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UPPER BOUNDS FOR THE LARGEST EIGENVALUE OF A BIPARTITE GRAPH*

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Abstract. Consider a finite, simple, undirected, and bipartite graph G with vertex sets $V = \{v_1, \dots, v_m\}$ and $W = \{w_1, \dots, w_n\}$, $m \le n$, $V \cap W = \emptyset$. Let the vertices of V have degrees $d_1 \ge d_2 \ge \cdots \ge d_m > 0$, respectively. Let N_i be the set of neighbors of v_i $(i = 1, \dots, m)$. Define $d_{ij} = |N_i \cap N_j|$ $(i, j = 1, \dots, m)$, where $|\,.\,|$ stands for the cardinality. Denote $e = d_1 + d_2 + \cdots + d_m$, $g = \sum_{i,j} d_{ij}^2$, and $f = d_1^2 + 3d_2^2 + 5d_3^2 + \cdots + (2m-1)d_m^2$. In this paper, it is proven that the largest eigenvalue λ of G satisfies

$$\lambda \le \sqrt{\frac{e}{m} + \sqrt{\frac{m-1}{m}\left(g - \frac{e^2}{m}\right)}} \le \sqrt{\frac{e}{m} + \sqrt{\frac{m-1}{m}\left(f - \frac{e^2}{m}\right)}} \le \sqrt{e}.$$

It is also proven that if $d_i \le d_1 - i + 1$ $(i = 2, \dots, m)$, then

$$\lambda \le \sqrt{\frac{M+1}{2} + \frac{M-1}{2}\sqrt{\frac{2M^2 + 3M + 1}{3}}} < \sqrt{\frac{M(M+1)}{2}},$$

where $M = \max(m, d_1)$.

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1. Introduction.

Let G be a finite, simple, and undirected graph with N vertices and e edges. Its largest eigenvalue λ (i.e., the largest eigenvalue of its adjacency matrix) has been widely studied. For a thorough review, see Cvetković and Rowlinson [2]. We recall some well-known upper bounds for λ. According to Stanley [7],

$$\lambda \le \frac{1}{2}\big(-1 + \sqrt{1 + 8e}\big). \tag{1.1}$$

Friedland [3] improved this to

$$\lambda \le \frac{1}{2}\big(p - 2 + \sqrt{p^2 + 4q}\big), \tag{1.2}$$

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where

$$p(p-1)+2q=2e, \quad 0 \le q < p. \tag{1.3}$$

Also, Yuan [9] has shown

$$\lambda \le \sqrt{2e-N+1} \tag{1.4}$$

if G has no isolated vertices.

We are interested in upper bounds for λ when G is bipartite with no isolated vertices. So, let G have vertex sets $V=\{v_1,\ldots,v_m\}$, $W=\{w_1,\ldots,w_n\}$, $m\le n$, $V\cap W=\emptyset$. Order $v_1,\ldots,v_m$ so that they have degrees $d_1\ge d_2\ge\cdots\ge d_m>0$, respectively. Order $w_1,\ldots,w_n$ so that they have degrees $d_1'\ge\cdots\ge d_n'>0$. We have then $e=d_1+\cdots+d_m=d_1'+\cdots+d_n'$. The adjacency matrix of G is an $(m+n)\times(m+n)$ matrix

$$\mathbf{A}=\begin{bmatrix}\mathbf{O} & \mathbf{P}\\ \mathbf{P}^T & \mathbf{O}\end{bmatrix},$$

where $\mathbf{P}$ is an $m\times n$ matrix whose e nonzero entries are one and the remaining entries are zero. The row sums of $\mathbf{P}$ are $d_1,\ldots,d_m$ and the column sums are $d_1',\ldots,d_n'$. We use all these notations throughout.

Since λ is the largest singular value of $\mathbf{P}$, we have

$$\lambda \le \sqrt{d_1 d_1'}, \tag{1.5}$$

see [5, (3.7.2)]. The singular values of $\mathbf{P}$ are the square roots of eigenvalues of $\mathbf{D}=(d_{ij})=\mathbf{P}\mathbf{P}^T$. As a result, we can find an upper bound for λ^2 by applying any upper bound for the largest eigenvalue of $\mathbf{D}$ and interpreting the result graph theoretically. Let us now consider the bound

$$\lambda^2 \le \frac{\operatorname{tr}\mathbf{D}}{m}+\sqrt{\frac{m-1}{m}\Big[\operatorname{tr}\mathbf{D}^2-\frac{(\operatorname{tr}\mathbf{D})^2}{m}\Big]} \tag{1.6}$$

(Wolkowicz and Styan [8, Theorem 2.1]), where equality holds if and only if the $m-1$ smallest eigenvalues of $\mathbf{D}$ are equal.

Let N_i denote the set of neighbors of v_i $(i=1,\ldots,m)$. Then

$$d_{ij}=|N_i\cap N_j| \quad (i,j=1,\ldots,m),$$

so $d_{ii}=d_i$ $(i=1,\ldots,m)$. Here $|\cdot|$ stands for the cardinality.

In Section 2 we present upper bounds for λ by applying (1.6). In Section 3, we will compare these bounds with bounds (1.1), (1.2), (1.4), and (1.5). Finally, in Section 4, we will draw conclusions.

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2. Upper bounds. Our first theorem applies the d_{ij}'s.

THEOREM 2.1. *Denote*

$$g = \sum_{i,j=1}^{m} d_{ij}^2 = d_1^2 + \cdots + d_m^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^{m} d_{ij}^2$$

and

$$f = d_1^2 + 3d_2^2 + 5d_3^2 + \cdots + (2m-1)d_m^2.$$

Then

$$\lambda \leq \sqrt{\frac{e}{m} + \sqrt{\frac{m-1}{m}\Big(g - \frac{e^2}{m}\Big)}} \leq \sqrt{\frac{e}{m} + \sqrt{\frac{m-1}{m}\Big(f - \frac{e^2}{m}\Big)}} \leq \sqrt{e}. \tag{2.1}$$

For $m = 1$, equality holds throughout. For $m \geq 2$, the first bound is exact if and only if $m-1$ smallest eigenvalues of $\mathbf{D}$ are equal. The second and third bounds are exact if and only if G is complete bipartite.

Proof. First inequality. Since

$$\operatorname{tr} \mathbf{D} = e, \quad \operatorname{tr} \mathbf{D}^2 = g,$$

this follows from (1.6).

Second inequality. We must show that $g \leq f$; in other words,

$$\sum_{i,j=1}^{m} d_{ij}^2 \leq d_1^2 + 3d_2^2 + \cdots + (2m-1)d_m^2.$$

We prove this by induction on m. The claim holds for $m = 2$. Suppose that it holds for m. Because $d_{ij} \leq d_j$ for all $i, j = 1, \ldots, m$, we have

$$\begin{aligned}
\sum_{i,j=1}^{m+1} d_{ij}^2 &= \sum_{i,j=1}^{m} d_{ij}^2 + 2\sum_{i=1}^{m} d_{i,m+1}^2 + d_{m+1}^2 \\
&\leq d_1^2 + 3d_2^2 + \cdots + (2m-1)d_m^2 + 2\sum_{i=1}^{m} d_{m+1}^2 + d_{m+1}^2 \\
&= d_1^2 + 3d_2^2 + \cdots + (2m-1)d_m^2 + (2m+1)d_{m+1}^2.
\end{aligned}$$

Hence, the claim holds for $m+1$.

Third inequality. The claim is equivalent to $f \leq e^2$; i.e.,

$$d_1^2 + 3d_2^2 + \cdots + (2m-1)d_m^2 \leq (d_1 + \cdots + d_m)^2.$$

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Again, we prove this by induction on m. We note that this holds for $m = 2$ and suppose that it holds for m. Then

$$\begin{aligned} & d_1^2 + 3d_2^2 + \cdots + (2m-1)d_m^2 + (2m+1)d_{m+1}^2 \\ &= d_1^2 + 3d_2^2 + \cdots + (2m-1)d_m^2 + 2\sum_{i=1}^{m} d_{m+1}^2 + d_{m+1}^2 \\ &\le (d_1 + \cdots + d_m)^2 + 2\sum_{i=1}^{m} d_i d_{m+1} + d_{m+1}^2 \\ &= (d_1 + \cdots + d_m)^2 + 2(d_1 + \cdots + d_m)d_{m+1} + d_{m+1}^2 = (d_1 + \cdots + d_{m+1})^2, \end{aligned}$$

so the claim holds for $m + 1$.

Equality conditions. The equality condition of the first inequality follows from that of (1.6). However, see Remark 4.3. The proofs of the second and third inequality imply that equality holds if and only if $d_{ij} = d_i$ for all $i, j = 1, \ldots, m$. This happens if and only if G is complete bipartite. Then $\operatorname{rank} \mathbf{D} = 1$, and thus, equality holds also in the first inequality. □

The *chain graph* corresponding to G is a bipartite graph $\tilde{G}$ with vertex sets V and $\{w_1, \ldots, w_{d_1}\}$ and edges

$$(v_1, w_1), \ldots, (v_1, w_{d_1}), (v_2, w_1), \ldots, (v_2, w_{d_2}), \ldots, (v_m, w_1), \ldots, (v_m, w_{d_m}).$$

Its largest eigenvalue $\tilde{\lambda}$ satisfies

$$\lambda \le \tilde{\lambda} \tag{2.2}$$

(Bhattacharya et al. [1, Theorem 3.1]).

Our second theorem concerns only $d_1, \ldots, d_m$, but requires certain assumptions.

THEOREM 2.2. *If*

$$d_2 \le d_1 - 1,\ d_3 \le d_1 - 2, \ldots, d_m \le d_1 - m + 1, \tag{2.3}$$

then

$$\lambda \le \sqrt{\frac{M+1}{2} + \frac{M-1}{2}\sqrt{\frac{2M^2+3M+1}{3}}} < \sqrt{\frac{M(M+1)}{2}}, \tag{2.4}$$

where $M = \max(m, d_1)$.

Proof. The adjacency matrix of $\tilde{G}$ is

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{O} & \tilde{\mathbf{P}} \\ \tilde{\mathbf{P}}^T & \mathbf{O} \end{bmatrix},$$

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where $\tilde{\mathbf{P}}$ is the $m \times d_1$ matrix obtained from $\mathbf{P}$ by moving in each row the ones to the beginning and the zeros to the end and then (in case of $d_1 < n$) deleting the zero columns. Evidently, the largest eigenvalue of

$$\tilde{\mathbf{P}}\tilde{\mathbf{P}}^T = \tilde{\mathbf{D}} = \begin{bmatrix} d_1 & d_2 & d_3 & \cdots & d_m \\ d_2 & d_2 & d_3 & \cdots & d_m \\ d_3 & d_3 & d_3 & \cdots & d_m \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d_m & d_m & d_m & \cdots & d_m \end{bmatrix}$$

is $\tilde{\lambda}^2$.

Consider the $M \times M$ matrix

$$\mathbf{T} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

Let $\tilde{\mathbf{P}}_0 = (p_{ij}^0)$ be the $M \times M$ matrix obtained from $\tilde{\mathbf{P}}$ by appending zero rows (if $m < d_1$) or zero columns (if $m > d_1$) appropriately. Since $i + j > d_1 + 1 \Rightarrow p_{ij}^0 = 0$ by (2.3), we have

$$\tilde{\mathbf{P}}_0 \leq \mathbf{T},$$

where $\leq$ is entrywise. If $m < d_1 (= M)$, append to $\tilde{\mathbf{D}}$ zero rows and zero columns to obtain an $M \times M$ matrix $\tilde{\mathbf{D}}_0$. Also the largest eigenvalue of $\tilde{\mathbf{D}}_0$ is $\tilde{\lambda}^2$. Because

$$\tilde{\mathbf{D}}_0 \leq \mathbf{T}^2 = \begin{bmatrix} M & M-1 & M-2 & \cdots & 3 & 2 & 1 \\ M-1 & M-1 & M-2 & \cdots & 3 & 2 & 1 \\ M-2 & M-2 & M-2 & \cdots & 3 & 2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 3 & 3 & 3 & \cdots & 3 & 2 & 1 \\ 2 & 2 & 2 & \cdots & 2 & 2 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix},$$

the largest eigenvalue μ of $\mathbf{T}$ satisfies

$$\tilde{\lambda}^2 \leq \mu^2. \tag{2.5}$$

This follows from the fact that if $\mathbf{B}$ and $\mathbf{C}$ are nonnegative square matrices of same size and $\mathbf{B} \leq \mathbf{C}$, then the Perron roots $\rho(\mathbf{B}) \leq \rho(\mathbf{C})$ (e.g., [4, Corollary 8.1.19]).

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Now

$$\operatorname{tr} \mathbf{T}^2 = M + (M-1) + \cdots + 1 = \frac{M(M+1)}{2}$$

and [6, Sequences A005900 and A006325]

$$\operatorname{tr} \mathbf{T}^4 = \sum_{i=1}^{M} [1^2 + 2^2 + \cdots + i^2 + (i-1)^2 + \cdots + 1^2] = \frac{(M+1)^5 - M^5 - 1}{30}.$$

Since

$$\frac{M-1}{M} \left\{ \frac{(M+1)^5 - M^5 - 1}{30} - \frac{1}{M} \left[\frac{M(M+1)}{2} \right]^2 \right\} = \frac{(M-1)^2(2M^2+3M+1)}{12},$$

applying (1.6) to $\mathbf{T}^2$ gives us the result

$$\mu^2 \leq \frac{M+1}{2} + \frac{M-1}{2} \sqrt{\frac{2M^2+3M+1}{3}}. \tag{2.6}$$

Furthermore,

$$\begin{aligned} \frac{M+1}{2} + \frac{M-1}{2} \sqrt{\frac{2M^2+3M+1}{3}} &< \frac{M+1}{2} + \frac{M-1}{2} \sqrt{M^2+2M+1} \\ &= \frac{M+1}{2} + \frac{(M-1)(M+1)}{2} = \frac{M(M+1)}{2}. \end{aligned} \tag{2.7}$$

Now (2.2), (2.5), (2.6), and (2.7) imply (2.4). ∎

Corollary 2.3. *If* (2.3) *holds, then*

$$\lambda \leq \sqrt{\frac{n+1}{2} + \frac{n-1}{2} \sqrt{\frac{2n^2+3n+1}{3}}} < \sqrt{\frac{n(n+1)}{2}}.$$

Proof. Since $m \leq n$ and $d_1 \leq n$, we have $M \leq n$. ∎

3. Comparisons and examples. We do some comparison between the bounds discussed above.

(2.1) *versus* (1.2). The inequality

$$\sqrt{e} \leq \frac{1}{2}\left(p - 2 + \sqrt{p^2+4q}\right),$$

where p and q satisfy (1.3), is equivalent to

$$p - 2 \leq (p-2)\sqrt{p^2+4q}.$$

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Since $p \geq 2$ and $q \geq 0$, this always holds, so (2.1) is better than (1.2). Since (1.2) improves (1.1), then (2.1) also improves (1.1).

(2.1) *versus* (1.4). The inequality

$$\sqrt{e} \leq \sqrt{2e - m - n + 1}$$

is equivalent to

$$m + n - 1 \leq e,$$

which is true if G is connected. Then (2.1) improves (1.4). This may change if G is not connected, as we see in the following example.

Example 3.1. Consider G with

$$\mathbf{P} = \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $\lambda = 1$, $m = n = 3$, $e = g = 3$, $f = 9$. The bound (1.4), $\lambda \leq 1$, improves the second bound of (2.1), $\lambda \leq 1.732$, but equals the first bound. The question whether (1.4) can improve the first bound of (2.1) remains open.

(2.1) *versus* (1.5). We study two examples.

Example 3.2. Consider G with

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then $\lambda = 2.247$, $m = n = 3$, $e = 6$, $d_1 = d_1' = 3$. The third bound of (2.1), $\lambda \leq 2.449$, is better than (1.5), $\lambda \leq 3$.

Example 3.3. To see that (1.5) may be better, consider G with

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $\lambda = 2$, $m = n = 3$, $e = 5$, $g = 17$, $d_1 = d_1' = 2$. Now (1.5) yields $\lambda \leq 2$, while the first bound of (2.1) yields $\lambda \leq 2.018$.

(2.1) *versus* (2.4). The proof of Theorem 2.2 implies that the first bound of (2.1) is better than the first bound of (2.4). We show that the third bound of (2.1) is better than the second bound of (2.4). Assuming (2.3), we have

$$\begin{aligned} e &\leq d_1 + (d_1 - 1) + \cdots + (d_1 - (m-1)) \\ &\leq M + (M-1) + \cdots + (M - (m-1)) = m\,\frac{2M - m + 1}{2}. \end{aligned}$$

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Hence,

$$\frac{M(M+1)}{2} - e \geq \frac{1}{2}[M(M+1) - m(2M-m+1)] = \frac{1}{2}[(M-m)^2 + M - m] \geq 0,$$

and the claim follows. If $\mathbf{P} = \mathbf{T}$, then the first bound of (2.4) is equal to the first and second bounds of (2.1), and the second bound of (2.4) is equal to the third bound of (2.1).

Example 3.4. Let G be as in Example 3.2. The first bound of (2.4) and the first and second bounds of (2.1) yield $\lambda \leq 2.248$. The second bound of (2.4) and the third bound of (2.1) yield $\lambda \leq 2.449$. The best of the other bounds is (1.4), $\lambda \leq 2.646$.

4. Conclusions and remarks. We presented upper bounds for the largest eigenvalue of a bipartite graph and compared them with certain upper bounds that work more generally. We conclude our paper with three remarks.

Remark 4.1. The third bound of (2.1) is well-known (e.g., [1, Proposition 2.1]).

Remark 4.2. If we proceed as in the proof of the first bound of (2.1) but study $\mathbf{P}^T\mathbf{P}$ instead of $\mathbf{D} = \mathbf{P}\mathbf{P}^T$, we obtain

$$\lambda^2 \leq \frac{e}{n} + \sqrt{\frac{n-1}{n}\Big(g - \frac{e^2}{n}\Big)}, \tag{4.1}$$

but this is weaker than the first bound of (2.1). Namely, adding $n - m$ zeros to relevant places, we have

$$\begin{aligned}
&\text{square of first bound of (2.1)}\\
&= \max\{\,\mu_1 \mid \mu_1 + \cdots + \mu_m = e,\ \mu_1^2 + \cdots + \mu_m^2 = g,\ \mu_1 \geq \mu_2, \ldots, \mu_m\,\}\\
&= \max\{\,\mu_1 \mid \mu_1 + \cdots + \mu_m + 0 + \cdots + 0 = e,\\
&\qquad\qquad \mu_1^2 + \cdots + \mu_m^2 + 0^2 + \cdots + 0^2 = g,\ \mu_1 \geq \mu_2, \ldots, \mu_m, 0, \ldots, 0\,\}\\
&\leq \max\{\,\mu_1 \mid \mu_1 + \cdots + \mu_n = e,\ \mu_1^2 + \cdots + \mu_n^2 = g,\ \mu_1 \geq \mu_2, \ldots, \mu_n\,\}\\
&= \text{right-hand side of (4.1).}
\end{aligned}$$

Remark 4.3. The equality condition of the first bound of (2.1) depends on the eigenvalues of $\mathbf{D}$ and thus also on those of G. A natural further question is to characterize equality without using eigenvalues but using only the structure of G and related quantities. We claim that equality holds if

(i) $|N_1| = \cdots = |N_m|$,
(ii) $|N_i \cap N_j| = |N_k \cap N_l|$ for all $i, j, k, l \in \{1, \ldots, m\}$ with $i \neq j$, $k \neq l$,

in other words if

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(i) $d_1 = \cdots = d_m$,
(ii) $d_{ij} = d_{kl}$ for all $i, j, k, l \in \{1, \ldots, m\}$ with $i \neq j$, $k \neq l$.

For the proof, let d and δ denote the common value of the d_i's and d_{ij}'s, respectively. Then

$$\mathbf{D} = \begin{bmatrix} d & \delta & \delta & \cdots & \delta \\ \delta & d & \delta & \cdots & \delta \\ \delta & \delta & d & \cdots & \delta \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \delta & \delta & \delta & \cdots & d \end{bmatrix}.$$

All row sums of $\mathbf{D}$ are $d + (m-1)\delta$. Since the largest eigenvalue of $\mathbf{D}$ is between the smallest and largest row sums (e.g., [4, Theorem 8.1.22]), we have $\lambda^2 = d + (m-1)\delta$. Substituting

$$e = md, \quad g = md^2 + m(m-1)\delta^2,$$

a simple computation shows that also the square of the first bound of (2.1) is $d + (m-1)\delta$. Hence, equality holds.

We conjecture the converse: Equality holds only if (i) and (ii) are satisfied.

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