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ON THE IRREDUCIBILITY, SELF-DUALITY, AND NON-HOMOGENEITY OF COMPLETELY POSITIVE CONES*

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Abstract. For a closed cone $\mathcal{C}$ in $\mathbb{R}^n$, the completely positive cone of $\mathcal{C}$ is the convex cone $\mathcal{K}_\mathcal{C}$ in $\mathcal{S}^n$ generated by $\{uu^T : u \in \mathcal{C}\}$. Such a cone arises, for example, in the conic LP reformulation of a nonconvex quadratic minimization problem over an arbitrary set with linear and binary constraints. Motivated by the useful and desirable properties of the nonnegative orthant and the positive semidefinite cone (and more generally of symmetric cones in Euclidean Jordan algebras), this paper investigates when (or whether) $\mathcal{K}_\mathcal{C}$ can be irreducible, self-dual, or homogeneous.

Key words. Copositive cones, Completely positive cones, Self-dual, Irreducible cone, Homogeneous cone.

AMS subject classifications. 15B48, 90C20.

1. Introduction. Given a closed cone $\mathcal{C}$ in $\mathbb{R}^n$ that is not necessarily convex, we consider two related cones in the space $\mathcal{S}^n$ of all $n \times n$ real symmetric matrices:

The *completely positive cone of* $\mathcal{C}$ defined by

$$\mathcal{K}_\mathcal{C} := \left\{ \sum uu^T : u \in \mathcal{C} \right\}, \tag{1.1}$$

where the sum denotes a finite sum, and the *copositive cone of* $\mathcal{C}$ given by

$$\mathcal{E}_\mathcal{C} := \{A \in \mathcal{S}^n : x^T Ax \geq 0, \forall\, x \in \mathcal{C}\}. \tag{1.2}$$

Such cones or their generalizations have been previously studied in the literature, see for instance, [13], [14], [21] and the references therein. We note that copositive cones are also called set-semidefinite cones [15] and completely positive cones are also called generalized completely positive cones [7].

If $\mathcal{C}$ is the cone generated by the standard unit vectors $e_1, e_2, \ldots, e_n$ in $\mathbb{R}^n$, then $\mathcal{K}_\mathcal{C}$ is the cone of nonnegative diagonal matrices in $\mathcal{S}^n$ (which is isomorphic to the nonnegative orthant); the corresponding $\mathcal{E}_\mathcal{C}$ is the cone of matrices in $\mathcal{S}^n$ with nonnegative

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diagonal. When $\mathcal{C} = \mathbb{R}^n$, both cones $\mathcal{K}_\mathcal{C}$ and $\mathcal{E}_\mathcal{C}$ are equal to $\mathcal{S}^n_+$ (the cone of positive semidefinite matrices), which is the underlying cone in semidefinite programming [23] and semidefinite linear complementarity problems [17], [18]. In the case of $\mathcal{C} = \mathbb{R}^n_+$ (the nonnegative orthant), these cones reduce, respectively, to the cones of completely positive matrices and copositive matrices which have appeared prominently in statistical and graph theoretic literature [4] and in copositive programming [11]. In a path-breaking work, Burer [5] showed that a nonconvex quadratic minimization problem over the nonnegative orthant with some additional linear and binary constraints can be reformulated as a linear program over the cone of completely positive matrices. Since then, a number of authors have investigated the properties of the cone of completely positive matrices, specifically describing the interior and facial structure of the cones of completely positive and copositive matrices, see [8], [9], [12].

The work of Burer has been recently extended to the case of an arbitrary closed convex cone (in place of the nonnegative orthant) by Burer [6] and more generally to an arbitrary nonempty set by Eichfelder and Povh [15] (with corrections in Dickinson, Eichfelder and Povh [10]). To elaborate, let $M \in \mathcal{S}^n$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, G be an arbitrary nonempty set in $\mathbb{R}^n$, and let $J \subseteq \{1, 2, \ldots, n\}$. Then, it is shown that under certain conditions, the quadratic optimization problem

$$\begin{gathered} \min\, x^T M x + 2c^T x \\ \text{such that} \\ Ax = b, \\ x_j \in \{0,1\} \text{ for all } j \in J, \\ x \in G, \end{gathered}$$

can be reformulated as a linear programming problem over a completely positive cone in $\mathcal{S}^{n+1}$:

$$\begin{gathered} \min\, \langle \widehat{M}, Y\rangle \\ L(Y) = B \\ Y \in \mathcal{K}_\mathcal{C}, \end{gathered}$$

where $B \in \mathcal{S}^{n+1}$, L is linear on $\mathcal{S}^{n+1}$,

$$\widehat{M} = \begin{bmatrix} 0 & c^T \\ c & M \end{bmatrix}, \quad \mathcal{C} = \overline{\text{cone}\,(\{1\}\times G)}, \quad \text{and} \quad \mathcal{K}_\mathcal{C} = \left\{\sum uu^T : u \in \mathcal{C}\right\}.$$

The above reformulation demonstrates the importance of studying completely positive cones $\mathcal{K}_\mathcal{C}$ that come from (general) closed cones $\mathcal{C}$ in $\mathbb{R}^n$. Motivated by the useful and desirable properties (such as self-duality and homogeneity) of the nonnegative orthant and the positive semidefinite cone (and more generally of symmetric cones in Euclidean Jordan algebras [16]), in this paper, we address the questions of

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when or whether $\mathcal{K}_{\mathcal{C}}$ can be irreducible, self-dual, or homogeneous. We show, for example,

- $\mathcal{K}_{\mathcal{C}}$ is irreducible when $\mathcal{C}$ has nonempty interior or $\mathcal{C} \setminus \{0\}$ is connected,
- $\mathcal{K}_{\mathcal{C}}$ is self-dual in $\mathcal{S}^n$ if and only if $\mathbb{R}^n = \mathcal{C} \cup -\mathcal{C}$, or equivalently, $\mathcal{K}_{\mathcal{C}} = \mathcal{S}^n_+$, and
- $\mathcal{K}_{\mathcal{C}}$ is non-homogeneous when $\mathcal{C}$ is a proper (convex) cone.

We prove similar results for the copositive cone $\mathcal{E}_{\mathcal{C}}$.

Here is an outline of this paper. In the next section, we cover basic definitions, examples, and results. Section 3 deals with some elementary properties of completely positive cones, including the description of extreme vectors and interior. In Sections 4 and 5, we discuss, respectively, the irreducibility and self-duality properties of $\mathcal{K}_{\mathcal{C}}$. Section 6 deals with the non-homogeneity property of $\mathcal{K}_{\mathcal{C}}$. Our final section deals with properties of the copositive cone $\mathcal{E}_{\mathcal{C}}$.

2. Preliminaries. Throughout this paper, H denotes either $\mathbb{R}^n$ or $\mathcal{S}^n$. In the case of $\mathbb{R}^n$, vectors are regarded as column vectors and the usual inner product is written as $\langle x, y\rangle$ or as $x^T y$. The standard unit vectors in $\mathbb{R}^n$ are denoted by $e_1, e_2, \ldots, e_n$; thus, e_i has one in the ith slot and zeros elsewhere. The space $\mathcal{S}^n$ – consisting of all real $n \times n$ symmetric matrices – carries the trace inner product $\langle X, Y\rangle = trace(XY)$, where the trace of a matrix is the sum of its diagonal elements. $\mathbb{R}^n_+$ denotes the nonnegative orthant in $\mathbb{R}^n$ and $\mathcal{S}^n_+$ denotes the set of all positive semidefinite matrices in $\mathcal{S}^n$.

For a set K in H, $\operatorname{int}(K)$, $\overline{K}$, and $K^\perp$ denote, respectively, the interior, closure, and orthogonal complement of K. The subspace generated by K is denoted by $\operatorname{span}(K)$. We let

$$\operatorname{cone}(K) = \{\lambda x : \lambda \geq 0,\, x \in K\}.$$

The dual of K is given by

$$K^* := \{y \in H : \langle y, x\rangle \geq 0 \ \forall x \in K\}.$$

We say that two sets U and V in H are *separated* if $\overline{U} \cap V = \emptyset = U \cap \overline{V}$. We recall the well known definition: A set in H is *connected* if it cannot be written as the union of two nonempty separated sets.

With $\mathcal{L}(H)$ denoting the Banach space of all (bounded) linear transformations on H with operator norm, we let, for any nonempty set K in H,

- $\pi(K) := \{L \in \mathcal{L}(H) : L(K) \subseteq K\}$.
- $\operatorname{Aut}(K) := \{L \in \mathcal{L}(H) : L \text{ is invertible and } L(K) = K\}$.

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We denote the closure of $\mathrm{Aut}\,(K)$ in $\mathcal{L}(H)$ by $\overline{\mathrm{Aut}\,(K)}$. It is easy to see that

$$L(K) \subseteq K \Rightarrow L^T(K^*) \subseteq K^* \quad \text{and} \quad L \in \mathrm{Aut}\,(K) \Rightarrow L^T \in \mathrm{Aut}\,(K^*).$$

A nonempty set K is a *cone* if $K = \mathrm{cone}\,(K)$. A closed cone K in H is said to be

(a) *pointed* if $K \cap -K = \{0\}$;
(b) *proper* if K is convex, pointed, and has nonempty interior;
(c) *self-dual* if $K = K^*$;
(d) *reducible* if there exist closed cones $K_1 \neq \{0\}$ and $K_2 \neq \{0\}$ such that

$$K = K_1 + K_2, \quad \mathrm{span}(K_1) \cap \mathrm{span}(K_2) = \{0\}.$$

If K is not reducible, we say that it is *irreducible.* In the literature on convex cones, terms like 'decomposable cone' and 'indecomposable cone' are also used;
(e) *homogeneous* if it is *proper* and for any two elements $x, y \in \mathrm{int}(K)$, there exists $L \in \mathrm{Aut}\,(K)$ such that $L(x) = y$;
(f) *symmetric* if it is self-dual and homogeneous.

For a convex cone K, we denote the set of all extreme vectors by $\mathrm{Ext}(K)$. Recall that a nonzero vector x in K is an *extreme vector* if the equality $x = y + z$ with $y, z \in K$ holds only when y and z are nonnegative multiples of x.

Throughout this paper, we assume that

- K is an arbitrary nonempty set in H,
- $\mathcal{C}$ is a closed cone in $\mathbb{R}^n$ that is not necessarily convex, and
- the associated cones $\mathcal{K}_{\mathcal{C}}$ and $\mathcal{E}_{\mathcal{C}}$ in $\mathcal{S}^n$ are given, respectively, by (1.1) and (1.2).

Our results in the paper are obtained under various conditions on $\mathcal{C}$ such as (i) $\mathcal{C}$ is pointed, (ii) $\mathcal{C}$ has interior, (iii) $\mathcal{C}^*$ has interior, (iv) $\mathcal{C}\backslash\{0\}$ is connected, (v) $\mathrm{int}(\mathcal{C})$ is connected. *Note that when $\mathcal{C}$ is a proper cone, all the above conditions hold.* In what follows, we provide a few examples of cones having some of the above properties. Our examples are of the form $\mathcal{C} = \overline{\mathrm{cone}\,(\{1\} \times G)}$, for some closed nonempty set G in $\mathbb{R}^{n-1}$. For such cones, we have a basic result from [1], Lemma 2.1.1: For every nonempty closed set G in $\mathbb{R}^n$ it holds that,

$$\overline{\mathrm{cone}\,(\{1\} \times G)} = \mathrm{cone}\,(\{1\} \times G) \cup (\{0\} \times G_\infty), \tag{2.1}$$

where G_∞ is the asymptotic cone of G (defined as the set of all x for which there exist sequences x_k in G, λ_k in $\mathbb{R}_+$ such that $\lambda_k \to 0^+$ and $\lambda_k x_k \to x$).

An immediate consequence of (2.1) is that when G is a closed cone in $\mathbb{R}^n$, we have $\mathcal{C} = \overline{\mathrm{cone}\,(\{1\} \times G)} = \mathbb{R}_+ \times G$. In this setting, $\mathrm{int}(\mathcal{C}) = \mathbb{R}_{++} \times \mathrm{int}(G)$, $\mathcal{C}^* = \mathbb{R}_+ \times G^*$,

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and $\mathrm{int}(C^*) = \mathbb{R}_{++} \times \mathrm{int}(G^*)$, where $\mathbb{R}_{++}$ denotes the set of all positive real numbers. It follows that $\mathcal{C}$ inherits certain properties of G. For example, if G is pointed, then so is $\mathcal{C}$; if G (G^*) has nonempty interior, then so does $\mathcal{C}$ (respectively, $\mathcal{C}^*$); if $\mathrm{int}(G)$ is connected, then so is $\mathrm{int}(\mathcal{C})$. Also, $\mathcal{C}\backslash\{0\}$ and $\mathcal{C}^*\backslash\{0\}$ are always (path) connected.

Example 2.1. Let G (inside $\mathbb{R}^2_+$) be the union of two closed convex cones G_1 and G_2, where G_1 is generated by $(1,0)$ and $(2,1)$, and G_2 is generated by $(0,1)$ and $(1,2)$. Then $\mathcal{C} = \overline{\mathrm{cone}\,(\{1\} \times G)} = \mathbb{R}_+ \times G$ is pointed, has nonempty interior, $\mathcal{C}\backslash\{0\}$ is connected, and $\mathrm{int}(\mathcal{C}^*) = \mathbb{R}_{++} \times \mathrm{int}(G^*)$ is nonempty.

Another consequence of (2.1) is:

When G is compact in $\mathbb{R}^n$, we have $\mathcal{C} = \overline{\mathrm{cone}\,(\{1\} \times G)} = \mathrm{cone}\,(\{1\} \times G)$.

Example 2.2. Let G be the closed unit ball in $\mathbb{R}^n$ (with respect to the Euclidean norm). Then $\mathcal{C} = \mathrm{cone}\,(\{1\} \times G)$ is the so-called ice-cream cone (or the second order cone) in $\mathbb{R}^{n+1}$.

Example 2.3. Let G in $\mathbb{R}^2$ be the union of the closed unit disc (centered at the origin) and a nonempty finite set of points outside this disc. In this case, $\mathcal{C} = \mathrm{cone}\,(\{1\} \times G)$ is pointed, has nonempty interior, $\mathcal{C}\backslash\{0\}$ is not connected, $\mathcal{C}^*$ has nonempty interior, and $\mathrm{int}(\mathcal{C})$ is connected.

Example 2.4. Let G in $\mathbb{R}^2$ be the union of closed unit disc and a finite nonempty set of rays emanating from the origin. In this case, for an appropriate choice of rays of G, $\mathcal{C} = \overline{\mathrm{cone}\,(\{1\} \times G)}$ is pointed, has nonempty interior, $\mathcal{C}\backslash\{0\}$ is connected, $\mathcal{C}^*$ (which is just a ray) has empty interior, and $\mathrm{int}(C)$ is connected.

2.1. Some basic lemmas. In this section, we present some lemmas that are needed in the paper. Although these lemmas are specialized for $\mathbb{R}^n$ and $\mathcal{S}^n$, they are valid in any finite dimensional real Hilbert space. The first lemma is well known and easy to prove (see the proof of Theorem 2.2 in [8]) and is similar to Lemma 5.6 in [13].

Lemma 2.5. *Suppose $\mathcal{C}$ is a closed cone in $\mathbb{R}^n$ with nonempty interior and $A \in \mathcal{E}_{\mathcal{C}}$. Let $u \in \mathrm{int}(\mathcal{C})$ with $u^TAu = 0$. Then $A \in \mathcal{S}^n_+$ and $Au = 0$.*

Lemma 2.6. *Suppose K is a closed pointed cone in H with nonempty interior and $L \in \pi(K)$. If $L(u) = 0$ for some $u \in \mathrm{int}(K)$, then $L = 0$.*

Proof. Let $x \in H$ and $u \in \mathrm{int}(K)$ with $L(u) = 0$. Then for all small $\varepsilon > 0$, $u + \varepsilon\, x, u - \varepsilon\, x \in K$. Since $L \in \pi(K)$, we must have $\varepsilon L(x) = L(u + \varepsilon\, x) \in K$ and $-\varepsilon L(x) = L(u - \varepsilon\, x) \in K$. Thus, $L(x) \in K \cap -K = \{0\}$. Since x is arbitrary, we see

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that $L = 0$. $\square$

Lemma 2.7. *Let K be closed cone in H such that K and K^* have nonempty interiors. Suppose $L \in \overline{\mathrm{Aut}\,(K)}$ such that for some $d \in \mathrm{int}(K)$, we have $L(d) \in \mathrm{int}(K)$. Then $L \in \mathrm{Aut}\,(K)$. In particular, this assertion holds if K is a proper cone.*

Proof. Let L_k be a sequence in $\mathrm{Aut}\,(K)$ such that $L_k \to L$ in $\mathcal{L}(H)$. Then $L_k^T \to L^T$. Note that $L_k \in \mathrm{Aut}\,(K) \Rightarrow L_k^T \in \mathrm{Aut}\,(K^*)$. Fix $u \in \mathrm{int}(K^*)$ and let $x_k := (L_k^T)^{-1}(u)$ for all $k = 1, 2, \ldots$. Then $x_k \in K^*$ and $L_k^T(x_k) = u$ for all $k = 1, 2, \ldots$. We claim that the sequence x_k is bounded. Assuming the contrary, let, without loss of generality, $||x_k|| \to \infty$ and $\lim \frac{x_k}{||x_k||} = y \in K^*$. Then $L^T(y) = 0$ and $0 = \langle L^T(y), d\rangle = \langle y, L(d)\rangle > 0$ (the last inequality holds since K is a closed cone, $0 \neq y \in K^*$ and $L(d) \in \mathrm{int}(K)$), which is a contradiction. Now, as x_k is bounded, we may assume that $x_k \to x \in K^*$. Then $L^T(x) = u$. Since u is arbitrary in $\mathrm{int}(K^*)$, this means that the range of L^T contains an open set and, consequently, equals H. Thus, L^T is an onto transformation on the finite dimensional space H and hence invertible. It follows that L is invertible. From $L_k \to L$ we have $L_k^{-1} \to L^{-1}$ (see [2], p. 11). Since for all k, we have $L_k(K) \subseteq K$ and $L_k^{-1}(K) \subseteq K$, it follows that $L(K) \subseteq K$ and $L^{-1}(K) \subseteq K$. Thus, $L \in \mathrm{Aut}\,(K)$. $\square$

As a simple consequence, we have:

Corollary 2.8. *Suppose $\mathcal{C}$ is a closed cone in $\mathbb{R}^n$ $(n > 1)$, such that $\mathcal{C}$ and $\mathcal{C}^*$ have nonempty interiors. Then for any $v \in \mathrm{int}(\mathcal{C})$, $vv^T \notin \overline{\mathrm{Aut}\,(\mathcal{C})}$.*

Remark 2.9. The above lemma and its corollary may not hold for a closed cone whose dual has empty interior. For example, when K is the closed upper-half plane in $\mathbb{R}^2$, every element of $\mathrm{Aut}\,(K)$ is of the form (see Example 4 in [19])

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix},$$

where $a \neq 0$ and $c > 0$. Clearly, the standard unit vector $e_2 \in \mathrm{int}(K)$ and $e_2e_2^T \in \overline{\mathrm{Aut}\,(K)}$, but not invertible.

3. Some elementary properties of completely positive and copositive cones. In this section, we collect some elementary properties of cones $\mathcal{K}_\mathcal{C}$ and $\mathcal{E}_\mathcal{C}$.

Proposition 3.1. *Let $\mathcal{C}$ be a closed cone in $\mathbb{R}^n$. Then the following statements hold:*

(i) $\mathcal{E}_\mathcal{C}$ *and* $\mathcal{K}_\mathcal{C}$ *are closed convex cones in* $\mathcal{S}^n$, *and* $\mathcal{K}_\mathcal{C} \subseteq \mathcal{S}^n_+ \subseteq \mathcal{E}_\mathcal{C}$.
(ii) $\mathcal{K}_\mathcal{C}$ *is pointed.*
(iii) $\mathcal{E}_\mathcal{C}$ *is the dual of* $\mathcal{K}_\mathcal{C}$.
(iv) $\mathrm{Ext}(\mathcal{K}_\mathcal{C}) = \left\{uu^T : 0 \neq u \in \mathcal{C}\right\}$.

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(v) $\mathrm{int}(\mathcal{C}) \neq \emptyset \Rightarrow \mathcal{K}_\mathcal{C}$ *(equivalently,* $\mathcal{E}_\mathcal{C}$*) is proper* $\Rightarrow \mathrm{span}(\mathcal{C}) = \mathbb{R}^n$.

(vi) *If* $\mathcal{C}$ *is also convex, then the reverse implications in* (v) *hold.*

Proof. Statements $(i) - (iv)$ and the first implication in (v) are covered in [19], Propositions 5 and 7. Suppose that $\mathcal{K}_\mathcal{C}$ is proper. (Since a closed convex cone is proper if and only if its dual is proper [3], we see that this is equivalent to $\mathcal{E}_\mathcal{C}$ being proper.) If (the subspace) $\mathrm{span}(\mathcal{C}) \neq \mathbb{R}^n$, then there exists a nonzero v in $\mathrm{span}(\mathcal{C})^\perp$. Let $A := vv^T$. Then A is nonzero and $\langle A, uu^T \rangle = 0$ for all $u \in \mathcal{C}$. This implies that $A, -A \in (\mathcal{K}_\mathcal{C})^* = \mathcal{E}_\mathcal{C}$ contradicting the properness of $\mathcal{E}_\mathcal{C}$. Hence, $\mathrm{span}(\mathcal{C}) = \mathbb{R}^n$ proving (v). Now, to see (vi), assume that $\mathcal{C}$ is also convex and suppose that $\mathrm{span}(\mathcal{C}) = \mathbb{R}^n$. Then $\mathcal{C}$ will have nonempty relative interior in $\mathrm{span}(\mathcal{C}) = \mathbb{R}^n$. This means that $\mathcal{C}$ has nonempty interior. $\square$

REMARK 3.2. Here we provide examples to show that the reverse implications in (v) may not hold for a general closed cone. (i) For $\mathcal{C} = \mathrm{cone}\,\{e_1, e_2, \ldots, e_n\}$ in $\mathbb{R}^n$, $\mathcal{K}_\mathcal{C}$ is the cone of all nonnegative diagonal matrices in $\mathcal{S}^n$. Although, $\mathrm{span}(\mathcal{C}) = \mathbb{R}^n$, $\mathcal{K}_\mathcal{C}$ is not proper in $\mathcal{S}^n$. (ii) Let $\mathcal{C}$ be the boundary of $\mathbb{R}^n_+$ in $\mathbb{R}^n$. The interior of this cone is empty. Clearly, $\mathcal{K}_\mathcal{C}$ is pointed (see statement (ii) in the above proposition). As $\mathcal{K}_\mathcal{C}$ is convex and contains the basis $\{e_i e_i^T : 1 \leq i \leq n\} \cup \{(e_i + e_j)(e_i + e_j)^T : 1 \leq i < j \leq n\}$ of $\mathcal{S}^n$, $\mathcal{K}_\mathcal{C}$ has nonempty interior. Thus, $\mathcal{K}_\mathcal{C}$ is proper.

The proof of the following result is similar to (and an extension of) the one given for the completely positive cone of $\mathbb{R}^n_+$, see [4], [8],[12].

THEOREM 3.3. *Let* $\mathcal{C}$ *be a closed cone with nonempty interior. Then* $\mathrm{int}(\mathcal{K}_\mathcal{C}) = \mathcal{M}$*, where*

$$\mathcal{M} = \left\{ \sum_{i=1}^{N} u_i u_i^T : \mathrm{span}\{u_1, \ldots, u_N\} = \mathbb{R}^n, u_i \in \mathcal{C}\ \forall\, i \text{ and } u_j \in \mathrm{int}(\mathcal{C}) \text{ for some } j \right\}.$$

Proof. Clearly, $\mathcal{M} \subseteq \mathcal{K}_\mathcal{C}$. Consider any nonzero $A \in \mathcal{E}_\mathcal{C}$. Then $\langle A, X \rangle \geq 0$ for all $X \in \mathcal{M}$. If $\langle A, X \rangle = 0$, say, for some $X = \sum_{i=1}^N u_i u_i^T \in \mathcal{M}$, then $u_i^T A u_i = 0$ for all i; As some $u_j \in \mathrm{int}(\mathcal{C})$, by Lemma 2.5, $A \in \mathcal{S}^n_+$ and hence (whether u_i belongs to $\mathrm{int}(\mathcal{C})$ or not), $Au_i = 0$ for all i. Since the vectors u_i span $\mathbb{R}^n$, we must have $A = 0$, contradicting our choice of A. Thus, for any nonzero $A \in \mathcal{E}_\mathcal{C}$, $\langle A, X \rangle > 0$ for all $X \in \mathcal{M}$. As $\mathcal{K}_\mathcal{C}^* = \mathcal{E}_\mathcal{C}$ and both cones are proper (by the previous result), we must have (see [3], p. 3)

$$\mathrm{int}(\mathcal{K}_\mathcal{C}) = \{Z \in \mathcal{S}^n : \langle A, Z \rangle > 0\ \forall\ A \in \mathcal{E}_\mathcal{C} \setminus \{0\}\}$$

and so $\mathcal{M} \subseteq \mathrm{int}(\mathcal{K}_\mathcal{C})$. Now to see the reverse inclusion, let $Y \in \mathrm{int}(\mathcal{K}_\mathcal{C})$, $X \in \mathcal{M}$, $X \neq Y$. Since $\mathcal{K}_\mathcal{C}$ is convex, we can extend the line segment joining X and Y (slightly) beyond Y to get $Z \in \mathcal{K}_\mathcal{C}$ such that Y is a convex combination of X and Z.

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Because of the form of X and Z, this convex combination is in $\mathcal{M}$. This completes the proof. ☐

REMARK 3.4. By using Lemma 3.7 in [8], we can state the following:

When $\mathcal{C}$ is a closed cone with nonempty interior and $\mathcal{C} = \overline{\text{int}(\mathcal{C})}$ (which holds, for instance, if $\mathcal{C}$ is also convex),

$$\text{int}(\mathcal{K}_{\mathcal{C}}) = \left\{ \sum_{i=1}^{N} u_i u_i^T : \text{span}\{u_1, u_2, \ldots, u_N\} = \mathbb{R}^n,\ u_i \in \text{int}(\mathcal{C}) \text{ for all } i \right\}.$$

As noted by a referee, the above equality may not hold if $\mathcal{C} \neq \overline{\text{int}(\mathcal{C})}$. For example, let $\mathcal{C} = \mathbb{R}^2_+ \cup \text{cone}\,(e_1 - e_2)$ in $\mathbb{R}^2$, $u = e_1 + e_2$ and $v = e_1 - e_2$. Then $uu^T + vv^T$ belongs to $\text{int}(\mathcal{K}_{\mathcal{C}})$ (by the previous theorem), but not to the set on the right given above.

REMARK 3.5. Since $\mathcal{K}_{\mathcal{C}}$ is pointed, $\mathcal{E}_{\mathcal{C}} = \mathcal{K}^*_{\mathcal{C}}$ has nonempty interior ([3], p. 2) and so, from ([3], p. 3), $\text{int}(\mathcal{E}_{\mathcal{C}}) = \{Z \in \mathcal{S}^n : \langle A, Z\rangle > 0\ \forall\ A \in \mathcal{K}_{\mathcal{C}} \setminus \{0\}\}$. Since every A in $\mathcal{K}_{\mathcal{C}} \setminus \{0\}$ is a finite sum of the form $\sum uu^T$ with $0 \neq u \in \mathcal{C}$, it follows that

$$Z \in \text{int}(\mathcal{E}_{\mathcal{C}}) \Leftrightarrow u^T Z u > 0\ \forall u \in \mathcal{C} \setminus \{0\}.$$

In other words, Z belongs to the interior of the copositive cone of $\mathcal{C}$ if and only if Z is strictly-copositive on $\mathcal{C}$.

Motivated by a result in [22], that for proper cones K_1 and K_2,

$$\pi(K_1) = \pi(K_2) \Rightarrow K_1 = \pm K_2,$$

one referee raises the following question:

If $\mathcal{C}_1$ and $\mathcal{C}_2$ are closed pointed convex cones such that $\mathcal{K}_{\mathcal{C}_1} = \mathcal{K}_{\mathcal{C}_2}$, does it follow that $\mathcal{C}_1 = \pm\mathcal{C}_2$?

Below, we provide a positive answer.

PROPOSITION 3.6. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be closed cones in $\mathbb{R}^n$ with corresponding completely positive cones $\mathcal{K}_{\mathcal{C}_1}$ and $\mathcal{K}_{\mathcal{C}_2}$. Then the following statements hold:*

(*i*) $\mathcal{K}_{\mathcal{C}_1} = \mathcal{K}_{\mathcal{C}_2} \Leftrightarrow \mathcal{C}_1 \cup -\mathcal{C}_1 = \mathcal{C}_2 \cup -\mathcal{C}_2$.
(*ii*) *If $\mathcal{C}_1 \backslash \{0\}$ is connected and $\mathcal{C}_2$ is pointed* (*in particular, $\mathcal{C}_1$ and $\mathcal{C}_2$ are closed pointed convex cones*), *then $\mathcal{K}_{\mathcal{C}_1} = \mathcal{K}_{\mathcal{C}_2} \Leftrightarrow \mathcal{C}_1 = \pm\mathcal{C}_2$.*

Proof. (*i*) It is easy to verify that $\mathcal{C}_1 \cup -\mathcal{C}_1 = \mathcal{C}_2 \cup -\mathcal{C}_2 \Rightarrow \mathcal{K}_{\mathcal{C}_1} = \mathcal{K}_{\mathcal{C}_2}$. The reverse implication follows easily from $\text{Ext}(\mathcal{K}_{\mathcal{C}_1}) = \text{Ext}(\mathcal{K}_{\mathcal{C}_2})$, Proposition 3.1(*iv*) and the observation that $uu^T = vv^T \Rightarrow u = \pm v$ (see for instance, Proposition 6 in [19]).

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(ii) Assume the specified conditions on $\mathcal{C}_1$ and $\mathcal{C}_2$, and let $\mathcal{K}_{\mathcal{C}_1} = \mathcal{K}_{\mathcal{C}_2}$. From (i), we have $\mathcal{C}_1 \cup -\mathcal{C}_1 = \mathcal{C}_2 \cup -\mathcal{C}_2$. Then

$$\mathcal{C}_1\backslash\{0\} \subseteq \mathcal{C}_2\backslash\{0\} \cup -(\mathcal{C}_2\backslash\{0\}).$$

As $\mathcal{C}_2$ is pointed, $\mathcal{C}_2\backslash\{0\}$ and $-(\mathcal{C}_2\backslash\{0\})$ are separated (in the sense that the closure of one is disjoint from the other). Since $\mathcal{C}_1\backslash\{0\}$ is connected, we must have $\mathcal{C}_1\backslash\{0\} \subseteq \mathcal{C}_2\backslash\{0\}$ or $\mathcal{C}_1\backslash\{0\} \subseteq -(\mathcal{C}_2\backslash\{0\})$. Taking closures, we get $\mathcal{C}_1 \subseteq \mathcal{C}_2$ or $\mathcal{C}_1 \subseteq -\mathcal{C}_2$. Without loss of generality, let $\mathcal{C}_1 \subseteq \mathcal{C}_2$. We claim that $\mathcal{C}_1 = \mathcal{C}_2$. To get a contradiction, suppose that there is a y with $0 \neq y \in \mathcal{C}_2$, $y \notin \mathcal{C}_1$. Then, $y \in \mathcal{C}_1 \cup -\mathcal{C}_1$ and so $y \in -\mathcal{C}_1 \subseteq -\mathcal{C}_2$. Thus, $0 \neq y \in -\mathcal{C}_2 \cap \mathcal{C}_2$ contradicting the pointedness of $\mathcal{C}_2$. This proves that $\mathcal{C}_1 = \mathcal{C}_2$. $\square$

4. Irreducibility. In this section, we address the irreducibility property of $\mathcal{K}_{\mathcal{C}}$.

THEOREM 4.1. *Let $\mathcal{C}$ be a closed cone in $\mathbb{R}^n$. Then $\mathcal{K}_{\mathcal{C}}$ is irreducible under one of the following conditions:*

(i) *$\mathcal{C}$ has nonempty interior.*
(ii) *$\mathcal{C} \setminus \{0\}$ is connected.*

Proof. For the sake of contradiction, suppose $\mathcal{K}_{\mathcal{C}}$ is reducible. For $i = 1, 2$, let $\mathcal{K}_i \neq \{0\}$ be closed cones in $\mathcal{S}^n$ such that $\mathcal{K}_{\mathcal{C}} = \mathcal{K}_1 + \mathcal{K}_2$ and $\mathrm{span}(\mathcal{K}_1) \cap \mathrm{span}(\mathcal{K}_2) = \{0\}$. Let

$$\mathcal{C}_1 := \{u \in \mathcal{C} : uu^T \in \mathcal{K}_1\} \quad \text{and} \quad \mathcal{C}_2 := \{u \in \mathcal{C} : uu^T \in \mathcal{K}_2\}.$$

Clearly, $\mathcal{C}_1$ and $\mathcal{C}_2$ are closed cones in $\mathbb{R}^n$, $\mathcal{C}_1 \cup \mathcal{C}_2 \subseteq \mathcal{C}$ and $\mathcal{C}_1 \cap \mathcal{C}_2 = \{0\}$. Now, for any nonzero $u \in \mathcal{C}$, $uu^T \in \mathcal{K}_{\mathcal{C}} = \mathcal{K}_1 + \mathcal{K}_2$; hence $uu^T = x_1 + x_2$, where $x_i \in \mathcal{K}_i$, $i = 1, 2$. Since uu^T is an extreme vector of $\mathcal{K}_{\mathcal{C}}$ and $\mathcal{K}_1 \cap \mathcal{K}_2 = \{0\}$, we must have $x_1 = 0$ or $x_2 = 0$. Thus, uu^T belongs to $\mathcal{K}_1$ or $\mathcal{K}_2$. Hence $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$.

(i) Now suppose that $\mathcal{C}$ has nonempty interior. Then, one of the sets, say, $\mathcal{C}_1$ has nonempty interior in $\mathcal{C}$ as well as in $\mathbb{R}^n$. (This follows from, for example, the Baire Category Theorem.) Then the completely positive cone $\mathcal{K}_{\mathcal{C}1}$ generated by $\mathcal{C}_1$ within $\mathcal{S}^n$ is proper (by Proposition 3.1), and, in particular, $\mathrm{span}(\mathcal{K}_1) = \mathcal{S}^n$. This implies that $\mathrm{span}(\mathcal{K}_2) = \{0\}$, a contradiction.

(ii) Now suppose that $\mathcal{C} \setminus \{0\}$ is connected. From $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, we see that

$$\mathcal{C} \setminus \{0\} = (\mathcal{C}_1 \setminus \{0\}) \cup (\mathcal{C}_2 \setminus \{0\}).$$

As $\mathcal{C}_1, \mathcal{C}_2$ are closed and $\mathcal{C}_1 \cap \mathcal{C}_2 = \{0\}$, the sets $\mathcal{C}_1 \setminus \{0\}$ and $\mathcal{C}_2 \setminus \{0\}$ are separated. By the connectedness of $\mathcal{C} \setminus \{0\}$, we must have (without loss of generality), $\mathcal{C} \setminus \{0\} = \mathcal{C}_1 \setminus \{0\}$, and hence $\mathcal{C} = \mathcal{C}_1$. But then, $\{uu^T : u \in \mathcal{C}\} \subseteq \mathcal{K}_1$ and so, $\mathcal{K}_{\mathcal{C}} \subseteq \mathcal{K}_1$. This

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implies that $\mathcal{K}_2 = \{0\}$, yielding a contradiction. This completes the proof of the theorem. ◻

Remark 4.2. If $\mathcal{C}$ has empty interior and $\mathcal{C} \setminus \{0\}$ is not connected, $\mathcal{K}_{\mathcal{C}}$ may or may not be irreducible. This can be seen as follows. In $\mathbb{R}^2$, consider the standard unit vectors e_1 and e_2 and let $\mathcal{C} = \text{cone}\,\{e_1, e_2\}$ so that the corresponding completely positive cone is the set of all nonnegative diagonal matrices in $\mathcal{S}^2$. Clearly, $\mathcal{K}_{\mathcal{C}}$ is reducible. If on the other hand, $\mathcal{C} = \text{cone}\,\{e_1, e_2, e_1 + e_2, e_1 - e_2\}$, then $\mathcal{C}$ has empty interior, while $\mathcal{K}_{\mathcal{C}}$ is irreducible. (This last example is due to a referee.) The boundary of $\mathbb{R}^3_+$ is an example of $\mathcal{C}$ for which the above result applies.

5. Self-duality. Recall that a cone K in H is *self-dual* if $K^* = K$ and a *symmetric cone* if it is self-dual and homogeneous.

Theorem 5.1. *The following are equivalent for a closed cone $\mathcal{C}$ in $\mathbb{R}^n$:*

(a) $\mathbb{R}^n = \mathcal{C} \cup -\mathcal{C}$.
(b) $\mathcal{K}_{\mathcal{C}} = \mathcal{S}^n_+$.
(c) *$\mathcal{K}_{\mathcal{C}}$ is a symmetric cone in $\mathcal{S}^n$.*
(d) *$\mathcal{K}_{\mathcal{C}}$ is a self-dual cone in $\mathcal{S}^n$.*

If $\mathcal{C}$ is also convex, then the above conditions are further equivalent to:

(e) *$\mathcal{C} = \mathbb{R}^n$ or a closed half-space.*

Proof. The implication (a) $\Rightarrow$ (b) follows from the spectral theorem for real symmetric matrices. That $\mathcal{S}^n_+$ is a symmetric cone in $\mathcal{S}^n$ is well-known [16] and the implication (c) $\Rightarrow$ (d) is obvious. Now suppose (d) holds. Since $\mathcal{E}_{\mathcal{C}}$ is the dual of $\mathcal{K}_{\mathcal{C}}$ in $\mathcal{S}^n$, the inclusions $\mathcal{K}_{\mathcal{C}} \subseteq \mathcal{S}^n_+ \subseteq \mathcal{E}_{\mathcal{C}}$ imply that $\mathcal{K}_{\mathcal{C}} = \mathcal{S}^n_+ = \mathcal{E}_{\mathcal{C}}$. Now consider any nonzero $x \in \mathbb{R}^n$. Then $xx^T \in \text{Ext}(\mathcal{S}^n_+) = \text{Ext}(\mathcal{K}_{\mathcal{C}})$. By Proposition 3.1, $xx^T = uu^T$ for some $0 \neq u \in \mathcal{C}$. It follows that (see for instance, Proposition 6 in [19]), $x = \pm u \in \mathcal{C}$. Thus, every x in $\mathbb{R}^n$ belongs to $\mathcal{C} \cup -\mathcal{C}$ proving the implication (d) $\Rightarrow$ (a).

Now suppose that $\mathcal{C}$ is a closed convex cone. Since the implication (e) $\Rightarrow$ (a) is obvious, we prove (a) $\Rightarrow$ (e). If the origin is an interior point of $\mathcal{C}$, then $\mathcal{C} = \mathbb{R}^n$. Now suppose that the origin is a boundary point of $\mathcal{C}$ so that there is a supporting hyperplane induced by a nonzero vector $d \in \mathbb{R}^n$: $\mathcal{C} \subseteq \{x \in \mathbb{R}^n : \langle x, d\rangle \geq 0\}$. Now for any $y \in \mathbb{R}^n$ with $\langle y, d\rangle > 0$, $-y \notin \{x \in \mathbb{R}^n : \langle x, d\rangle \geq 0\}$; thus, $-y \notin \mathcal{C}$, and so $y \notin -\mathcal{C}$. As $\mathcal{C} \cup -\mathcal{C} = \mathbb{R}^n$, we must have $y \in \mathcal{C}$. Hence

$$\{x \in \mathbb{R}^n : \langle x, d\rangle > 0\} \subseteq \mathcal{C} \subseteq \{x \in \mathbb{R}^n : \langle x, d\rangle \geq 0\}.$$

As $\mathcal{C}$ is closed, we see that $\mathcal{C} = \{x \in \mathbb{R}^n : \langle x, d\rangle \geq 0\}$. ◻

Corollary 5.2. *Suppose $n > 1$ and $\mathcal{C}$ is a closed pointed cone. Then, $\mathcal{K}_{\mathcal{C}}$ cannot be self-dual in $\mathcal{S}^n$.*

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Proof. If $\mathcal{K}_{\mathcal{C}}$ is self-dual in $\mathcal{S}^n$, then $\mathbb{R}^n = \mathcal{C} \cup -\mathcal{C}$ and so $\mathbb{R}^n \setminus \{0\} = \mathcal{C} \setminus \{0\} \cup -(\mathcal{C} \setminus \{0\})$. Since we assume that $\mathcal{C}$ is a closed pointed cone, the sets $\mathcal{C} \setminus \{0\}$ and $-(\mathcal{C} \setminus \{0\})$ are separated. As $\mathbb{R}^n \setminus \{0\}$ is connected for $n > 1$, we reach a contradiction. $\square$

6. Homogeneity. Recall that a proper cone K in H is *homogeneous* if for every $x, y \in \text{int}(K)$, there exists $L \in \text{Aut}\,(K)$ such that $L(x) = y$. Two standard examples are: (i) $\mathcal{S}^n_+$ in $\mathcal{S}^n$, which is the completely positive cone of $\mathbb{R}^n$ and (ii) $\mathbb{R}^n_+$ in $\mathbb{R}^n$, which is isomorphic to the completely positive cone of $\mathcal{C} = \text{cone}\,\{e_1, e_2, \ldots, e_n\}$. More generally, every symmetric cone [16] is homogeneous. For a detailed study of homogeneous cones, see [24].

Falling short of a characterization of (non)homogeneous completely positive cones, in this section, we show that completely positive cones coming from certain $\mathcal{C}$ are nonhomogeneous. The non-homogeneity of $\mathcal{K}_{\mathcal{C}}$ is proved via a recent result in [19] where it is shown that under certain conditions on $\mathcal{C}$ (for example, $\mathcal{C}$ is a proper cone), every automorphism of $\mathcal{K}_{\mathcal{C}}$ is of the form

$$X \mapsto QXQ^T,$$

with $Q \in \text{Aut}\,(\mathcal{C})$.

THEOREM 6.1. *Suppose $\mathcal{C}$ is a closed pointed cone in $\mathbb{R}^n$ $(n > 1)$ such that $\mathcal{C}$ and $\mathcal{C}^*$ have nonempty interiors and $\mathcal{C}\backslash\{0\}$ is connected. Then $\mathcal{K}_{\mathcal{C}}$ cannot be homogeneous.*

In particular, this conclusion holds if $\mathcal{C}$ is a proper cone.

Proof. Suppose that $\mathcal{K}_{\mathcal{C}}$ is homogeneous. Pick $u_1, u_2, \ldots, u_n$ and v in $\text{int}(\mathcal{C})$ such that $\{u_1, u_2, \ldots, u_n\}$ and $\{v, u_2, \ldots, u_n\}$ are bases in $\mathbb{R}^n$. Put $X := u_1u_1^T + u_2u_2^T + \cdots + u_nu_n^T$ and for any natural number k, $Y_k := vv^T + \frac{1}{k}(u_2u_2^T + \cdots + u_nu_n^T)$. Then, by Theorem 3.3, X and Y_k are in $\text{int}(\mathcal{K}_{\mathcal{C}})$. By assumption, there exists $L_k \in \text{Aut}\,(\mathcal{K}_{\mathcal{C}})$ such that $L_k(X) = Y_k$ for all k. Since $\mathcal{C}$ is a closed pointed cone such that $\text{int}(\mathcal{C})$ is nonempty and $\mathcal{C}\backslash\{0\}$ is connected, by Theorem 2 in [19], there exists $Q_k \in \text{Aut}\,(\mathcal{C})$ such that $L_k(Z) = Q_kZQ_k^T$ for all $Z \in \mathcal{S}^n$; in particular, $L_k(X) = Q_kXQ_k^T$. This implies

$$Q_k(u_1u_1^T + u_2u_2^T + \cdots + u_nu_n^T)Q_k^T = vv^T + \frac{1}{k}(u_2u_2^T + \cdots + u_nu_n^T)$$

for all k. We now consider two cases.

Case (i) : Suppose that the sequence Q_k is unbounded. In this case, we may let $||Q_k|| \to \infty$ and $\frac{Q_k}{||Q_k||} \to Q \in \overline{\text{Aut}\,(\mathcal{C})}$. This leads to

$$Q(u_1u_1^T + u_2u_2^T + \cdots + u_nu_n^T)Q^T = 0$$

and, upon simplification, to $Qu_i = 0$ for all i. As $\{u_1, u_2, \ldots, u_n\}$ spans $\mathbb{R}^n$, we see

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that $Q = 0$ leading to a contradiction (as norm of Q is one). Thus, this case cannot happen.

Case (ii): Suppose that the sequence Q_k is bounded. In this case, we may assume that $Q_k \to Q \in \overline{\mathrm{Aut}\,(\mathcal{C})}$. This leads to

$$Q(u_1u_1^T + u_2u_2^T + \cdots + u_nu_n^T)Q^T = vv^T.$$

Now, $vv^T \in \mathrm{Ext}(\mathcal{K}_\mathcal{C})$ (see Proposition 3.1) and $Qu_i \in \mathcal{C}$ for every i. (Note that $Q_k(u_i) \in \mathcal{C}$ for each i.) Thus, by definition of extreme vector, Qu_i is a multiple of v for each i. Since $Q \neq 0$ and $\{u_1, u_2, \ldots, u_n\}$ spans $\mathbb{R}^n$, the range of Q is one-dimensional and so Q is of rank one. Let $Qu_1 = \lambda v$. Then $\lambda \neq 0$ by Lemma 2.6 (applied to $\mathcal{C}$ and Q in place of K and L). Also, the pointedness of $\mathcal{C}$ implies that λ cannot be negative. Thus, $Qu_1 \in \mathrm{int}(\mathcal{C})$ and $Q \in \overline{\mathrm{Aut}\,(\mathcal{C})}$. As $u_1 \in \mathrm{int}(\mathcal{C})$, by Lemma 2.7 (applied to $\mathcal{C}$ and Q in place of K and L), Q is invertible. But this cannot happen as Q has rank one and $n > 1$. Thus, even this case cannot happen. We conclude that $\mathcal{K}_\mathcal{C}$ is not homogeneous. ◻

The following corollary is immediate from the above theorem. However, we give an independent and slightly different proof.

Corollary 6.2. *For any $n > 1$, the completely positive cone of $\mathbb{R}^n_+$ is not homogeneous.*

Proof. Let $X = [x_{ij}]$ and $Y = [y_{ij}]$ be in $\mathrm{int}(\mathcal{K}_\mathcal{C})$, where $\mathcal{K}_\mathcal{C}$ is the completely positive cone of $\mathbb{R}^n_+$ and assume that there is an automorphism $L \in \mathrm{Aut}\,(\mathcal{K}_\mathcal{C})$ such that $L(X) = Y$. By Theorem 2 in [19], there is a $Q \in \mathrm{Aut}\,(\mathbb{R}^n_+)$ such that $L(Z) = QZQ^T$ for all $Z \in \mathcal{S}^n$ and so $Y = L(X) = QXQ^T$. Since every element of $\mathrm{Aut}\,(\mathbb{R}^n_+)$ is a product of a permutation and a diagonal matrix with positive diagonals, we must have, for some $i \neq j$ and positive numbers r_i and r_j,

$$\begin{bmatrix} r_i^2 x_{ii} & r_ir_jx_{ij} \\ r_ir_jx_{ij} & r_j^2x_{jj} \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{12} & y_{22} \end{bmatrix}.$$

This implies (as all entries of X and Y are positive) that

$$\frac{y_{11}y_{22}}{y_{12}^2} \in \left\{ \frac{x_{ii}x_{jj}}{x_{ij}^2} : i \neq j \right\}. \tag{6.1}$$

Now, we construct specific X and Y violating this property.

Recall that $e_1, e_2, \ldots, e_n$ denote the standard unit vectors in $\mathbb{R}^n$; let e be the vector of all ones. Let

$$X = ee^T + \sum_{i=1}^n e_ie_i^T \quad \text{and} \quad Y = ee^T + 2\sum_{i=1}^n e_ie_i^T.$$

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Then, by Theorem 3.3, $X, Y \in \text{int}(\mathcal{K}_{\mathcal{C}})$, $\{\frac{x_{ii}x_{jj}}{x_{ij}^2} : i \neq j\} = \{4\}$ and $\frac{y_{11}y_{22}}{y_{12}^2} = \{9\}$. For the above X and Y, (6.1) is violated and hence X cannot be mapped onto Y by any automorphism of $\mathcal{K}_{\mathcal{C}}$. Thus, $\mathcal{K}_{\mathcal{C}}$ is not homogeneous. $\square$

Consider a closed cone K in H with interior. For each $x \in \text{int}(K)$, let

$$[x] := \{L(x) : L \in \text{Aut}\,(K)\}$$

(which is a subset of $\text{int}(K)$) denote the orbit of x under the automorphism group $\text{Aut}\,(K)$. Note that $\text{int}(K)$ is a disjoint union of such orbits and K is homogeneous if and only if there is only one orbit in $\text{int}(K)$. The following result, perhaps known, sheds some light on the nature and number of orbits.

Proposition 6.3. *Let K be a closed pointed cone in H such that K and K^* have nonempty interiors. Then, for any $x \in \text{int}(K)$, $[x]$ is a closed subset of* $\text{int}(K)$. *Moreover, if K is not homogeneous and* $\text{int}(K)$ *is connected, then there are an uncountable number of orbits in* $\text{int}(K)$. *In particular, this conclusion holds when K is a proper cone.*

Proof. Fix $x \in \text{int}(K)$ and a sequence $x_k \in [x]$ with $\lim x_k = y \in \text{int}(K)$. We show that $y \in [x]$.

By definition, there exist $L_k \in \text{Aut}\,(K)$ such that $L_k(x) = x_k$ and so $y = \lim\, L_k(x)$. We consider two cases.

Case 1: Suppose that the sequence L_k is bounded and let $L_k \to L \in \overline{\text{Aut}\,(K)}$. Then, $y = L(x)$ with $x, y \in \text{int}(K)$ and $L \in \overline{\text{Aut}\,(K)}$. By Lemma 2.3, $L \in \text{Aut}\,(K)$. Thus, $y \in [x]$.

Case 2: Suppose that the sequence L_k is unbounded. Then we may assume that $||L_k|| \to \infty$ and $\frac{L_k}{||L_k||} \to L \in \overline{\text{Aut}\,(K)} \subseteq \pi(K)$. Then $L(x) = 0$. By Lemma 2.2, $L = 0$, which is clearly a contradiction. Thus, this case is not possible, and hence $[x]$ is closed in $\text{int}(K)$.

Now, suppose that K is not homogeneous and there are a countable number of orbits. Then, $\text{int}(K)$ can be written as a disjoint union of countable number (more than one) closed sets (orbits) within $\text{int}(K)$. Since $\text{int}(K)$ is locally compact, by the Baire Category Theorem (see [20], Theorem 2.2), there is one orbit whose interior is nonempty in $\text{int}(K)$. By considering the union of images of this interior under various automorphisms, we conclude that this orbit is also open in $\text{int}(K)$. Thus, this orbit is both open and closed, contradicting the connectedness of $\text{int}(K)$. This proves that there must be an uncountable number of orbits in $\text{int}(K)$. Finally, when K is a proper cone which is not homogeneous, all the conditions listed in the proposition hold and the result follows. $\square$

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The following corollary is immediate.

Corollary 6.4. *For any proper cone $\mathcal{C}$ in $\mathbb{R}^n$ $(n > 1)$, the number of orbits in* $\mathrm{int}(\mathcal{K}_{\mathcal{C}})$ (*induced by* $\mathrm{Aut}\,(\mathcal{K}_{\mathcal{C}})$) *is uncountable.*

7. The copositive cone of $\mathcal{C}$. Based on the results we have obtained so far, we can record some properties of the copositive cone $\mathcal{E}_{\mathcal{C}}$ corresponding to a closed cone $\mathcal{C}$.

Theorem 7.1.

(*i*) *$\mathcal{E}_{\mathcal{C}}$ is self-dual in $\mathcal{S}^n$ if and only if $\mathbb{R}^n = \mathcal{C} \cup -\mathcal{C}$, or equivalently, $\mathcal{E}_{\mathcal{C}} = \mathcal{S}^n_+$.*
(*ii*) *If $\mathcal{C}$ has nonempty interior or $\mathcal{C} \setminus \{0\}$ is connected, then $\mathcal{E}_{\mathcal{C}}$ is irreducible.*
(*iii*) *If $\mathcal{C}$ is a proper cone in $\mathbb{R}^n$ $(n > 1)$, then $\mathcal{E}_{\mathcal{C}}$ is not homogeneous and* $\mathrm{int}(\mathcal{E}_{\mathcal{C}})$ *contains uncountable number of orbits* (*induced by* $\mathrm{Aut}\,(\mathcal{E}_{\mathcal{C}})$).

Proof. (*i*) Since $\mathcal{E}_{\mathcal{C}}$ is self-dual if and only if $\mathcal{K}_{\mathcal{C}}$ is self-dual, the result follows from Theorem 5.1.

(*ii*) Suppose $\mathcal{C}$ is a closed cone such that either $\mathrm{int}(\mathcal{C})$ is nonempty or $\mathcal{C} \setminus \{0\}$ is connected. Then $\mathcal{K}_{\mathcal{C}}$ is irreducible from Theorem 4.1. Hence its dual $\mathcal{E}_{\mathcal{C}}$ is also irreducible ([3], Page 20).

(*iii*) Suppose $\mathcal{C}$ is a proper cone. Then, by Theorem 6.1, $\mathcal{K}_{\mathcal{C}}$ is not homogeneous. Then its dual $\mathcal{E}_{\mathcal{C}}$ is also not homogeneous, by a result of Vinberg (Proposition 9 in [24]). The uncountability of the orbits come from the previous proposition. ∎

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