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# NON-SINGULAR CIRCULANT GRAPHS AND DIGRAPHS*

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**Abstract.** For a fixed positive integer $n$, let $W_n$ be the permutation matrix corresponding to the permutation $\begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix}$. In this article, it is shown that a symmetric matrix with rational entries is circulant if, and only if, it lies in the subalgebra of $\mathbb{Q}[x]/\langle x^n - 1\rangle$ generated by $W_n + W_n^{-1}$. On the basis of this, the singularity of graphs on $n$-vertices is characterized algebraically. This characterization is then extended to characterize the singularity of directed circulant graphs. The $k$th power matrix $W_n^k + W_n^{-k}$ defines a circulant graph $C_n^k$. The results above are then applied to characterize its singularity, and that of its complement graph. The digraph $C_{r,s,t}$ is defined as that whose adjacency matrix is circulant $circ(\mathbf{a})$, where $\mathbf{a}$ is a vector with the first $r$-components equal to 1, and the next $s, t$ and $n - (r + s + t)$ components equal to zero, one, and zero respectively. The singularity of this digraph (graph), under certain conditions, is also shown to depend algebraically upon these parameters. A slight generalization of these graphs are also studied.

**Key words.** Graphs, Digraphs, Circulant matrices, Primitive roots.

**AMS subject classifications.** 05C50, 15A18.

## 1. Introduction and preliminaries.

Let $\mathbb{Q}$ denote the set of rational numbers and let $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1}) \in \mathbb{Q}^n$, for a fixed positive integer $n$. Consider the shift operator $T : \mathbb{Q}^n \rightarrow \mathbb{Q}^n$, defined by $T(a_0, a_1, \ldots, a_{n-1}) = (a_{n-1}, a_0, \ldots, a_{n-2})$. The circulant matrix, denoted $circ(\mathbf{a})$, associated with $\mathbf{a}$ is the $n \times n$ matrix whose $k$th row is $T^{k-1}(\mathbf{a})$, for $k = 1, 2, \ldots, n$. For example, the identity matrix, denoted $I$, equals $circ(1, 0, \ldots, 0)$, the matrix of all 1's, denoted $\mathbf{J}$, equals $circ(1, 1, \ldots, 1)$ and $W_n = cicr(0, 1, \ldots, 0)$ is the permutation matrix corresponding to the permutation $\begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix}$. It is well known that $W_n$ is the generator of the algebra $\mathbb{Q}[x]/\langle x^n - 1\rangle$ (see, for instance, [6]), where $\mathbb{Q}[x]$ is the polynomial ring over $\mathbb{Q}$ and $\langle x^n - 1\rangle$ denotes the principal ideal of $\mathbb{Q}[x]$ generated by $x^n - 1$. Or equivalently, every circulant matrix of order $n$ is a polynomial in $W_n$.

Lemma 1.1. [2] *Let $A$ be an $n \times n$ matrix with rational entries. Then $A$ is circulant if, and only if, it is a polynomial over $\mathbb{Q}$ in $W_n$.*

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For $\mathbf{a} \in \mathbb{Q}^n$, let $A = circ(\mathbf{a})$. Then Lemma 1.1 ensures the existence of the polynomial $\gamma_A(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} \in \mathbb{Q}[x]$, called the representer polynomial, with $A = \gamma_A(W_n)$. One also obtains the following result about circulant matrices.

LEMMA 1.2. *Let $A = circ(\mathbf{a})$ be a circulant matrix. Then the eigenvalues of $A$ equal $\gamma_A(\zeta_n^k) = a_0 + a_1\zeta_n^k + \cdots + a_{n-1}(\zeta_n^k)^{n-1}$, for $k = 0, 1, \ldots, n-1$, where $\zeta_n$ denotes a primitive $n$-th root of unity.*

The reader is referred to [4] and/or [1] for standard concepts and results used in this article without explicit references.

Let $G = (V, E)$ be a simple graph (has no loops), directed or not, on $n = |V|$ vertices, where for a finite set $S$, $|S|$ denotes the number of elements in $S$. Given an ordering $V = \{v_1, v_2, \ldots, v_n\}$ of the vertices, the adjacency matrix of $A(G) = [a_{ij}]$ of $G$, is by definition the $n \times n$ matrix whose $ij$th entry equals the number of edges $(v_i, v_j)$, from the vertex $v_i$ to the vertex $v_j$. Observe that, the adjacency matrix $A(G)$ of a graph $G$ is symmetric, whereas the adjacency matrix of a directed graph need not be symmetric. The graph $G$ is said to be nonsingular if $A(G)$ is a nonsingular matrix and is called circulant if $A(G)$ is a circulant matrix. For example, the adjacency matrix of the cycle graph $C_n$ on $n$ vertices equals $circ(0, 1, 0, \ldots, 1) = W_n + W_n^{-1}$ and $\gamma(x) = x + x^{n-1}$ is its representer polynomial. Thus, using Lemma 1.2, the graph $C_n$ is singular if, and only if, $n$ is a multiple of 4.

Recall that the $n$-th cyclotomic polynomial, denoted $\Phi_n(x)$, is an element of $\mathbb{Z}[x]$ and equals $\prod\limits_{\substack{\gcd(k,n)=1 \\ 1 \le k \le n}} (x - \zeta_n^k)$. Also, $x^n - 1 = \prod\limits_{d|n} \Phi_d(x)$ (here $a \mid b$ means $a$ 'divides' $b$). Moreover, $\Phi_n(x)$ is the minimal polynomial of $\zeta_n$ and hence, $f(\zeta_n) = 0$ for some $f(x) \in \mathbb{Z}[x]$ implies $\Phi_n(x)$ divides $f(x) \in \mathbb{Z}[x]$. The next result appears in [8].

LEMMA 1.3. [8] *Let $p$ be a prime and let $n$ be a positive integer. Then*

$$\Phi_{pn}(x) = \begin{cases} \Phi_n(x^p), & \text{if } p \mid n, \\ \frac{\Phi_n(x^p)}{\Phi_n(x)}, & \text{if } p \nmid n. \end{cases}$$

*In particular, for each positive integer $k$, $\Phi_{p^k}(x) = 1 + x^{p^{k-1}} + x^{2p^{k-1}} + \cdots + x^{(p-1)p^{k-1}}$.*

The following result is an application of Lemma 1.2. This result also appears in [5] (also see Corollary 10 in [6]).

LEMMA 1.4. [5, 6] *Let $A = circ(\mathbf{a})$. Then $A$ is singular if, and only if, $\deg(\gcd(\gamma_A(x), x^n - 1)) \ge 1$.*

Lemma 1.4 gives us the following useful remarks.

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Remark 1.5.

1. Let $A = circ(\mathbf{a})$ with $\mathbf{a} = (\underbrace{1, \ldots, 1}_{s \text{ times}}, 0, \ldots, 0)$. Then $A$ is singular if, and only if, $\gcd(s, n) > 1$.
2. Let $A = circ(\mathbf{a})$ with $a_i = 0$, for $0 \le i \le k-1$ and $a_k \neq 0$. Then $\gamma_A(x) = x^k \Gamma_A(x)$, for some polynomial $\Gamma_A(x) \in \mathbb{Q}[x]$. Thus, using Lemma 1.4, $A$ is singular (nonsingular) if, and only if, $W_n^k A$ is singular (nonsingular), for some $k = 0, 1, \ldots, n-1$. That is, to study singularity (nonsingularity) of $A$, it is enough to study $\Gamma_A(x)$.

Using Remark 1.5.2 and Lemma 1.4, the following result is immediate, and hence, the proof is omitted.

Lemma 1.6. *Let $G$ be a circulant digraph of order $n$. Then $G$ is singular if, and only if, $\Phi_d(x) \mid \Gamma_{A(G)}(x)$, for some divisor $d \neq 1$ of $n$.*

As an immediate corollary of Lemma 1.6 and 1.3, the following result follows.

Corollary 1.7. *Let $p$ be a prime and let $k, \ell$ be positive integers with $p \nmid k$. Also, let $G$ be a $k$-regular circulant graph/digraph on $p^\ell$ vertices. Then $G$ is nonsingular.*

*Proof.* Using Lemma 1.6, it is enough to show that $\Phi_d(x) \nmid \Gamma_{A(G)}(x)$, for every $d \mid p^\ell$ with $d \neq 1$. On the contrary, assume that $\Gamma_{A(G)}(x) = \Phi_d(x)g(x)$, for some $g(x) \in \mathbb{Z}[x]$. Then using Lemma 1.3, $\Phi_d(1) = p$, for every $d \mid p^\ell$ with $d \neq 1$. As $g(x) \in \mathbb{Z}[x], g(1) \in \mathbb{Z}$ and hence $k = \Gamma_{A(G)}(1) = \Phi_d(1)g(1) = p\, g(1)$. A contradiction to the assumption that $p \nmid k$. $\square$

The remaining part of this paper consists of two more sections that are mainly concerned with applications of Lemma 1.6. Section 2 gives necessary and sufficient conditions for a few classes of circulant graphs to be nonsingular and Section 3 gives possible generalization of the results studied in Section 2. Before proceeding to Section 2, recall that for a graph $G = (V, E_1)$, the complement graph of $G$, denoted $G^c = (V, E_2)$, is a graph in which $(u, v) \in E_2$ whenever $(u, v) \notin E_1$ and vice versa, for every $u \neq v \in V$. Note that as $G$ is a simple graph, $G^c$ is also simple and $A(G^c) = \mathbf{J} - A(G) - I$. Moreover, $G$ is circulant if, and only if, $G^c$ is circulant.

**2. Some singular circulant graphs.** We start this section by showing that the adjacency matrix of a circulant graph on $n$ vertices is a polynomial in the adjacency matrix of $C_n$.

For $1 \le i \le \lfloor \frac{n}{2} \rfloor$, let $C_n^i$ denote a graph whose vertex set is same as the vertex set of the cycle graph $C_n$ and $(x, y)$ is an edge in $C_n^i$ whenever the distance of $x$ from $y$ in $C_n$ is exactly $i$. Also, define the matrices $A_i$ (commonly called the distance matrices

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of $C_n$) by

$$A_i = W_n^i + W_n^{n-i} \text{ and } A_{\lfloor \frac{n}{2} \rfloor} = \begin{cases} W_n^{\lfloor \frac{n}{2} \rfloor}, & \text{if } n \text{ is even,} \\ W_n^{\lfloor \frac{n}{2} \rfloor} + W_n^{n-\lfloor \frac{n}{2} \rfloor}, & \text{if } n \text{ is odd.} \end{cases}$$

Then note that $A_i = A(C_n^i)$, for $1 \le i \le \lfloor \frac{n}{2} \rfloor$.

The identity $(x^k + x^{-k}) = (x + x^{-1})(x^{k-1} + x^{1-k}) - (x^{k-2} + x^{2-k})$ enables us to readily establish, by mathematical induction, that $x^k + x^{-k}$ is a monic polynomial in $x + x^{-1}$ of degree $k$. Consequently, $A_i$'s, for $1 \le i \le \lfloor \frac{n}{2} \rfloor$, are polynomials of degree $\le i$, in the adjacency matrix of $C_n$ over $\mathbb{Q}$. Thus, every circulant symmetric matrix with rational entries lies in the subalgebra of $\mathbb{Q}[x]/\langle x^n - 1\rangle$ generated by $W_n + W_n^{-1}$ (for $n$ even, note that $2W_n^{\lfloor \frac{n}{2} \rfloor} = W_n^{\frac{n}{2}} + W_n^{-\frac{n}{2}}$).

THEOREM 2.1. *Let $B = circ(\mathbf{q})$, with $\mathbf{q} \in \mathbb{Q}^n$. Then $B$ is a symmetric if, and only if, $B$ is a polynomial in $W_n + W_n^{-1}$ with rational coefficients.*

*Proof.* One part is immediate, as every polynomial in $W_n + W_n^{-1}$ is a symmetric circulant matrix.

Conversely, let $B = circ(\mathbf{q})$ be symmetric. Then $B = \sum\limits_{i=0}^{n-1} q_i W_n^i$ and $B^t = \sum\limits_{i=0}^{n-1} q_i W_n^{n-i}$. But $B = B^t$ implies $q_i = q_{n-i}$, for $1 \le i \le n-1$. Thus, $B = \sum\limits_{i=0}^{\lfloor \frac{n}{2} \rfloor} q_i A_i$. $\square$

Observe that for $1 \le i < \lfloor \frac{n}{2} \rfloor$, $\gamma_{A_i}(x) = x^i \left(1 + x^{n-2i}\right)$ and $\gamma_{A_{\lfloor \frac{n}{2} \rfloor}}(x) = x^{\lfloor \frac{n}{2} \rfloor}$, if $n$ is even, and $\gamma_{A_{\lfloor \frac{n}{2} \rfloor}}(x) = x^{\lfloor \frac{n}{2} \rfloor}(1+x)$, if $n$ is odd. Using these, the next result gives a necessary and sufficient condition for the graphs $C_n^i$, for $1 \le i \le \lfloor \frac{n}{2} \rfloor$, to be singular.

PROPOSITION 2.2. *Let $n \ge 3$ and $1 \le i \le \lfloor \frac{n}{2} \rfloor$. Then the graph $C_n^i$ is singular if, and only if, $n$ is a multiple of 4 and $\gcd(i, \frac{n}{2}) \mid \frac{n}{4}$.*

*Proof.* Note that $\Gamma_{A_{\lfloor \frac{n}{2} \rfloor}}(x) = 1 + x$ if $n$ is odd and $\Gamma_{A_{\lfloor \frac{n}{2} \rfloor}}(x) = 1$ if $n$ is even. Hence, $A_{\lfloor \frac{n}{2} \rfloor}$ is nonsingular for all $n$. For $1 \le i < \lfloor \frac{n}{2} \rfloor$, $\Gamma_{A_i}(x) = 1 + x^{n-2i}$ and hence $A_i$ is singular if, and only if, $\Gamma_{A_i}(\zeta_n^k) = 0$, for some $k, 1 \le k \le n-1$. That is, one needs $\left(\zeta_n^k\right)^{n-2i} = -1$. This holds if, and only if, 4 divides $n$ and $\gcd(i, \frac{n}{2}) \mid \frac{n}{4}$. $\square$

As an immediate consequence of Proposition 2.2, the graph $C_n$ is singular if, and only if, $4 \mid n$. The next result gives a necessary and sufficient condition for the complement graph $(C_n^i)^c$, of $C_n^i$, to be singular.

PROPOSITION 2.3. *Let $n \ge 4$. Then the graph*

1. *$(C_n^{\lfloor \frac{n}{2} \rfloor})^c$ is singular if, and only if, $n$ is even or $n \equiv 3 \pmod 6$.*
2. *$(C_n^i)^c$, for $1 \le i < \lfloor \frac{n}{2} \rfloor$, is singular if, and only if, $3 \mid n$ and $\gcd(i, n) \mid \frac{n}{3}$.*

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*Proof.* Note that $A = A((C_n^{\lfloor \frac{n}{2} \rfloor})^c) = \mathbf{J} - A_{\lfloor \frac{n}{2} \rfloor} - I$ and $\gamma_A(x) = \dfrac{x^n - 1}{x - 1} - (1 + x^{\lfloor \frac{n}{2} \rfloor})$, for $n$ even. Thus, using Lemma 1.2, the graph $(C_n^{\lfloor \frac{n}{2} \rfloor})^c$ is singular, whenever $n$ is even. And for $n$ odd, $\gamma_A(x) = \dfrac{x^n - 1}{x - 1} - (1 + x^{\lfloor \frac{n}{2} \rfloor} + x^{\lfloor \frac{n}{2} \rfloor + 1})$. Consequently, $(C_n^{\lfloor \frac{n}{2} \rfloor})^c$ is singular if, and only if, $1 + (\zeta_n^k)^{\lfloor \frac{n}{2} \rfloor} + (\zeta_n^k)^{\lfloor \frac{n}{2} \rfloor + 1} = 0$, for some $k, 1 \le k \le n-1$. Or equivalently, $\zeta_n^{k \lfloor \frac{n}{2} \rfloor}$ is a primitive 3-rd root of unity. Thus, $n \equiv 3 \pmod 6$.

Now assume that $1 \le i < \lfloor \frac{n}{2} \rfloor$. Then $A((C_n^i)^c) = \mathbf{J} - A_i - I$, and hence, $\gamma_{A((C_n^i)^c)}(x) = \dfrac{x^n - 1}{x - 1} - (1 + x^i + x^{n-i})$. Thus, $(C_n^i)^c$ is singular if, and only if, $1 + \zeta_n^{ki} + \zeta_n^{-ki} = 0$, for some $k$, $1 \le k \le n-1$. That is, $\zeta_n^{ki}$ is a primitive 3-rd root of unity. Thus, $(C_n^i)^c$ is singular if, and only if, $3 \mid n$ and $\gcd(i, n) \mid \frac{n}{3}$. ◻

As an immediate consequence of Proposition 2.3, it follows that the complement graph $C_n^c$, of $C_n$, is singular if, and only if, $3 \mid n$.

We now obtain necessary and sufficient conditions for nonsingularity of circulant graphs that appeared in [9]. For the sake of clarity, his notations have been modified. Fix a positive integer $n \ge 3$ and let $1 \le r < \lfloor \frac{n}{2} \rfloor$. The first class of circulant graphs, denoted $C_n^{(r)}$, has the same vertex set as the vertex set of the cycle $C_n$ and $(x, y)$ is an edge whenever the length of the shortest path from $x$ to $y$ in $C_n$ is at most $r$. Note that $C_n^{(\lfloor \frac{n}{2} \rfloor)}$ is the complete graph. The second class of graphs, denoted $C(2n, r)$ is a graph on $2n$ vertices and its adjacency matrix is the sum of the adjacency matrices of $C_{2n}^{(r)}$ and $C_{2n}^n$, where $1 \le r < n$. A necessary and sufficient condition for the graph $C(2n, r)$ to be singular appears later in Theorem 2.7. A similar result on $C_n^{(r)}$ appears as Theorem 2.2 of [9]. A separate proof is given for the sake of completeness.

THEOREM 2.4. [9] *Let $n \ge 3$ and let $1 \le r < \lfloor \frac{n}{2} \rfloor$. Then the graph $C_n^{(r)}$ is singular if, and only if, one of the following conditions hold:*

1. $\gcd(n, r) > 1$,
2. $\gcd(n, r) = 1$, *$n$ is even and* $\gcd(r + 1, n)$ *divides* $\frac{n}{2}$.

*Proof.* Note that $A = A(C_n^{(r)}) = circ(0, \underbrace{1, \ldots, 1}_{r}\ \underbrace{0, \ldots, 0}_{n-2r-1}\ \underbrace{1, \ldots, 1}_{r})$ and hence $(x - 1)\Gamma_A(x) = (x^r - 1)(1 + x^{n-r-1})$. Therefore, $C_n^{(r)}$ is singular if, and only if, $\Gamma_A(\zeta_n^d) = 0$, for some $d, 1 \le d \le n - 1$. Or equivalently either $(\zeta_n^d)^r - 1 = 0$ or $1 + (\zeta_n^d)^{n-r-1} = 0$.

But $(\zeta_n^d)^r - 1 = 0$ whenever $\gcd(r, n) > 1$. If $\gcd(r, n) = 1$ then $1 + (\zeta_n^d)^{n-r-1} = 0$ and this implies that $n$ is even and $\gcd(r + 1, n)$ divides $\frac{n}{2}$. ◻

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The following result follows from Remark 1.5.1.

Corollary 2.5. *Let* $n \geq 3$ *and let* $1 \leq r < \lfloor \frac{n}{2} \rfloor$. *Then the graph* $(C_n^{(r)})^c$ *is nonsingular if, and only if,* $\gcd(n, 2r+1) = 1$.

*Proof.* Note that $A = A((C_n^{(r)})^c) = circ(\underbrace{0,\ldots,0}_{r+1}\ \underbrace{1,\ldots,1}_{n-2r-1}\ \underbrace{0,\ldots,0}_{r})$ and hence $(x-1)\Gamma_A(x) = x^{n-2r-1} - 1$. Thus, using Remark 1.5, $(C_n^{(r)})^c$ is singular if, and only if, $\gcd(2r+1, n) > 1$. ▯

Before proceeding with the next result that gives a necessary and sufficient condition for the graph $C(2n, r)$ to be singular, we state a result that appears as Proposition 1 in Kurshan & Odlyzko [7]

Lemma 2.6. [7] *Let* $m$ *and* $n$ *be positive integers with* $m \neq n$ *and let* $\zeta_n$ *be a primitive root of unity. Then there exists a unit* $u \in \mathbb{Z}[\zeta_n]$ *dependent on* $m, n$ *and* $\zeta_n$ *such that*

$$\Phi_m(\zeta_n) = \begin{cases} pu, & \text{if } \frac{m}{n} = p^{\alpha}, \quad p \text{ a prime}, \quad \alpha > 0; \\ (1-\zeta_{p^{\alpha}})u, & \text{if } \frac{m}{n} = p^{-\alpha}, \quad p \text{ a prime}, \quad \alpha > 0; \quad p \nmid m; \\ (1-\zeta_{p^{\alpha+1}})^{p-1}u, & \text{if } \frac{m}{n} = p^{-\alpha}, \quad p \text{ a prime}, \quad \alpha > 0; \quad p \mid m; \\ u, & \text{otherwise.} \end{cases}$$

Theorem 2.7. *Let* $n$ *and* $r$ *be positive integers such that the circulant graph* $C(2n, r)$ *is well defined. Then the circulant graph* $C(2n, r)$ *is singular if, and only if,* $\gcd(n, 2r+1) \geq 3$.

*Proof.* Note that $A = A(C(2n, r)) = circ(0, \underbrace{1,\ldots,1}_{r}, \underbrace{0,\ldots,0}_{n-r-1}, 1, \underbrace{0,\ldots,0}_{n-r-1}, \underbrace{1,\ldots,1}_{r})$. Consequently, $\gamma_A(x) = x + x^2 + \cdots + x^r + x^n + x^{2n-r} + \cdots + x^{2n-1} = x\Gamma_A(x)$ and

$$\begin{aligned}(x-1)\Gamma_A(x) &= x^r - 1 + x^{n-1}(x-1) + x^{2n-r-1}(x^r - 1) \\ &= x^r(1 - x^{2n-2r-1}) + (x^n - 1) - (x^{n-1} - x^{2n-1}) \\ &= (x^n - 1)(x^{n-1} + 1) - x^r(x^{2n-2r-1} - 1).\end{aligned}$$

Now, let us assume that $\gcd(n, 2r+1) = d \geq 3$. Then $(\zeta_{2n}^{2n/d} - 1)\Gamma_A(\zeta_{2n}^{2n/d}) = 0$ as

$$\left(\zeta_{2n}^{2n/d}\right)^n = \left(\zeta_{2n}^{2n}\right)^{n/d} = 1 = \left(\zeta_{2n}^{2n}\right)^{(2r+1)/d} = \left(\zeta_{2n}^{2n/d}\right)^{2r+1} = \left(\zeta_{2n}^{2n/d}\right)^{2n-2r-1}.$$

Hence, the circulant graph $C(2n, r)$ is singular.

Conversely, let us assume that the graph $C(2n, r)$ is singular. This implies that there exists an eigenvalue of $C(2n, r)$ that equals zero. That is, there exists a $k$, $1 \leq k \leq 2n-1$, such that $\gamma_A(\zeta_{2n}^k) = 0$. We will now show that if $\gcd(n, 2r+1) = 1$

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then the expression $(x-1)\Gamma_A(x)$ evaluated at $x=\zeta_{2n}^k$ can never equal zero, for any $k,\ 1\le k\le 2n-1$, and this will complete the proof of the result.

We need to consider two cases depending on whether $k$ is odd or $k$ is even. Let $k=2m,\ \ 1\le m<n$. Then, using $\gcd(n,2r+1)=1$, $(\zeta_{2n}^{2m}-1)\Gamma_A(\zeta_{2n}^{2m})$ equals $-\left(\zeta_{2n}^{2m}\right)^r\left[\left(\zeta_{2n}^{2m}\right)^{-(2r+1)}-1\right]\neq 0$.

Now, let $k=2m+1$, $0\le m\le n-1$. Then $(\zeta_{2n}^{2m+1}-1)\Gamma_A(\zeta_{2n}^{2m+1})$ equals

$$\begin{aligned}
&\left[\left(\zeta_{2n}^{2m+1}\right)^n-1\right]\left[\left(\zeta_{2n}^{2m+1}\right)^{(n-1)}+1\right]-\left(\zeta_{2n}^{2m+1}\right)^r\left[\left(\zeta_{2n}^{2m+1}\right)^{2n-2r-1}-1\right]\\
&\qquad=-2\left[-\zeta_{2n}^{-(2m+1)}+1\right]-\zeta_{2n}^{-(2m+1)(r+1)}\left[1-\zeta_{2n}^{(2m+1)(2r+1)}\right]\\
&\qquad=-\frac{\zeta_{2n}^{2m+1}-1}{\zeta_{2n}^{(2m+1)(r+1)}}\left[-2\zeta_{2n}^{(2m+1)r}+\frac{\zeta_{2n}^{(2m+1)(2r+1)}-1}{\zeta_{2n}^{(2m+1)}-1}\right]\\
&\qquad=-\frac{\zeta_{2n}^{2m+1}-1}{\zeta_{2n}^{(2m+1)(r+1)}}\left[-2\zeta_{2n}^{(2m+1)r}+\prod_{\ell|(2r+1),\ell\neq 1}\Phi_\ell(\zeta_{2n}^{2m+1})\right]. \qquad (2.1)
\end{aligned}$$

Note that, $\zeta_{2n}^{2m+1}$ is a $d$-th primitive root of unity, for some $d$ dividing $2n$. As $\gcd(2r+1,2n)=1$, $\gcd(2r+1,d)=1$ and hence using Lemma 2.6, $\prod\limits_{\ell|(2r+1),\ell\neq 1}\Phi_\ell(\zeta_{2n}^{2m+1})$ is a unit in $\mathbb{Z}[\zeta_d]$. That is, $\left|\prod\limits_{\ell|(2r+1),\ell\neq 1}\Phi_\ell(\zeta_{2n}^{2m+1})\right|=1$. Hence, in equation (2.1), the term in the parenthesis cannot be zero. Thus, the result for the odd case as well. ☐

Note that the necessary part of Theorem 2.7 was stated and proved as Theorem 2.1 in [9]. We will now try to understand the complement graph $C(2n,r)^c$ of $C(2n,r)$.

Proposition 2.8. *Let $n$ and $r$ be positive integers such that the circulant graph $C(2n,r)$ is well defined. Then $C(2n,r)^c$ is nonsingular if, and only if,*

1. *$n$ and $r$ have the same parity,*
2. *$\gcd(n,r+1)=1$, and*
3. *the highest power of 2 dividing $n$ is strictly smaller than the highest power of 2 dividing $n-r$.*

*Proof.* Note that $A=A(C(2n,r)^c)=circ(\underbrace{0,\dots,0}_{r+1}\ \underbrace{1,\dots,1}_{n-r-1}\ 0\ \underbrace{1,\dots,1}_{n-r-1}\ \underbrace{0,\dots,0}_{r})$. Thus, $(x-1)\Gamma_A(x)=(1+x^{n-r})(x^{n-r-1}-1)$. Let us now assume that the graph $C(2n,r)^c$ is nonsingular. That is, $\Gamma_A(\zeta_{2n}^k)\neq 0$, for any $k=1,2,\dots,2n-1$.

Note that if $n$ and $r$ have opposite parity then $\gcd(2n,n-r-1)=d\ge 2$ and hence $\Gamma_A(\zeta_{2n}^{2n/d})=0$. Also, if $n$ and $r$ have the same parity and $\gcd(n,r+1)=d>2$ then $n-r-1$ is odd and $\gcd(2n,n-r-1)=\gcd(n,n-r-1)=\gcd(n,r+1)=d$.

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Hence, in this case again, $\Gamma_A(\zeta_{2n}^{2n/d}) = 0$.

Now, the only case that we need to check is "$n$ and $r$ have the same parity, $\gcd(n, r+1) = 1$ and the highest power of 2 dividing $n$ is greater than or equal to the highest power of 2 dividing $n - r$".

As $n$ and $r$ have the same parity and $\gcd(n, r+1) = 1$, we get $\gcd(2n, n-r-1) = 1$ and thus $(\zeta_{2n}^k)^{n-r-1} - 1 \neq 0$, for any $k = 1, 2, \ldots, 2n-1$. Therefore, we need to find a condition on $k$ so that $1 + (\zeta_{2n}^k)^{n-r} = 0$. But this is true if, and only if, $\gcd(2n, n-r) \mid n$, or equivalently, the highest power of 2 dividing $n$ is greater than or equal to the highest power of 2 dividing $n - r$. ∎

REMARK 2.9. Observe that using Proposition 2.8, the graph $C(2n, r)^c$ is non-singular, whenever $n$ and $r$ are both odd and $\gcd(n, r+1) = 1$. Such numbers can be easily computed. For example, a class of such graphs can be obtained by choosing two positive integers $s$ and $t$ with $s > t$ and defining $n = 2^s - 2^t + 1$ and $r = 2^t - 1$.

## 3. Generalizations.

In this section, we look at a few classes of graphs/digraphs, which are generalizations of the graphs that appear in Section 2. We first start with a class of circulant digraphs.

Let $A = circ(\mathbf{q})$, where $\mathbf{q} = (\underbrace{1, \ldots, 1}_{r}, \underbrace{0, \ldots, 0}_{s}, \underbrace{1, \ldots, 1}_{t}, \underbrace{0, \ldots, 0}_{n-(r+t+s)})$, for certain non-negative integers $r, s, t$ and $n$. Let the corresponding digraphs be denoted $C_{r,s,t}$. Then a few conditions under which the $C_{r,s,t}$ digraphs are singular appears next.

PROPOSITION 3.1. *The $C_{r,s,t}$ digraph is singular if*

1. $\gcd(n, t, r) > 1$, *or*
2. $\gcd(n, t) = 1$ *and one of the following condition holds:*
   (a) *there exists $d \geq 2$ such that $d \mid s$ and $t = \ell r$, for some positive integer $\ell \equiv -1 \pmod{d}$.*
   (b) *$n$ is even, there exists an even integer $d$ such that $(r + s)$ is an odd multiple of $\frac{d}{2}$ and $t = \ell r$, for some positive integer $\ell \equiv 1 \pmod{d}$.*

*Proof. Part 1:* Observe that if $A = A(C_{r,s,t})$ then

$$(x-1)\gamma_A(x) = (x^r - 1) + x^{r+s}(x^t - 1). \tag{3.1}$$

If $\gcd(n, r, t) = k > 1$, then $\zeta_n^{n/k}$ is a root of equation (3.1), and hence, the $C_{r,s,t}$ digraph is singular. This completes the proof of the first part.

*Part 2a:* Let $\gcd(n, t) = 1$ and suppose that there exists a positive integer $d \geq 2$ such that $d \mid s$ and $t = \ell r$, for some positive integer $\ell \equiv -1 \pmod{d}$. So, there exists $\beta \in \mathbb{Z}$ such that $\ell = \beta d - 1$. In this case, the RHS of equation (3.1) evaluated at $\zeta_n^{n/d}$ reduces to 0. Thus, the required result holds.

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*Part 2b:* Let $\gcd(n,t) = 1$ and $n = 2m$. Also, suppose that there exists an even positive integer $d$ such that $r+s$ is an odd multiple of $\frac{d}{2}$ and $t = \ell r$, for some positive integer $\ell \equiv 1 \pmod{d}$. Then there exists $\beta \in \mathbb{Z}$ such that $\ell = \beta d + 1$. In this case, the RHS of equation (3.1) evaluated at $\zeta_n^{n/d}$ reduces to $(\zeta_n^{rn/d} - 1)\left(1 + \zeta_n^{n/2)}\right)$. Thus, under the given conditions, the corresponding digraph $C_{r,s,t}$ is singular.

Hence, the proof of the lemma is complete. □

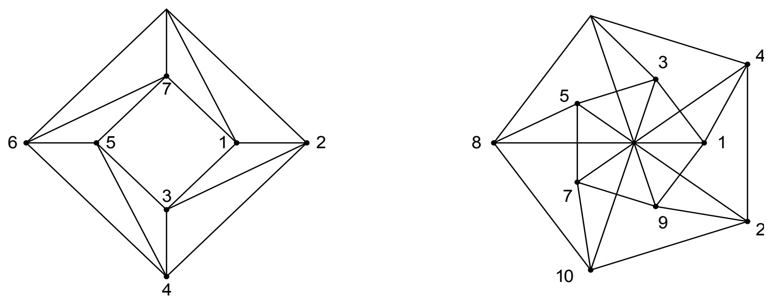


FIG. 3.1. *Singular circulant* $(2,1,2)$*-graph on* 8 *vertices and* $(2,3,2)$*-graph on* 10 *vertices.*

Thus, the above result gives conditions under which the $C_{r,s,t}$-digraphs, for non-negative values of $r, s$ and $t$, are singular. For certain values of $r, s$ and $t$, these digraphs correspond to graphs. For example, for a fixed positive integer $r$, the

1. $C_{r,1,r}$-digraph is singular, whenever $n$ is chosen so that $\gcd(n,r) \geq 2$ and $n \geq 2r+1$. Using Remark 1.5.2, one obtains a circulant graph with adjacency matrix $circ(0, \underbrace{1,\ldots,1}_{r}, \underbrace{0,\ldots,0}_{n-(2r+1)}, \underbrace{1,\ldots,1}_{r})$. In particular, for $r = 2$ and $n = 8$, the $C_{2,1,2}$-graph shown in Figure 3.1 is singular.
2. $C_{2r,2r+1,2r}$-digraph is singular, whenever $n$ is even and $n \geq 6r+1$. In this case, $circ(\underbrace{0,\ldots,0}_{r+1}, \underbrace{1,\ldots,1}_{2r}, \underbrace{0,\ldots,0}_{n-(6r+1)}, \underbrace{1,\ldots,1}_{2r}, \underbrace{0,\ldots,0}_{r})$ is the adjacency matrix of a circulant graph. In particular, for $r = 1$ and $n = 10$, the $C_{2,3,2}$-graph shown in Figure 3.1 is singular.

We now define another class of circulant digraphs and obtain conditions under which the circulant digraphs are singular. Let $i, j, k$ and $\ell$ be non-negative integers such that $j > \ell$ and $kj + i + \ell < n$ and let $C_{i,j,k,\ell}$ be a circulant digraph on $n$ vertices with $\gamma_{A(C_{i,j,k,\ell})}(x) = \sum\limits_{t=0}^{k} \sum\limits_{s=i}^{i+\ell} x^{s+tj} = x^i \prod_{s|\ell+1, s\neq 1} \Phi_s(x) \cdot \prod_{t|(k+1)j, t\nmid j} \Phi_t(x)$ as its representer polynomial. Thus, we have the following theorem which we state without proof.

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THEOREM 3.2. *The circulant digraph $C_{i,j,k,\ell}$, defined above, is singular if, and only if, either* $\gcd(\ell+1,n) \geq 2$ *or* $\gcd\left(k+1, \frac{n}{\gcd(n,j)}\right) \geq 2$.

REMARK 3.3. Note that we can vary the non-negative integers $i, j, k$ and $\ell$ to define quite a few classes of circulant digraphs. For example, the graphs $G(r,t)$ in [3] are a particular case of $C_{i,j,k,\ell}$ and Theorem 3.2 is a generalization of Remark 1.5.1.

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REFERENCES

[1] J.A. Bondy and U.S.R. Murty. *Graph Theory with Applications.* The Macmillan Press Ltd., London, 1976.
[2] Ph.J. Davis. *Circulant Matrices.* Wiley, New York, 1979.
[3] M. Doob. Circulant graphs with $\det(-A(G)) = -\deg(G)$: Codeterminants with $K_n$. *Linear Algebra and its Applications*, 340:87–96, 2002.
[4] D.S. Dummit and R.M. Foote. *Abstract Algebra*, second edition. John Wiley and Sons, 1999.
[5] D. Geller, I. Kra, S. Popescu, and S. Simanca. On circulant matrices. Availiable at `http://www.math.sunysb.edu/~sorin/eprints/circulant.pdf`.
[6] I. Kra and S. Simanca. On circulant matrices. *Notices of the American Mathematical Society*, 59(3):368–377, 2012.
[7] R.P. Kurshan and A.M. Odlyzko. Values of cyclotomic polynomials at roots of unity. *Mathematica Scandinavica*, 49:15–35, 1981.
[8] V.V. Prasolov. *Polynomials.* Springer, 2001.
[9] L. Aquino-Ruivivar. Singular and nonsingular circulant graphs. *Journal of Research in Science, Computing and Engineering*, 3(3), 2006.