely that the graph is a loop graph, even if there are no loops. In a loop graph, every vertex is specified as being looped or unlooped.

DEFINITION 6.7. The *set of symmetric matrices with entries in a field F described by a loop graph $\widehat{G}$* is

$$\mathcal{S}^F(\widehat{G}) = \{A = [a_{ij}] \mid A^T = A, a_{ij} \in F, \text{ and } a_{ij} \neq 0 \text{ if and only if } \{i, j\} \in E_{\widehat{G}}\}.$$

and the *maximum nullity* of $\widehat{G}$ over F is

$$M^F(\widehat{G}) = \max\{\text{nullity } A \mid A \in \mathcal{S}^F(\widehat{G})\}$$

DEFINITION 6.8 (Color-change rule for a loop graph). Let $\widehat{G}$ be a loop graph with each vertex colored white or black. If exactly one neighbor u of v is white, then change the color of u to black.

Note that the color-change rule for a loop graph is quite similar to the color-change rule for a simple graph. The only difference is that when using a loop graph, two additional coloring forces are valid. First, a looped white vertex that has no other white neighbors may be colored black. Second if an unlooped white vertex has only one white neighbor u , u may be colored black. By $Z(\widehat{G})$, we mean the same thing as in Definition 6.1, except we use the color-change rule for a loop graph. (We distinguish the two cases by whether or not the graph is a loop graph.)

The following results are from [2]. We note that the results were stated without reference to a field but the proofs hold for any field.

THEOREM 6.9. For any loop graph $\widehat{G}$ and any field F , $M^F(\widehat{G}) \leq Z(\widehat{G})$.

DEFINITION 6.10. The *enhanced zero forcing number* of a graph G denoted by $\widehat{Z}(G)$, is the maximum of $Z(\widehat{G})$ over all loop graphs $\widehat{G}$ such that the underlying simple graph of $\widehat{G}$ is G .

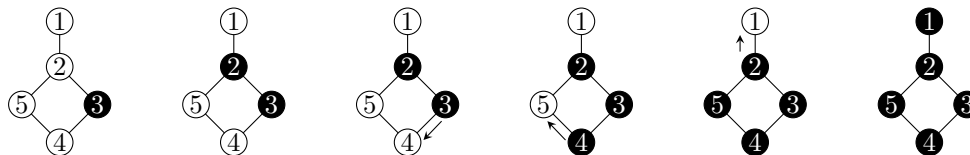
COROLLARY 6.11. For any graph G and any field F , $M^F(G) \leq \widehat{Z}(G) \leq Z(G)$.

The following example illustrates the coloring rules defined above.

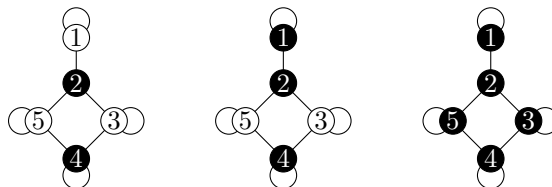
EXAMPLE 6.12. Let $\widehat{G}_1$ and $\widehat{G}_2$ be the following loop graphs:



First consider $\widehat{G}_1$. Color vertex 3 black (see illustration below). Since 1 is an unlooped vertex and only has one white neighbor 2, 2 can be colored black. Then 3 forces 4, 4 forces 5, and 2 forces 1. Thus, $Z(\widehat{G}_1) \leq 1$. It is straightforward to see that $Z(\widehat{G}_1) \geq 1$. Hence, $Z(\widehat{G}_1) = 1$.



Now consider $\widehat{G}_2$. Color vertices 2 and 4 black (see illustration below). Since 1 is a looped vertex that has no white neighbors, it may be colored black. Similarly, 3 and 5 may be colored black. Thus, $Z(\widehat{G}_2) \leq 2$. It is straightforward to see that $Z(\widehat{G}_2) \geq 2$. Hence, $Z(\widehat{G}_2) = 2$.



Since $\widehat{Z}(G)$ is the maximum of $Z(\widehat{G})$ over all loop graphs $\widehat{G}$, $\widehat{Z}(G) \geq Z(\widehat{G}_2) = 2$. It is straightforward to verify $Z(G) = 2$. By Corollary 6.11, $\widehat{Z}(G) \leq Z(G) = 2$, and thus, $\widehat{Z}(G) = 2$.

The ideas and results that have been given for the zero forcing number and the enhanced zero forcing number can be combined in a way that can determine nil and nonzero vertices.

THEOREM 6.13 (The $\widehat{Z}$ Method: Nil). *Let G be a graph, v be a vertex of G , and F a field. Let $\mathcal{G}_v$ be the set of all loop graphs with underlying simple graph G such that v is looped. If $Z(\widehat{G}) < M^F(G)$ for all $\widehat{G} \in \mathcal{G}_v$ then v is a nil vertex.*

Proof. Let $\widehat{G}$ be an arbitrary graph in $\mathcal{G}_v$. By hypothesis, $Z(\widehat{G}) < M(G)$ and thus by Theorem 6.9, $M(\widehat{G}) \leq Z(\widehat{G}) < M(G)$. Thus, no matrix in $\mathcal{S}(\widehat{G})$ has nullity equal to $M(G)$. Note that by the definition of $\mathcal{S}(\widehat{G})$, the condition that v is looped corresponds to the condition that the diagonal entry corresponding to v in every matrix in $\mathcal{S}(\widehat{G})$ is nonzero. Since $\widehat{G}$ was an arbitrary loop graph with the condition that v is looped, no matrix with a nonzero diagonal entry corresponding to v achieves $M(G)$ (or equivalently $\text{mr}(G)$). Therefore v is a nil vertex. $\square$

THEOREM 6.14 (The $\widehat{Z}$ Method: Nonzero). *Let G be a graph, v be a vertex of G , and F a field. Let $\mathcal{G}_v$ be the set of all loop graphs with underlying simple graph G such that v is unlooped. If $Z(\widehat{G}) < M^F(G)$ for all $\widehat{G} \in \mathcal{G}_v$ then v is a nonzero vertex.*

Proof. The proof is similar to the proof of Theorem 6.13. $\square$

For some graphs the following corollaries can be used to determine nil and nonzero vertices. Note that they reduce the number of loop graphs to consider to one.

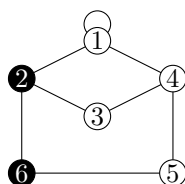
COROLLARY 6.15. *Let G be a graph. Let $\hat{G}$ be the graph G where a vertex v is looped and no other vertices are specified looped or unlooped. Let F be a field. If there exists a set of less than $M^F(G)$ vertices of $\hat{G}$ such that starting with these vertices colored black, every vertex in $\hat{G}$ can be colored black by following the color-change rule for a simple graph (see Definition 6.1) and the additional rule that the looped vertex v may be colored black if it has no white neighbors, then v is a nil vertex of G .*

Proof. Let $\hat{G}$ be an arbitrary loop graph with underlying simple graph G such that v has a loop. Since there exists a set of less than $M(G)$ vertices such that starting with these vertices colored black, every vertex in $\hat{G}$ can be colored black by following the color-change rule for a simple graph and the rule that the looped vertex v may be colored black if it has no white neighbors, these same forcing moves will color every vertex of $\hat{G}$ black. Thus, $Z(\hat{G}) \leq |Z| < M(G)$. By Theorem 6.13, v is a nil vertex. $\square$

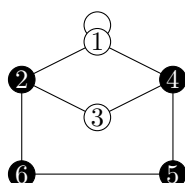
COROLLARY 6.16. *Let G be a graph. Let $\hat{G}$ be the graph G where a vertex v is unlooped and no other vertices are specified looped or unlooped. Let F be a field. If there exists a set of less than $M^F(G)$ vertices of $\hat{G}$ such that starting with these vertices colored black, every vertex in $\hat{G}$ can be colored black by following the color-change rule for a simple graph and the additional rule that if v has only one white neighbor u , u may be colored black, then v is a nonzero vertex of G .*

Proof. The proof is similar to the proof of Corollary 6.15. $\square$

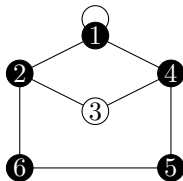
EXAMPLE 6.17. We consider the graph G used in Examples 5.12 and 6.5. We will show that vertices 1 and 3 are nil and vertices 5 and 6 are nonzero. From Example 5.12, $M^F(G) = 3$ for any field F . We begin by placing a loop on 1 and coloring 2 and 6 black.



Using the color-change rule for a simple graph, 6 can force 5 and then 5 can force 4.

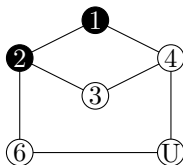


Now the looped vertex has no white neighbors and thus can be colored black.

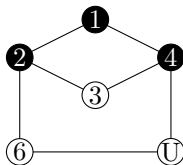


Lastly, 4 can force 3. We used a set of fewer than $M(G) = 3$ vertices to color the entire graph black. Hence, by Corollary 6.15, vertex 1 is a nil vertex. By symmetry, vertex 3 is a nil vertex.

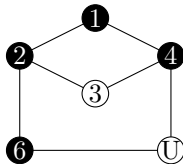
We now mark 5 as an unlooped vertex by labeling it with a U . We also color 1 and 2 black.



Using the color-change rule for a simple graph 1 can force 4.



Now the unlooped white vertex has only one white neighbor 6, and thus, 6 may be colored black.



Now by the color-change rule for a simple graph, 6 can force 5 black and 2 can force 3 black. We used a set of less than 3 vertices to color the entire graph black. Hence, by Corollary 6.16, vertex 5 is nonzero. By symmetry, vertex 6 is nonzero.

The above method of using the concepts of zero forcing and enhanced zero forcing can also be used to make conclusions about the rank-spread of certain vertices.

THEOREM 6.18. *Let F be a field and assume that for a graph G on n vertices,*

$\widehat{Z}(G) = M^F(G)$. Let $\widehat{G}$ be the graph where a vertex v is looped and no other vertices are specified looped or unlooped. If there exists a set $\mathcal{Z}$ of less than $M^F(G)$ vertices of $\widehat{G}$ such that every vertex in $\widehat{G}$ can be colored black by following the color-change rule for a simple graph and the additional rule that the looped vertex v may be colored black if it has no white neighbors, then $r_v^F(G) = 0$.

Proof. We claim that during the forcing process that started with $\mathcal{Z}$ and ended with all vertices black, the additional rule that v may be colored black if it has no white neighbors was used. When this additional forcing rule is used to color v black we say v “died alone”. Thus, we claim that v died alone in the forcing process. Suppose by way of contradiction that in the zero forcing process, the vertex v did not die alone. Thus, only the color-change rule for a simple graph was used and so $\mathcal{Z}$ is a zero forcing set for G . Hence, by Corollary 6.11, $M(G) \leq \widehat{Z}(G) \leq Z(G) \leq |\mathcal{Z}| < M(G)$, a contradiction.

Since v died alone, $\mathcal{Z}$ is a zero forcing set for $G - v$. Thus,

$$Z(G - v) \leq |\mathcal{Z}| < \widehat{Z}(G).$$

By Theorem 6.2, $M(G - v) \leq Z(G - v) < \widehat{Z}(G) = M(G)$. This fact along with the facts that $\text{mr}(G - v) \leq \text{mr}(G)$ (by Remark 2.11), $\text{mr}(G - v) + M(G - v) = n - 1$ and $\text{mr}(G) + M(G) = n$ imply $\text{mr}(G) = \text{mr}(G - v)$. As a consequence, $r_v(G) = 0$. $\square$

EXAMPLE 6.19. Consider the graph G in Example 6.17. As seen in this example, for any field F , $M^F(G) = Z(G)$. By Corollary 6.11, $M^F(G) = \widehat{Z}(G)$. Thus, by Theorem 6.18, Example 6.17 shows that vertices 1 and 3 in G have rank-spread 0.

7. Tree classification. We begin by classifying the vertices of the simplest type of tree—the path.

PROPOSITION 7.1. *For any field, P_n has all neutral vertices if $n > 2$.*

Proof. Let $n = 3$. Consider $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \in \mathcal{S}(P_3)$. Each matrix has rank 2, hence each is in $\mathcal{MR}(P_3)$. Thus, P_3 has all neutral vertices.

Assume $n > 3$, and label the vertices of P_n consecutively. By Lemma 5.1, the interior vertices of P_n are neutral since they have rank-spread two. Take the minimum rank cover $\{H_1, H_2\}$ of G where H_1 is the subgraph of G induced on vertices 1, 2, 3, and H_2 is the subgraph of G induced on vertices 3, 4, $\dots$, n . Let $\widetilde{A}_i \in \mathcal{MR}(\widetilde{H}_i)$ for $i = 1, 2$. Then $M = \widetilde{A}_1 + \widetilde{A}_2 \in \mathcal{MR}(P_n)$ and m_{11} could have been chosen to be zero or nonzero since P_3 has all neutral vertices. Thus, the pendant vertices of P_n are

neutral. $\square$

REMARK 7.2. For paths on fewer than 3 vertices we note $P_2 = K_2$ has all nonzero vertices and that K_1 is an isolated vertex and hence is nil.

Recall the following facts, which will be used extensively throughout the remainder of this section.

- $\text{mr}^F(K_n) = 1$ for $n \geq 2$.
- $\text{mr}^F(S_n) = 2$ for $n \geq 3$.
- Pendant vertices of stars on $n \geq 4$ vertices are nil. (See Lemma 5.6 or Example 2.13.)
- The dominating vertex of a star on $n \geq 3$ vertices is neutral. (See Lemma 5.1 or Example 2.13.)
- All vertices of $S_3 = P_3$ are neutral. (See Proposition 7.1.)

The following result appears as Corollary 3.15 in [12].

THEOREM 7.3. *If T is a tree and F is any field, then there is a K_2 -star cover of T whose rank sum is $\text{mr}^F(T)$.*

Note that given a minimum rank K_2 -star cover of T that is not edge-disjoint, the only way two elements could overlap would be two stars overlapping on a single edge. Replacing one of those stars in the cover with a star on one less vertex that covers the same edges as previously except for the edge that was covered by both stars will result in a new minimum rank cover of T with one less overlap. In this manner, we can always obtain an edge-disjoint minimum rank K_2 -star cover of T .

LEMMA 7.4. *Let F be a field and let v be a vertex of a tree T such that $r_v^F(T) = 2$. There exists a minimum rank K_2 -star cover where v is the center vertex of a star.*

Proof. Since $r_v(T) = 2$, $\text{mr}(T - v) = \text{mr}(T) - 2$. By Theorem 7.3, there is a minimum rank K_2 -star cover for $T - v$ with rank sum $\text{mr}(T) - 2$. Adding a star centered at v to this cover results in a minimum rank K_2 -star cover of T . $\square$

LEMMA 7.5. *Let F be a field, T be a tree, and v a vertex of T with $r_v^F(T) = 0$. Then, for any vertex w adjacent to v , $r_w^F(T) = 2$.*

Proof. By Theorem 7.3, there exists a minimum rank K_2 -star cover $\mathcal{C}$. In $\mathcal{C}$, the edge vw is covered by a K_2 or a star. If K_2 covers vw , then deleting v (and the edges adjacent to v) removes the K_2 , resulting in a cover of $T - v$ with rank sum at most $\text{mr}(T) - 1$, contradicting $r_v(T) = 0$. Thus, a star covers vw . This star is not centered at v since if so, deleting v and the star would result in a cover of $T - v$ with rank sum at most $\text{mr}(T) - 2$. Thus, w is the center of the star. Deleting w and the star centered at w gives a cover for $T - w$ with rank sum at most $\text{mr}(T) - 2$. Thus,

$r_w(T) \geq 2$. By Remark 2.11, $r_w(T) = 2$. $\square$

LEMMA 7.6. *Let v be a pendant vertex of a tree T . Then for any field, v is a nonzero vertex if and only if it is adjacent to a rank-spread 1 vertex.*

Proof. Assume that v is a nonzero vertex. Since v is a pendant vertex, by Remark 2.11 $r_v(T) = 0$ or 1. By Lemma 5.6, $r_v(T) = 1$. Let w be the vertex adjacent to v . By Lemma 7.5, $r_w(T) \neq 0$. Suppose that $r_w(T) = 2$. By Lemma 7.4, there is a minimum rank cover of T with a star centered at w . Let $T_1, \dots, T_m$ be the elements in the cover and let T_1 be the star centered at w . Let $\widetilde{A}_i \in \mathcal{MR}(\widetilde{T}_i)$, $i = 1, \dots, m$. Since pendant vertices of stars are nil or neutral, the diagonal entry in $\widetilde{A}_1$ corresponding to v may be chosen to be zero. Thus, the diagonal entry corresponding to v in $A = \sum_{i=1}^m \widetilde{A}_i \in \mathcal{MR}(T)$ is zero, contradicting that v is a nonzero vertex. Therefore $r_w(T) = 1$.

Assume that v is adjacent to a rank-spread 1 vertex; call it w . Since v is a pendant vertex, $r_v(T) = 0$ or 1. By Lemma 7.5, $r_v(T) = 1$. Let H be K_2 with vertex set $\{v, w\}$. Since $r_w(H) = 1$, Theorem 3.2 implies that $r_w(T - v) = 0$. By Theorem 3.5, $\mathcal{MR}(T) = \mathcal{MR}(\widetilde{H}) + \mathcal{MR}(\widetilde{T - v})$. Since H is a complete graph, v is a nonzero vertex in H . It follows that v is a nonzero vertex in T . $\square$

We now give the result that classifies the vertices of trees.

THEOREM 7.7. *Let F be a field and let v be a vertex of a tree T . Then*

- *v is a nil vertex if and only if $r_v^F(T) = 0$.*
- *v is a nonzero vertex if and only if $r_v^F(T) = 1$ and a vertex adjacent to v has rank-spread 1.*
- *v is a neutral vertex if and only if $r_v^F(T) = 2$, or $r_v^F(T) = 1$ and no vertex adjacent to v has rank-spread 1.*

Proof. The first statement that v is a nil vertex if and only if the rank-spread of v is 0 is exactly Corollary 5.9.

By Lemma 5.1, rank-spread 2 vertices are always neutral. Thus, for trees, nonzero vertices have rank-spread 1. To prove the second statement it suffices to show that a rank-spread 1 vertex is nonzero if and only if it's adjacent to a rank-spread 1 vertex.

Let T be a tree and v be a vertex such that $r_v(T) = 1$. If v is a pendant vertex, then the proof is complete by Lemma 7.6. Otherwise T is a vertex-sum of $T_1, \dots, T_m$ at v where v is a pendant vertex in each T_i . By Theorem 3.2, $r_v(T_i) = 1$ for exactly one $i \in \{1, 2, \dots, m\}$ and $r_v(T_i) = 0$ otherwise. Without loss of generality, $r_v(T_1) = 1$. For each $i \in \{1, 2, \dots, m\}$ let the neighbor of v in T_i be labeled w_i . By Lemma 7.5, $r_{w_i}(T_i) = 2$ for all $i \in \{2, \dots, m\}$. By Lemma 27 of [14], $r_{w_i}(T_i) = r_{w_i}(T)$. Thus, $r_{w_i}(T) = 2$ for all $i \in \{2, \dots, m\}$. Having accounted for all the neighbors of v except

w_1 it follows that v is adjacent to a vertex with rank-spread 1 if and only if $r_{w_1}(T) = 1$. Lemma 27 of [14] also implies $r_{w_1}(T) = 1$ if and only if $r_{w_1}(T_1) = 1$. By Lemma 7.6, $r_{w_1}(T_1) = 1$ if and only if v is nonzero in T_1 . Since $r_v(T_i) = 0$ for all $i \in \{2, \dots, m\}$, the first statement implies that v is nil in T_i for $i \in \{2, \dots, m\}$. By Theorem 3.5, $\mathcal{MR}(T) = \mathcal{MR}(\widetilde{T}_1) + \dots + \mathcal{MR}(\widetilde{T}_m)$. Thus, v is nonzero in T_1 if and only if v is nonzero in T .

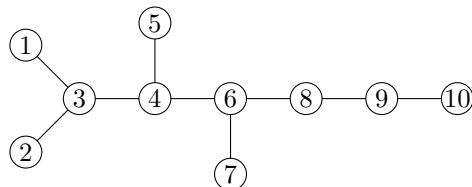
The third statement follows logically having proven the first two statements. $\square$

REMARK 7.8. The second and third statements in Theorem 7.7 could have been written as:

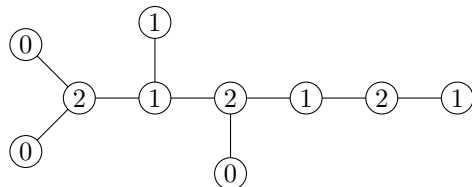
- v is a nonzero vertex if and only if $r_v^F(T) = 1$ and exactly one vertex adjacent to v has rank-spread 1.
- v is a neutral vertex if and only if $r_v^F(T) = 2$ or $r_v^F(T) = 1$ and every vertex adjacent to v has rank-spread 2.

The second statement follows from the proof given above and the third follows from Lemma 7.5 which implies that if $r_v(T) = 1$ then no vertex adjacent to v has rank-spread 0.

EXAMPLE 7.9. Consider the following tree T :



It is straightforward to calculate the rank spreads of each vertex and they are listed below.



By Theorem 7.7, vertices 1, 2 and 7 are nil vertices, vertices 4 and 5 are nonzero vertices, and the remaining vertices are neutral vertices.

8. Nullity two classification. Theorem 3.7 classifies the vertices of minimum rank two graphs over the real field, and Example 2.14 classifies the vertices of connected graphs with minimum rank one. It is well known that a graph has maximum

nullity one if and only if it is a path. Thus, Proposition 7.1 and Remark 7.2 classify the vertices of maximum nullity one graphs. We now proceed to classify the nil, nonzero and neutral vertices of graphs with maximum nullity equal to 2 over the real field.

Some results hold for more fields than $\mathbb{R}$ and their statements indicate so. Otherwise, we assume the field is $\mathbb{R}$.

LEMMA 8.1. *Let F be a field.*

1. *Each vertex of C_n , $n > 3$, is neutral.*
2. *For $F \neq F_2$, each vertex of a double cycle is neutral.*

Proof.

1. Let $n = 4$. Consider $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & -1 \end{bmatrix} \in \mathcal{S}(C_4)$. Since the matrix has

rank 2, it is in $\mathcal{MR}(C_4)$. Since the vertices of a cycle are indistinguishable, C_4 has all neutral vertices.

Let $n > 4$ and assume the statement is true for all $4 \leq k < n$. We use the standard labeling: $E(C_{n-1}) = \{12, 23, \dots, n-11\}$. Let $A \in \mathcal{MR}(C_{n-1})$ and let $B = A \oplus [0]$. Then $M = B + (0_{n-3} \oplus -b_{n-2, n-1} J_3) \in \mathcal{MR}(C_n)$. Since for $i < n-2$, b_{ii} could have been chosen to be either zero or nonzero and since $m_{ii} = b_{ii}$, vertex i is neutral in C_n . Thus, all vertices of C_n are neutral.

2. We first show that each vertex of the diamond is neutral. Throughout the proof we will label the vertices of the diamond as 

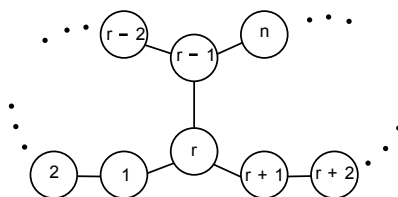
Note that unlike cycles, we must consider both degree 2 and degree 3 vertices when showing each vertex of a double cycle is neutral. Since F is a field with at least 3 elements, we may let $a \neq 0, 1$.

Consider the matrices $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 1 & a & 1 & 0 \\ a & a^2 - 1 & a - 1 & -1 \\ 1 & a - 1 & 0 & -1 \\ 0 & -1 & -1 & -1 \end{bmatrix}$ both of which

are in $\mathcal{S}(\text{diamond})$. Each has rank 2, hence each is in $\mathcal{MR}(\text{diamond})$. Thus, the vertices of the diamond are neutral.

We now consider the case of a double cycle G formed by overlapping C_r and C_s

on an edge, $r, s > 3$, $n = r + s - 2$. Label the vertices as in



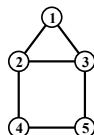
By Statement 1, each vertex of both C_r and C_s is neutral. Let $A \in \mathcal{MR}(C_r)$ and $B \in \mathcal{MR}(C_s)$ such that $a_{r-1,r} \neq -b_{1,2}$. This is possible since $F \neq F_2$. Consider the matrix $M = (A \oplus 0_{s-2}) + (0_{r-2} \oplus B) \in \mathcal{S}(G)$. Since $\text{rank } M \leq \text{rank } A + \text{rank } B = r - 2 + s - 2 = (r + s - 2) - 2 = n - 2 = \text{mr}(G)$, $M \in \mathcal{MR}(G)$. Since the vertices of C_r and C_s are neutral, each diagonal entry of M corresponding to a degree two vertex of G can be either zero or nonzero. Since each diagonal entry of M corresponding to a degree three vertex of G is a sum of diagonal entries corresponding to neutral vertices, the summands can be chosen to obtain a zero or nonzero sum. Thus, each vertex of G is neutral in this case.

Lastly, we consider the case of a double cycle G formed by overlapping C_r and K_3 on an edge, $r > 3$, $n = r + 1$. The proof of the case for $r, s > 3$ is sufficient both to construct an $M \in \mathcal{MR}(G)$ and to prove that the degree two vertices of G corresponding to C_r are neutral. In the construction of M , each diagonal entry corresponding to a degree three vertex of G is a sum of diagonal entries corresponding to a neutral vertex and a nonzero vertex. The diagonal entry corresponding to the neutral vertex can be chosen to be the negative of the diagonal entry corresponding to the nonzero vertex. Hence, even in this case the proof shows the degree three vertices of G are neutral and the degree two vertex corresponding to the K_3 is not nil.

To show that the degree two vertex of G corresponding to the K_3 is neutral, we use a different construction. Let $A \in \mathcal{MR}(C_{r-1})$. (Since $r > 3$, C_{r-1} is a proper cycle.) We use the standard labeling as previously. Let $B \in \mathcal{MR}(\text{diamond})$ such that $b_{1,2} = -a_{r-2,r-1}$ and $b_{4,4} = 0$. Then $M = (A \oplus 0_2) + (0_{r-3} \oplus B) \in \mathcal{S}(G)$ and since $\text{rank } M \leq \text{rank } A + \text{rank } B = (r - 1) - 2 + 2 = (r + 1) - 2 = n - 2 = \text{mr}(G)$, $M \in \mathcal{MR}(G)$. Thus, the degree two vertex corresponding to the K_3 is not nonzero. Thus, all vertices of G are neutral. $\square$

EXAMPLE 8.2. The *house* is the double cycle formed by overlapping K_3 and C_4 on an edge. Over F_2 , this minimum rank 3 graph has nil, nonzero, and neutral

vertices. We label the vertices of the house as



. Let

$$A = \begin{bmatrix} d_1 & 1 & 1 & 0 & 0 \\ 1 & d_2 & 1 & 1 & 0 \\ 1 & 1 & d_3 & 0 & 1 \\ 0 & 1 & 0 & d_4 & 1 \\ 0 & 0 & 1 & 1 & d_5 \end{bmatrix} \in \mathcal{S}^{F_2}(\text{house}),$$

where for $i \in \{1, 2, 3, 4, 5\}$, each d_i is the entry corresponding to vertex i . First, we show vertex 1 is nil. Suppose $d_1 = 1$ and consider the submatrix

$$A(3, 2) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & d_4 & 1 \\ 0 & 1 & 1 & d_5 \end{bmatrix}$$

formed by deleting the third row and second column. Since $\det A(3, 2) = 1$, A cannot be a minimum rank matrix unless $d_1 = 0$. We show vertices 2 and 3 are neutral by considering

$$B = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & d_2 & 1 & 1 & 0 \\ 1 & 1 & d_3 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \in \mathcal{S}^{F_2}(\text{house}).$$

If $d_2 = 0$ and $d_3 = 1$, then $\text{rank } B = 3 = \text{mr}^{F_2}(\text{house})$. Also if $d_2 = 1$ and $d_3 = 0$, $\text{rank } B = 3 = \text{mr}^{F_2}(\text{house})$. Lastly, we show vertices 4 and 5 are nonzero. Suppose $d_4 = 0$ and again consider the submatrix

$$A(3, 2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & d_5 \end{bmatrix}.$$

Then $\det A(3, 2) = 1$ so A cannot be a minimum rank matrix unless $d_4 \neq 0$. Similarly, $d_5 \neq 0$.

THEOREM 8.3. *Let G be a 2-connected graph with $M(G) = 2$. Then G has no nil vertices, and if $G \neq K_3$, G has no nonzero vertices.*

Proof. Since G is 2-connected and $M(G) = 2$, Lemma 4.8 of [11] implies that G is a chain of induced cycles $C_1, C_2, \dots, C_k$ with consecutive cycles overlapping on a

single edge. While such graphs have been given many names, i.e., linear 2-trees and LSEAC graphs, we shall refer to them as 2-connected partial 2-paths. Proceed by induction on k , the number of induced cycles. If $k \leq 2$, then G is a cycle or double cycle. By Lemma 8.1, all the vertices of G are neutral unless G is K_3 in which case they are all nonzero. Thus, we will assume that $k \geq 3$ and that for all 2-connected partial 2-paths G with less than k induced cycles, every vertex is neutral, unless G is K_3 .

Since G has at least 3 induced cycles, it is possible to consider G as the union of two smaller 2-connected partial 2-paths. The simplest example is decomposing a double cycle as the union of two cycles. Let v be a vertex of G .

Assume first that there exists an induced cycle C_i such that v does not belong to the vertex set of C_i . Then G is the union of two smaller 2-connected partial 2-paths, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, such that $v \notin V_1 \cap V_2$. Note that since there are at least 3 induced cycles, it is possible to stipulate that if v is in V_i , the number of induced cycles in G_i is at least 2. Without loss of generality, renaming if necessary, let $v \in V_1$. Since G_1 has at least 2 induced cycles it is not K_3 . Thus, by the inductive hypothesis v is neutral in G_1 . Let $A_i \in \mathcal{MR}(\widetilde{G}_i)$. Since $M(G_i) = 2$, $\text{rank } A_i = \text{mr}(G_i) = |V_i| - 2$. Then $\text{rank}(A_1 + A_2) \leq \text{rank } A_1 + \text{rank } A_2 = |V_1| - 2 + |V_2| - 2 = |V| - 2 = \text{mr}(G)$. The second equality follows from the fact that $V_1 \cap V_2$ consists of the two vertices of the overlapping edge. Since v was neutral in G_1 , the diagonal entry $a_{vv}^{(1)}$ in A_1 can be chosen to be zero or nonzero. Also since $v \notin V_2$, $a_{vv}^{(2)} = 0$. If necessary, we multiply A_2 by a nonzero constant to ensure that the sum of the nonzero off-diagonal entries corresponding to the overlapping edge is not zero. Note that this does not affect $a_{vv}^{(2)}$ since it is zero. Thus, $A_1 + A_2 = A \in \mathcal{S}(G)$, $\text{rank } A = \text{mr}(G)$, and the diagonal entry corresponding to v may be zero or nonzero. Thus, v is neutral in G .

Assume now that v belongs to every induced cycle in G . We consider the cases where $k = 3$ and $k \geq 4$ separately. First, if $k \geq 4$, then we may split G into two 2-connected partial 2-paths, G_1 and G_2 , where each has at least 2 induced cycles. Then by the inductive hypothesis v is neutral in both G_1 and G_2 . Let $A_i \in \mathcal{MR}(\widetilde{G}_i)$ again noting that $\text{rank } A_i = \text{mr}(G_i) = |V_i| - 2$. Let $a_{vv}^{(1)} \neq 0$ and $a_{vv}^{(2)} = 0$. Then, multiplying A_2 by a constant if necessary, $A_1 + A_2 = A \in \mathcal{S}(G)$, with $\text{rank } A = \text{mr}(G)$, and $a_{vv} \neq 0$. Similarly, we may choose $a_{vv}^{(1)} = 0$, and construct a minimum rank matrix A for G , where $a_{vv} = 0$. Thus, v is neutral in G . Now consider the case where $k = 3$. This case is similar to the case above, where $k \geq 4$, except if G_1 or G_2 is K_3 . Without loss of generality, let $G_1 = K_3$. Since G_2 has two induced cycles, v is neutral in G_2 by the inductive hypothesis. Thus, it is possible to choose $a_{vv}^{(2)} = 0$ in a minimum rank matrix for $\widetilde{G}_2$. Since $G_1 = K_3$, $a_{vv}^{(1)} \neq 0$. Thus, it is possible to construct a minimum

rank matrix for G with $a_{vv} \neq 0$. Notice that

$$A = \begin{bmatrix} -1 & a & a \\ a & -a^2 & -a^2 \\ a & -a^2 & -a^2 \end{bmatrix} \in \mathcal{MR}(K_3),$$

where K_3 is labeled so that v corresponds to the first row and column of A . Appropriately embed A in a $|G| \times |G|$ matrix and call it A_1 . There exists $A_2 \in \mathcal{MR}(\widetilde{G}_2)$ such that $a_{vv}^{(2)} = 1$. Choosing an appropriate value for a so that the sum of the entries corresponding to the overlapping edge is not zero, $A_1 + A_2 \in \mathcal{S}(G)$. Further, $\text{rank}(A_1 + A_2) \leq \text{rank } A_1 + \text{rank } A_2 = \text{mr}(K_3) + \text{mr}(G_2) = 1 + |V_2| - 2 = |V| - 2$. Thus, $A_1 + A_2$ is a minimum rank matrix for G , where $a_{vv} = 0$. Thus, v is neutral in G . $\square$

LEMMA 8.4. $M_+(G) \geq 1$ for all graphs G .

Proof. The Laplacian matrix of a graph G is by construction a matrix in $\mathcal{S}_+(G)$ with nullity at least one. $\square$

THEOREM 8.5. *Let G be a connected graph with $M(G) = 2$. Then G has a nil vertex v if and only if G is a subdivision of S_4 and v is a pendant vertex adjacent to the degree three vertex.*

Proof. The reverse implication is clear, because by Corollary 5.9, the nil vertices of a tree are exactly the rank-spread zero vertices. Since G is a tree, $M(G) = P(G)$ by Theorem 3.1. Deleting v decreases $P(G)$ and $|G|$ by one each. Thus, $r_v(G) = 0$.

For the forward implication, assume G has a nil vertex. Then $\text{mr}(G) < \text{mr}_+(G)$, or $M(G) > M_+(G)$. Since $M(G) = 2$, by Lemma 8.4 $M_+(G) = 1$. By Theorem 3.4, G is a tree. By Theorem 3.1, $P(G) = M(G) = 2$. Then G must have 4 or fewer pendant vertices.

If G has 4 pendant vertices and $P(G) = 2$, then the 4 pendant vertices must form the endpoints of the two paths in any minimal path cover. Then, since G is a tree, G consists of two paths with exactly one edge joining the two (not incident to any of the endpoints of the paths, in which case G would have less than 4 pendant vertices). By Corollary 5.9, the nil vertices of a tree are the rank-spread zero vertices. However, if G is a tree of this form, G has no rank-spread zero vertices. If it did, deleting such a vertex would result in obtaining a path. Since G has 4 pendant vertices this is impossible. Thus, G must have 3 or fewer pendant vertices.

If G has 2 pendant vertices, then G is a path, in which case G has no nil vertices. Thus, G has exactly 3 pendant vertices. Since G is a tree, G must be a subdivision of S_4 . By Corollary 5.9, the nil vertices of G are the rank-spread zero vertices, which are exactly those pendant vertices that are adjacent to the degree three vertex. $\square$

The following theorem categorizes the nonzero vertices of connected graphs with maximum nullity 2. This completes our classification of nil, neutral, and nonzero vertices of “extreme” graphs, those with low or high minimum rank. The following proof shows that all nonzero vertices of these graphs come from subgraphs which are complete graphs on 2 or 3 vertices, and illustrates the utility and versatility of Theorem 3.5, the decomposition theorem for matrices of graphs with a cut-vertex.

THEOREM 8.6. *Let G be a connected graph on n vertices with $M(G) = 2$. Then v is a nonzero vertex of G if and only if one of the following is true:*

- v is a pendant vertex with neighbor u such that $G - u = P_{n-2} \cup K_1$;
- v is a degree 2 vertex such that $G - v = P_{n-2} \cup K_1$;
- v is a degree 2 simplicial vertex such that $G - v = P_{n-1}$;
- v and u are adjacent degree 2 simplicial vertices such that $G - v - u = P_{n-2}$.

Proof. We denote the vertex-sum of G_1 and G_2 at a common vertex u by $G_1 \oplus_u G_2$. We first prove the reverse implication, beginning with a general technique. If v is a pendant vertex with neighbor u , we may write $G = (G - v) \oplus_u K_2$, where the vertices of K_2 are u and v . If u is such that $r_u(G) < 2$, then $r_u(G - v) + r_u(K_2) < 2$ by Theorem 3.2. Thus, by Theorem 3.5, every minimum rank matrix for G comes from a minimum rank matrix for $G - v$ and a minimum rank matrix for K_2 , implying v is nonzero in G . Further, u is nonzero in G exactly when u is nil in $G - v$. Thus, to show a pendant vertex v is nonzero, it suffices to show the rank-spread of its neighbor is less than 2, and to show the neighbor u of a pendant vertex v is nonzero it suffices to show $r_u(G) < 2$ and that u is nil in $G - v$.

We now show that each of the four conditions of the Theorem imply that v is nonzero.

If v is a pendant vertex with neighbor u such that $G - u = P_{n-2} \cup K_1$, then $r_u(G) = \text{mr}(G) - \text{mr}(P_{n-2} \cup K_1) = (n - 2) - (n - 2 - 1) = 1$. Thus, v is nonzero.

If v is a degree 2 vertex such that $G - v = P_{n-2} \cup K_1$, then G is a subdivision of S_4 such that one edge of S_4 was subdivided exactly once, creating the vertex v . Let w be the pendant vertex adjacent to v . Then $G - w$ is exactly the tree described in Theorem 8.5 that has a nil vertex, and the nil vertex is v . Just as in the last case, $r_v(G) = 1$. Hence, v is nonzero in G .

Now suppose v is a degree 2 simplicial vertex such that $G - v = P_{n-1}$. Let u and w be the neighbors of v so that $\{u, v, w\}$ induces K_3 . Then either $G = K_3$, $G = K_3 \oplus_u P_{n-2}$, or $G = P_i \oplus_u (K_3 \oplus_w P_{n-i-1})$ (where u and w are pendant in the corresponding paths). In the first case v is obviously nonzero. In the second case, $r_u(G) = \text{mr}(G) - \text{mr}(P_{n-3} \cup K_2) = (n - 2) - (n - 4 + 1) = 1$ implies minimum rank

matrices for G come from minimum rank matrices for P_{n-2} and K_3 , and hence, v is nonzero. In the third case, applying the method of the second case twice we see v is nonzero.

Lastly, suppose that v and u are adjacent degree 2 simplicial vertices such that $G - v - u = P_{n-2}$. Let w be the common neighbor of v and u , so that $\{u, v, w\}$ induces K_3 . Then either $G = K_3$, or $G = K_3 \oplus P_{n-2}$ (where w may be pendant or interior in P_{n-2}). If $G = K_3$, v is nonzero. If $G = K_3 \oplus P_{n-2}$, by Theorem 3.5, matrices for G come from either matrices for K_3 and P_{n-2} or matrices for K_2 , a star, and one or two paths. In all cases v must be nonzero. This completes the proof of the reverse implication.

We prove the forward implication by showing that every vertex v of G meets one of the conditions listed in the theorem if v is nonzero.

First suppose G is a tree. From Theorem 3.1 either G is a subdivision of S_4 (allowing the set of all subdivisions of S_4 to contain S_4) or G is obtained by inserting one edge between an interior vertex of P_i and an interior vertex of P_j where $i + j = n$ and $i, j \geq 3$.

Assume G is a subdivision of S_4 . The central vertex has rank-spread 2, and hence is neutral. First consider a non-central vertex v of G that lies on a branch of G corresponding to an edge of S_4 that was not subdivided. By Theorem 8.5 v is nil. Now consider a non-central vertex v of G that lies on a branch of G created by subdividing an edge of S_4 at least once. Let u be the vertex of that branch adjacent to the central vertex. Then $r_u(G) = 1$. Thus, G decomposes into a path on 2 or more vertices and a subdivision of S_4 , where at least one branch has not been subdivided. If the path is on 2 vertices, then either $v = u$ or v is a pendant vertex adjacent to u . In either case v meets one of the conditions. If the path is on 3 or more vertices, since all vertices of a path on 3 or more vertices are neutral, by summing minimum rank matrices for the path and for the subdivision of S_4 we may obtain minimum rank matrices for G with zero and nonzero entries corresponding to both v and u . Thus, v and u are neutral in G , and so we have shown that if G is a subdivision of S_4 , every vertex either meets one of the conditions or is not nonzero.

Now assume G is obtained by inserting one edge between an interior vertex of P_i and an interior vertex of P_j where $i + j = n$. Since every non-pendant vertex has rank-spread 2, by Lemma 5.1 these vertices are neutral. Since the neighbor of each pendant vertex has rank-spread 2, by Lemma 7.6 the pendant vertices are not nonzero.

Now suppose G is not a tree. By Theorem 5.1 of [11], since $M(G) = 2$, either G has path cover number 2 or is an exceptional graph. If $P(G) = 2$, then G has exactly

one 2-connected block, since a 2-connected block requires two paths to cover it and any two separate 2-connected blocks could share at most one path in the path cover. So, G consists of a single 2-connected block with at most four paths extending from the block. If G is an exceptional graph, G consists of a single 2-connected block with at most five paths extending from the block (See Table B1 of [11]). Let H be the induced subgraph of the 2-connected block.

We will consider each vertex of G in four cases: vertices of $H \neq K_3$, cut-vertices, pendant vertices, and simplicial vertices of degree 2 where $P(G) = 2$ and $H = K_3$. These cases are exhaustive since the vertices of G that are not vertices of H are either cut-vertices or pendants, and if $H = K_3$ then the vertices of H are either cut-vertices or simplicial vertices of degree 2. Note that if G is an exceptional graph $H \neq K_3$ (see Table B1 of [11]). Thus, we need only consider simplicial vertices of degree 2 when $P(G) = 2$.

Case 1. Assume $H \neq K_3$ and let v be a vertex of H . Then by Theorem 8.3, all vertices of H are neutral in H . The set of H and the paths of appropriate lengths is a minimum rank cover for G . Thus, regardless of whether or not the individual paths contain nonzero vertices, every vertex of H is neutral in G .

Case 2. Assume v is a cut-vertex. Then we may write $G = G_1 \oplus_v G_2$. Note that G_1 and G_2 are connected. If v is nonzero, then the decomposition with the star given in Theorem 3.5 cannot occur. The only way for v to be nonzero in G then is for, without loss of generality, v to be nil in G_1 and nonzero in G_2 . Since nil vertices do not occur in paths, $M(G_1) \geq 2$. In terms of maximum nullity, Theorem 3.2 states

$$2 = M(G) = \max\{M(G_1) + M(G_2) - 1, M(G_1 - w) + M(G_2 - w) - 1\}.$$

Since maximum nullity is always positive, we must have $M(G_1) = 2$ and $M(G_2) = 1$. Thus, G_2 is a path. But the only paths that have nonzero vertices are $P_2 = K_2$. And by Theorem 8.5 the only connected maximum nullity 2 graphs that have nil vertices are subdivisions of S_4 , and the nil vertex must be a pendant vertex adjacent to the degree 3 vertex. Thus, v is a degree 2 vertex such that $G - v = P_{n-2} \cup K_1$.

Case 3. Assume v is a pendant vertex. Then its neighbor u is a cut-vertex, and we may write $G = G_1 \oplus_u K_2$. By Theorem 3.2, $r_u(G) \geq 1$ since $r_u(K_2) = 1$. If $r_u(G) = 2$, v is not nonzero. If $r_u(G) = 1$, then $r_u(G_1) = 0$. Then since $M(G_1) = 2$, $M(G_1 - u) = 1$. Thus, $G - u = P_{n-2} \cup K_1$.

Case 4. Assume v is a simplicial vertex of degree 2, $P(G) = 2$ and $H = K_3$. First suppose that v has a neighbor u such that u is also a simplicial vertex of degree 2. Let w be the common neighbor of v and u . If $G = K_3$ we are done. Otherwise, w must be a cut-vertex. We now consider the possible path covers of G , using the fact that $P(G) = 2$. Since u and v are both simplicial, each must be the endpoint

of a path. If they are covered by the same path, all remaining vertices of G must be covered by a single path. Then v and u are adjacent degree 2 simplicial vertices such that $G - v - u = P_{n-2}$. If they are covered by different paths, since w is a cut-vertex, we may assume v is the only vertex covered by its path. Thus, the remaining vertices of G are covered by a single path, and v is a degree 2 simplicial vertex such that $G - v = P_{n-1}$.

Now suppose that v does not have such a neighbor. Then the two neighbors of v , call them u and w , must both have degree at least 3. We again consider the path covers of G . If v is the only vertex covered by its path, as before, v is a degree 2 simplicial vertex such that $G - v = P_{n-1}$. Otherwise, we may assume u is covered by the same path as v . Then the remaining path either has w as an endpoint or passes through w . In the first case, since $H = K_3$, v is a degree 2 simplicial vertex such that $G - v = P_{n-1}$. In the second case, none of the conditions are met. Thus, we show that in this case v is not nonzero. Since $H = K_3$, w is a cut-vertex. Write $G = P_i \oplus_w G_2$, where v is in G_2 and w is of degree 2 in P_i . Since $r_w(G_2) + r_w(P_i) \geq 2$, the decomposition in Theorem 3.5 does occur with a star, and since the degree of w is 4 and the degree of u is 3, v is nil in the star and neutral in $G_2 - w$. Thus, v is neutral in G . This completes the proof. $\square$

9. Conclusion. In this paper, we investigated conditions under which diagonal entry restrictions exist on matrices that achieve the minimum rank of a specified graph, classifying the corresponding vertices as nil or nonzero. We also identified various conditions which guarantee that no restrictions exist. In Theorem 4.1, we gave a solution in terms of nil vertices to the question of when the minimum rank of a graph G equals the minimum rank of the graph obtained by subdividing an edge of G . In Section 6, we gave a complete classification of these vertices for trees. In Sections 7 and 8, we completed the classification of nil, nonzero and neutral vertices for graphs of extreme minimum ranks, that is, graphs whose minimum rank is 1, 2, $n-1$, or $n-2$. From the results, it appears that graphs with high or low connectivity tend to have more restrictions whereas for graphs with middling connectivity fewer restrictions exist. Example 7.9 illustrates the restrictions on trees, the graphs with the lowest connectivity excepting disconnected graphs. Complete graphs on at least two vertices have the highest connectivity, and every diagonal entry is restricted to be nonzero. However, a large class of 2-connected graphs (see Theorem 8.3) have no restrictions whatsoever. It remains to quantify these connections and determine if any results exist with regard to connectivity beyond what has been presented here.

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