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NUMERICAL RANGES OF QUADRATIC OPERATORS IN SPACES WITH AN INDEFINITE METRIC*

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Abstract. The numerical range of a quadratic operator acting on an indefinite inner product space is shown to have a hyperbolical shape. This result is extended to different kinds of indefinite numerical ranges, namely, indefinite higher rank numerical ranges and indefinite Davis-Wielandt shells.

Key words. Quadratic operators, Indefinite inner product spaces, Indefinite numerical range, Indefinite rank-k numerical range, Indefinite Davis-Wielandt shells, Hyperbolical range theorem.

AMS subject classifications. 15A60, 47B35.

1. Introduction. We denote by $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators acting on the Hilbert space $(\mathcal{H}, \langle , \rangle)$. If $\mathcal{H}$ has dimension n, we identify $\mathcal{B}(\mathcal{H})$ with the associative algebra of $n \times n$ complex matrices M_n. Any $A \in \mathcal{B}(\mathcal{H})$, may be uniquely written in the form $A = \Re^J A + i\Im^J A$, where $\Re^J A := (A+A^\#)/2$, $\Im^J A := (A-A^\#)/2i$, and $A^\#$ is the J-*adjoint* operator of A defined as $[Ax, y] = [x, A^\# y]$, for any $x, y \in \mathcal{H}$. For the most part of this note, finite dimensional spaces are considered.

Let $S : \mathcal{H} \to \mathcal{H}$ be a selfadjoint involution. Then $\mathcal{H}$ can be viewed as a (complex) *Krein space* with respect to the indefinite inner product $[x, y] = \langle Sx, y\rangle$. The *indefinite numerical range* of a linear operator $A : \mathcal{H} \to \mathcal{H}$ is defined as

$$W^S(A) = \{[Ax, x]/[x, x] : x \in \mathcal{H}, [x, x] \neq 0\}.$$

If S is the identity operator, this concept reduces to the well-known (classical) *numerical range* $W(A)$ of A, an important tool both in theoretical and applied research. There are several monographs devoted to the numerical range and its generalizations, see, for example, [12], [13, Chapter 1] and their references. Likewise, there is substancial interest in studying the indefinite numerical range (we refer the reader to [2, 4, 6, 14, 16, 17]). In addition to $W^S(A)$, we also consider the following two of its generalizations. Let P be a J-*selfadjoint projection*, that is, $P = P^\# = P^2$. The

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indefinite rank-k numerical range of $A \in M_N$ $(1 \leq k \leq N)$, $J = I_p \oplus -I_{N-p}$, is defined as

$$\Lambda_k^J(A) = \{\lambda \in \mathbb{C} : PAP = \lambda P \text{ for some rank-k } J\text{-selfadjoint projection}\}.$$

It is obvious that if $J = I_N$, then $\Lambda_k^J(T)$ reduces to the *rank-k numerical range* $\Lambda_k(T)$, a concept used in the study of quantum error correction (see [9] and references therein). It is also clear that if $k = 1$, then $\Lambda_1^J(T)$ reduces to the indefinite numerical range $W^J(T)$. Finally, the *indefinite shell* of $A \in M_n$, $J = I_p \oplus -I_{n-p}$, is

$$\mathcal{S}^J(A) = \left\{ \left(\frac{[\Re^J Ax, x]}{[x,x]}, \frac{[\Im^J Ax, x]}{[x,x]}, \frac{[Ax, Ax]}{[x,x]} \right) : x \in \mathbb{C}^n, [x,x] \neq 0 \right\} \subseteq \mathbb{C} \times \mathbb{R},$$

where we identify $\mathbb{C} \times \mathbb{R}$ with $\mathbb{R}^3$. If $J = I$, then $\mathcal{S}^J(A)$ turns to the *Davis-Wielandt shell*, which has been extensively studied, e.g., see [7, 10, 11].

In [1], it was proved that the Davis-Wielandt shell $W(A; A^*A)$ is convex if $\dim\mathcal{H} \geq 3$. In [20], the convexity of the higher numerical range was stated. Since similar results for indefinite inner products spaces are lacking, it seems of interest to treat such generalized numerical ranges in specific situations. So the investigation of the indefinite numerical range, rank-k numerical range and Davis-Wielandt shell of quadratic operators acting on spaces with an indefinite metric is the main objective of this note.

We recall that for arbitrary $\mathcal{H}$, operators satisfying the equation

$$A^2 - 2\mu A - \nu I = 0 \tag{1.1}$$

with some $\mu, \nu \in \mathbb{C}$ are called *quadratic operators* [8]. This class of operators includes idempotent and square-zero operators. According to Theorem 1.1 in [19], we may suppose that $\mathcal{H}$ has a decomposition $(\mathcal{H}_1 \oplus \mathcal{H}_1) \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$, such that A is unitarily similar to a matrix of the form

$$UAU^* = \begin{pmatrix} \lambda_1 I & 2X \\ 0 & \lambda_2 I \end{pmatrix} \oplus \lambda_1 I \oplus \lambda_2 I, \tag{1.2}$$

where $\dim \mathcal{H}_j$ (≥ 0) is uniquely defined by A and $X : \mathcal{H}_1 \to \mathcal{H}_1$ is a positive definite operator on $\mathcal{H}_1$, i.e., $\langle Xx, x\rangle \geq 0$ for all $x \in \mathcal{H}_1$ and $\langle Xx, x\rangle \neq 0$ for all nonzero $x \in \mathcal{H}_1$.

Our investigation strongly relies on the *Hyperbolical Range Theorem* [3, 4] which states that the indefinite numerical range of A when $\dim \mathcal{H} = 2$, is bounded by a two-component hyperbola with foci at the eigenvalues λ_1 and λ_2 of A and transverse axis of length $\left(\mathrm{Tr}\,(A^\# A) - |\lambda_1|^2 - |\lambda_2|^2\right)^{1/2}$. According to the Cayley-Hamilton theorem, A in this setting satisfies (1.1) with

$$\mu = (\lambda_1 + \lambda_2)/2, \;\; \nu = -\lambda_1\lambda_2. \tag{1.3}$$

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This note is organized as follows. In Section 2, the main results are presented. Namely, in Subsection 2.1 we characterize the indefinite numerical range of quadratic operators, extending Tso-Wu theorem on the numerical range [19] to the context of spaces with an indefinite metric. In Subsection 2.2, indefinite higher rank numerical ranges of finite dimensional quadratic operators are studied. In Subsection 2.3, the shapes of the indefinite Davis-Wielandt shells of these operators are characterized.

2. Main results.

2.1. Indefinite numerical range. We recall some concepts that are directly related with the subject of this note. Let

$$W_+^J(A) = \left\{ \frac{[Ax, Ax]}{[x, x]} : x \in \mathbb{C}^n, [x, x] > 0 \right\},$$

and

$$W_-^J(A) = \left\{ \frac{[Ax, Ax]}{[x, x]} : x \in \mathbb{C}^n, [x, x] < 0 \right\}.$$

Then

$$W^J(A) = W_+^J(A) \cup W_-^J(A).$$

From the definition it is clear that $W_-^J(A) = -W_+^{-J}(A)$, so we may concentrate in the investigation of one of these sets. In [17] it was proved that $W_+^J(A)$, $W_-^J(A)$ are convex and $W^J(A)$ is *pseudo-convex*, that is, for any pair of points x, y in this set, either the line segment $[x, y]$ or the half-lines $tx + (1-t)y$, $t \leq 0$ and $t \geq 1$, are contained in $W^J(A)$.

An operator $A \in \mathcal{B}(\mathcal{H})$ is called J*-unitary* if $AA^\# = I$. For a J-selfadjoint operator A, diagonalizable under a J-unitary similarity, define

$$\sigma_+^J(A) = \{\lambda \in \mathbb{R} : A\xi = \lambda\xi, \text{ for some } \xi \in \mathcal{H} \text{ with } [\xi, \xi] > 0\},$$

$$\sigma_-^J(A) = \{\lambda \in \mathbb{R} : A\xi = \lambda\xi, \text{ for some } \xi \in \mathcal{H} \text{ with } [\xi, \xi] < 0\}.$$

Throughout, $\|X\|$ denotes the largest eigenvalue of X.

In the proof of Theorem 2.1, we follow similar arguments to those in [18, Theorem 2.1].

THEOREM 2.1. *Let A be a quadratic operator satisfying (1.1) and (1.2) and let $J = USU^* = (I \oplus -I) \oplus I \oplus -I$ be acting on $(\mathcal{H}_1 \oplus \mathcal{H}_1) \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$. Then, we have:*

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(a) If $|\lambda_1 - \lambda_2| \geq 2\|X\|$, $W^J(A)$ *is the two-component hyperbola with foci* $\lambda_{1,2} = \mu \pm \sqrt{\mu^2+\nu}$, *transverse axis of length* $\sqrt{|\lambda_1-\lambda_2|^2 - 4\|X\|^2}$ *and conjugate axis of length* $2\|X\|$, *with the interior of each branch.*

Further, the set $W^J(A)$ *is closed when* $\|X\|$ *is attained and open otherwise.*

(b) If $|\lambda_1 - \lambda_2| < 2\|X\|$, *then* $W^J(A)$ *is the whole complex plane.*

Proof. (a) Without loss of generality, we may suppose that $\mu = 0$ and $\nu \geq 0$, because

$$W^J(\alpha\, A + \beta\, I) = \alpha\, W^J(A) + \beta,$$

for any $\alpha, \beta \in \mathbb{C}$. Hence, by (1.3),

$$\lambda_1 = -\lambda_2 = \lambda \geq 0, \;\; \lambda^2 = \nu.$$

The case $\mathcal{H}_1 = \{0\}$ corresponds to the case of A being a normal operator (in diagonal form) on the indefinite inner product space $\mathcal{H}$. We may conclude that $\overline{W^J(A)}$ is the union of the two half-rays $(-\infty, -\lambda]$ and $[\lambda, +\infty)$, e.g., see [2]. This agrees with (a), since in this case $\|X\| = 0$.

We now assume that $\|X\| < \lambda$ and dim $\mathcal{H}_1 > 0$. The (directed) distance from the origin to the support line ℓ_θ of $W^J(A)$ with the slope θ is an extremum point w_θ of the spectrum of $\Re^J(\mathrm{e}^{-i\theta}A)$, for $-\theta_0 < \theta < \theta_0$ with $\cos^2\theta_0 = \|X\|^2/\lambda^2$. Furthermore, ℓ_θ contains points of $W^J(A)$ if and only if w_θ belongs to the point spectrum of $\Re^J(\mathrm{e}^{-i\theta}A)$, for $-\theta_0 < \theta < \theta_0$. For A in the form (1.2) with λ_j as in (1.3), we find

$$\Re^J(\mathrm{e}^{-i\theta}A) = \begin{pmatrix} (\lambda\cos\theta)I & \mathrm{e}^{-i\theta}X \\ \mathrm{e}^{i\theta}X & -(\lambda\cos\theta)I \end{pmatrix} \oplus (\lambda\cos\theta)I \oplus (-\lambda\cos\theta)I.$$

Hence,

$$\Re^J(\mathrm{e}^{-i\theta}A) - wI = \begin{pmatrix} (\lambda\cos\theta - w)I & \mathrm{e}^{-i\theta}X \\ \mathrm{e}^{i\theta}X & -(\lambda\cos\theta + w)I \end{pmatrix} \oplus (\lambda\cos\theta - w)I \oplus (-\lambda\cos\theta - w)I.$$

The eigenvalues of the above first direct summand matrix are:

$$w = \pm\sqrt{\lambda^2\cos^2\theta - \|X\|^2}.$$

Therefore, $w_\theta = \sqrt{\lambda^2\cos^2\theta - \|X\|^2}$ is the leftmost point of $\sigma_+^J(\Re^J(\mathrm{e}^{-i\theta}A))$, while $w_\theta = -\sqrt{\lambda^2\cos^2\theta - \|X\|^2}$ is the rightmost point of $\sigma_-^J(\Re^J(\mathrm{e}^{-i\theta}A))$. Henceforth, the support lines of $W^J(A)$ are the same as the support lines of the indefinite numerical range of the 2×2 matrix

$$\begin{pmatrix} \lambda & 2\|X\| \\ 0 & -\lambda \end{pmatrix}.$$

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The theorem now follows from the Hyperbolical Range Theorem, having in mind the convexity of $W_+^J(A)$, $W_-^J(A)$ and the pseudo-convexity of $W^J(A)$. We notice that w_θ is an eigenvalue of $\Re^J(\mathrm{e}^{-i\theta}A)$ if and only if the norm of X is attained, so that this either occurs for all $-\theta_0 < \theta < \theta_0$ or for none of them. In the first case, every support line of $W^J(A)$ must contain at least one of its points, and the hyperbolical disc is closed, while in the second case the support lines do not intersect $W^J(A)$ and this set is open.

(b) The proof is left to the reader. □

2.2. Indefinite rank-k numerical range. Throughout this section, we assume that $\gamma \in \{\lambda_1, \lambda_2\}$, $\dim \mathcal{H}_1 = r, \dim \mathcal{H}_2 = s$. Moreover, we consider J acting on $(\mathcal{H}_1 \oplus \mathcal{H}_1) \oplus \mathcal{H}_2$ and having the form $J = USU^* = I_r \oplus -I_r \oplus I_s$ if $\gamma = \lambda_1$, and $J = USU^* = I_r \oplus -I_r \oplus -I_s$ if $\gamma = \lambda_2$. In Theorem 2.3 we characterize $\Lambda_k^J(A)$ for $\gamma = \lambda_1$, with the case $\gamma = \lambda_2$ similarly treated. Firstly, we recall the following result needed for its proof (cf. Theorem 4.3 in [5]).

Lemma 2.2. *Assume that $J = I_p \oplus -I_{N-p}$, $0 < p < N$. Let $A \in M_N$ be a J-selfadjoint matrix such that $\lambda_1 \geq \cdots \geq \lambda_p \in \sigma_J^+(A)$, $\lambda_{p+1} \geq \cdots \geq \lambda_N \in \sigma_J^-(A)$, and $\lambda_p > \lambda_{p+1}$. The following holds for $k \geq 1$ a fixed integer.*

(i) If $N - k + 1 > p$ and $N - k + 1 > N - p$ (and so $N + 2 > 2k$), then $\Lambda_k^J(A) = (-\infty, \lambda_{p+k}] \cup [\lambda_{p-k+1}, +\infty)$.
(ii) If $N - k + 1 \leq p$, $N - k + 1 > N - p$ and $\lambda_{p-k+1} \leq \lambda_{k-N+p}$, then $\Lambda_k^J(A) = [\lambda_{p-k+1}, \lambda_{k-N+p}]$, which reduces to an empty set or to a singleton, respectively when $\lambda_{p-k+1} > \lambda_{k-N+p}$ or $\lambda_{p-k+1} = \lambda_{k-N+p}$.
(iii) If $N - k + 1 > p$, $N - k + 1 \leq N - p$ and $\lambda_{N+p+1-k} \leq \lambda_{p+k}$, then $\Lambda_k^J(A) = [\lambda_{N+p+1-k}, \lambda_{p+k}]$, which reduces to an empty set or to a singleton, respectively when $\lambda_{k+p} < \lambda_{N+p+1-k}$ or $\lambda_{N+p+1-k} = \lambda_{k+p}$.

Theorem 2.3. *Let $A : \mathcal{H} \to \mathcal{H}$ be a non-scalar quadratic operator satisfying (1.1) and (1.2), with the eigenvalues of X arranged in non-increasing order $\sigma_1 \geq \cdots \geq \sigma_r \geq 0$. For k a positive integer not larger than* $\dim \mathcal{H} = 2r + s = n$*, the following hold:*

(a) If $k \leq r$ and $|\lambda_1 - \lambda_2| > 2\sigma_1$, then $\Lambda_k^J(A)$ is the hyperbolical disc with foci $\lambda_{1,2} = \mu \pm \sqrt{\mu^2 + \nu}$, and nontransverse axis of length $2\sigma_1$.

(b) If $k \leq r$ and $|\lambda_1 - \lambda_2| \leq 2\sigma_k$, then $\Lambda_k^J(A) = \mathbb{C}$.

(c) If $r < k \leq r + s$ and $|\lambda_1 - \lambda_2| > 2\sigma_1$, then $\Lambda_k^J(A) = \{\lambda_1\}$.

(d) If $r + s < k \leq n$ and $|\lambda_1 - \lambda_2| > 2\sigma_1$, then $\Lambda_k^J(A) = \emptyset$.

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Proof. (a) Under the J-unitary transformation $R \oplus R \oplus T$, where $R \in M_r$ and $T \in M_s$ are unitary matrices, UAU^* is J-unitarily similar to

$$Q(A_1 \oplus \cdots \oplus A_r \oplus \gamma I_s)Q^{-1},$$

where Q is a permutation, and

$$A_j = \begin{pmatrix} \lambda_1 & 2\sigma_j \\ 0 & \lambda_2 \end{pmatrix}, \quad j = 1, \ldots, r. \tag{2.1}$$

Further,

$$QJQ^{-1} = J_2 \oplus \cdots \oplus J_2 \oplus I_s \in M_{2r+s}, \quad J_2 = I_1 \oplus -I_1.$$

For $k \leq r$, we show that $\Lambda_k^J(A) = W^{J_2}(A_k)$, where A_k is as in (2.1). Let $z \in W^{J_2}(A_k)$. We have $W^{J_2}(A_k) = \mathcal{H}(\lambda_1, \lambda_2, 2\sigma_k)$, where $\mathcal{H}(\lambda_1, \lambda_2, \ell)$ denotes the hyperbolical disc with foci λ_1, λ_2 and nontransverse axis of length $\ell \leq |\lambda_1 - \lambda_2|$. It is easy to see that $\mathcal{H}(\lambda_1, \lambda_2, \ell_1) \subseteq \mathcal{H}(\lambda_1, \lambda_2, \ell_2)$ for $\ell_1 \leq \ell_2$ and so

$$\mathcal{H}(\lambda_1, \lambda_2, 2\sigma_r) \subseteq \cdots \subseteq \mathcal{H}(\lambda_1, \lambda_2, 2\sigma_2) \subseteq \mathcal{H}(\lambda_1, \lambda_2, 2\sigma_1). \tag{2.2}$$

Having in mind (2.2), there exist $x'_1, \ldots, x'_k \in \mathbb{C}^2$, such that

$$\frac{{x'}_1^* J_2 A_1 x'_1}{{x'}_1^* J_2 x'_1} = \cdots = \frac{{x'}_{k-1}^* J_2 A_{k-1} x'_{k-1}}{{x'}_{k-1}^* J_2 x'_{k-1}} = \frac{{x'}_k^* J_2 A_k x'_k}{{x'}_k^* J_2 x'_k} = z.$$

Now, we consider $x_1 = [{x'}_1^T, 0, \ldots, 0]^T \in \mathbb{C}^{2r+s}, \ldots, x_k = [0, \ldots, 0, {x'}_k^T, 0, \ldots, 0]^T \in \mathbb{C}^{2r+s}$ (x'_k provides the $(2k-1)$-th and the $2k$-th entries of x_k). Thus, $[x_i, x_j] = 0$ $(i \neq j)$ and we find

$$\frac{x_1^* J A x_1}{x_1^* J x_1} = \cdots = \frac{x_k^* J A x_k}{x_k^* J x_k} = z.$$

Let P be the projection operator on the subspace spanned by the J-orthogonal vectors $x_1, \ldots, x_k$:

$$P = \frac{x_1 x_1^* J}{x_1^* J x_1} + \cdots + \frac{x_k x_k^* J}{x_k^* J x_k}.$$

It clearly follows that $PAP = zP$. Hence, $\Lambda_k^J(A) \supseteq W^{J_2}(A_k) = \mathcal{H}(\lambda_1, \lambda_2, 2\sigma_k)$. In order to show that the reverse inclusion holds, we claim that for $\theta \in [\theta_1, \theta_2] \subset \mathbb{R}$ we have

$$\begin{aligned} \Lambda_k^J(\Re^J(\mathrm{e}^{-i\theta} A)) &= W^{J_2}(\Re^{J_2}(\mathrm{e}^{-i\theta} A_k)) \\ &= \left(-\infty, \lambda_2(\Re^{J_2}(\mathrm{e}^{-i\theta} A_1))\right] \cup \left[\lambda_1(\Re^{J_2}(\mathrm{e}^{-i\theta} A_1)), +\infty\right). \end{aligned}$$

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By the Hyperbolical Range Theorem we infer that $W^{J_2}(A_j) = \mathcal{H}(\lambda_1, \lambda_2, 2\sigma_j)$, and the claim follows for $k = 1$, because $\Lambda_1^{J_2}(A_j) = W^{J_2}(A_j)$. For $\theta \in [\theta_1, \theta_2] \subset \mathbb{R}$, we find

$$\lambda_2(\Re^{J_2}\,(\mathrm{e}^{-i\theta}A_j)) = \frac{1}{2}\left[\mathrm{Re}\,(\mathrm{e}^{-i\theta}(\lambda_1+\lambda_2)) - \sqrt{\left(\mathrm{Re}\,(\mathrm{e}^{-i\theta}(\lambda_1-\lambda_2))\right)^2 - 4\sigma_j^2}\,\right],$$

$$\lambda_1(\Re^{J_2}\,(\mathrm{e}^{-i\theta}A_j)) = \frac{1}{2}\left[\mathrm{Re}\,(\mathrm{e}^{-i\theta}(\lambda_1+\lambda_2)) + \sqrt{\left(\mathrm{Re}\,(\mathrm{e}^{-i\theta}(\lambda_1-\lambda_2))\right)^2 - 4\sigma_j^2}\,\right].$$

Thus,

$$\mathrm{Re}\;(\mathrm{e}^{-i\theta}\lambda_1) \geq \lambda_1(\Re^{J_2}(\mathrm{e}^{-i\theta}A_r)) \geq \cdots \geq \lambda_1(\Re^{J_2}\,(\mathrm{e}^{-i\theta}A_1))$$

$$\geq \lambda_2(\Re^{J_2}\,(\mathrm{e}^{-i\theta}A_1)) \geq \cdots \geq \lambda_2(\Re^{J_2}\,(\mathrm{e}^{-i\theta}A_r)).$$

It is obvious that

$$\lambda_1(\Re^{J}\,(\mathrm{e}^{-i\theta}A)), \ldots, \lambda_{r+s}(\Re^{J}\,(\mathrm{e}^{-i\theta}A)) \in \sigma_+^J(\Re^{J}\,(\mathrm{e}^{-i\theta}A)),$$

and

$$\lambda_{r+s+1}(\Re^{J}\,(\mathrm{e}^{-i\theta}A)), \ldots, \lambda_{n}(\Re^{J}\,(\mathrm{e}^{-i\theta}A)) \in \sigma_-^J(\Re^{J}\,(\mathrm{e}^{-i\theta}A)).$$

One can easily check that $\lambda_j(\Re^J\,(\mathrm{e}^{-i\theta}A))$, $j = 1, \ldots, 2r+s$, equals:

$$\begin{aligned}
&\lambda_1(\Re^{J}\,(\mathrm{e}^{-i\theta}A)) = \cdots = \lambda_s(\Re^{J}\,(\mathrm{e}^{-i\theta}A)) = \mathrm{Re}\;(\mathrm{e}^{-i\theta}\lambda_1),\\
&\lambda_{s+1}(\Re^{J}\,(\mathrm{e}^{-i\theta}A)) = \lambda_1(\Re^{J_2}\,(\mathrm{e}^{-i\theta}A_1)), \ldots, \lambda_{s+r}(\Re^{J}\,(\mathrm{e}^{-i\theta}A)) = \lambda_1(\Re^{J_2}\,(\mathrm{e}^{-i\theta}A_r)),\\
&\lambda_{s+r+1}(\Re^{J}\,(\mathrm{e}^{-i\theta}A)) = \lambda_2(\Re^{J_2}\,(\mathrm{e}^{-i\theta}A_r)), \ldots, \lambda_{s+2r}(\Re^{J}\,(\mathrm{e}^{-i\theta}A)) = \lambda_2(\Re^{J_2}\,(\mathrm{e}^{-i\theta}A_1)).
\end{aligned}$$

By Lemma 2.2 (i), we may conclude that the claim holds.

(b) Having in mind that A is permutationally similar to $A_1 \oplus \cdots \oplus A_r \oplus \gamma\, I_s$, where the A_j are as in (2.1), by Theorem 2.1 (b) we find

$$W^{J_2}(A_1) = \cdots = W^{J_2}(A_k) = \mathbb{C},$$

being

$$W^{J_2}(A_{k+1}) = \mathcal{H}(\lambda_1, \lambda_2, 2\sigma_{k+1}).$$

By similar arguments to those used in the proof of (a) we can easily show that $\mathbb{C} \subseteq \Lambda_k^J(A)$.

(c) For $r < k \leq r+s$, we prove that $\Lambda_k(A) \supseteq \{\lambda_1\}$. Let now

$$x = [x_1', 0, x_2', 0, \ldots, x_r', 0, x', 0, \ldots, 0]^T \in \mathbb{C}^{2r+s},$$

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with $x'_1, \dots, x'_r \in \mathbb{C}$ and $x' \in \mathbb{C}^{k-r}$. We get

$$\frac{x^* J A x}{x^* J x} = \lambda_1.$$

When the complex numbers $x'_1, \dots x'_r$ and the $(k-r)$-dimensional complex vector x' vary independently, x spans a k-dimensional subspace. Let P be the projection operator onto that subspace. Then $PAP = \lambda_1 P$. So, if $r < k \le r+s$, by Lemma 2.2 (ii), (iii), we may conclude that

$$\Lambda_k^J(A) = \{\lambda_1\},$$

and (c) follows.

Finally, if $r+s < k \le n$, by Lemma 2.2 (ii) or (iii), we get $\Lambda_k^J(A) = \emptyset$. □

Remark 2.4. Under the assumptions of Theorem 2.3, the characterization of $\Lambda_k^J(A)$ when $k > r$ and $|\lambda_1 - \lambda_2| \le 2\sigma_k$ is an open problem.

Next, we present an example that suggests the veracity of Theorem 2.3 (b) when $|\lambda_1 - \lambda_2| < 2\sigma_1$. Let

$$A = A_1 \oplus A_2, \quad A_1 = \begin{pmatrix} 1 & 2\sqrt{2} \\ 0 & -1 \end{pmatrix}, \quad A_2 = \mathrm{diag}\,(1,-1), \quad J = J_2 \oplus J_2, \quad J_2 = \mathrm{diag}\,(1,-1).$$

We show that $\Lambda_2^J(A) = W^{J_2}(A_2)$. Using the arguments in the proof of Theorem 2.3 (a), we may conclude that $\Lambda_2^J(A) \supseteq W^{J_2}(A_2)$. Hence, for $\theta \in (0, 2\pi)$ we get

$$\Lambda_2^J(\Re^J(\mathrm{e}^{-i\theta}A)) \supseteq W^{J_2}(\Re^{J_2}(\mathrm{e}^{-i\theta}A_2)) = (-\infty, -\cos\theta] \cup [\cos\theta, +\infty).$$

Considering adequate invariants of $\Re^J(\mathrm{e}^{-i\theta}A)$ (e.g., $\mathrm{Tr}C^{(k)}(\Re^J(\mathrm{e}^{-i\theta}A))$, $k = 1, \dots, n$, where $\mathrm{Tr}C^{(k)}(Y)$ denotes the trace of the kth compound of Y), it may be seen that $(-\cos\theta, \cos\theta) \cap \Lambda_2^J(\Re^J(\mathrm{e}^{-i\theta}A)) = \emptyset$, and so

$$\Lambda_2^J(\Re^J(\mathrm{e}^{-i\theta}A_2)) = W^{J_2}(\Re^{J_2}(\mathrm{e}^{-i\theta}A_2)) = (-\infty, -\cos\theta] \cup [\cos\theta, +\infty).$$

We do not present the complete proof that $(-\cos\theta, \cos\theta) \cap \Lambda_2^J(\Re^J(\mathrm{e}^{-i\theta}A)) = \emptyset$, because it is merely computational, but, in order to illustrate the involved techniques, we show that $0 \notin \Lambda_2^J(\Re^J A)$, where $\Re^J A = \begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & -1 \end{pmatrix} \oplus \mathrm{diag}\,(1,-1)$. We argue, by contradiction, that there does not exist a J-unitary matrix V such that

$$V^{\#}(\Re^J A)V = \begin{pmatrix} 0 & x & y & z \\ -\bar{x} & 0 & 0 & u \\ \bar{y} & 0 & 0 & v \\ -\bar{z} & \bar{u} & -\bar{v} & 0 \end{pmatrix}.$$

If such a V exists, then $\det(V^{\#}(\Re^J A)V) = |x|^2|v|^2 + |y|^2|u|^2 + xu\bar{y}\bar{v} + \bar{x}\bar{u}yv$, which is impossible because $\det(\Re^J A) = -1$.

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2.3. Indefinite Davis-Wielandt shells. The Davis-Wielandt shell of a quadratic operator can be a line segment, an ellipsoid with interior or the interior of an ellipsoid [14]. We characterize the indefinite Davis-Wielandt shells of quadratic operators in Theorem 2.3. We consider the following sets

$$\mathcal{S}_+^J(A) = \left\{ \left(\frac{[\Re^J Ax, x]}{[x,x]}, \frac{[\Im^J Ax, x]}{[x,x]}, \frac{[Ax, Ax]}{[x,x]} \right) : x \in \mathbb{C}^n, [x,x] > 0 \right\},$$

$$S_-^J(A) = \left\{ \left(\frac{[\Re^J Ax, x]}{[x,x]}, \frac{[\Im^J Ax, x]}{[x,x]}, \frac{[Ax, Ax]}{[x,x]} \right) : x \in \mathbb{C}^n, [x,x] < 0 \right\}.$$

From the definition it is clear that

$$\mathcal{S}^J(A) = \mathcal{S}_+^J(A) \cup S_-^J(A), \quad S_-^J(A) = -S_+^{-J}(A).$$

We remark that $\mathcal{S}_+^J(A)$, $\mathcal{S}_-^J(A)$ are connected sets since the sets $\{x \in \mathbb{C}^n : [x,x] > 0\}$, $\{x \in \mathbb{C}^n : [x,x] < 0\}$ are connected.

The following lemma, here included for the sake of completeness, will be used in the proof of Theorem 2.7 (cf. [6]).

Lemma 2.5. *If* $A = \begin{pmatrix} \lambda_1 & 2\sigma \\ 0 & \lambda_2 \end{pmatrix}$ *and* $J = \mathrm{diag}\,(1,-1)$*, then one of the following hold:*

(1) If $|\lambda_1 - \lambda_2| > 2|\sigma| > 0$*, then* $\mathcal{S}^J(A)$ *is the union of the two sheets of an hyperboloid (without interior) centered at*

$$\left(\frac{\lambda_1 + \lambda_2}{2}, \frac{|\lambda_1|^2 + |\lambda_2|^2}{2} - 2|\sigma|^2 \right).$$

Moreover, $(x,y,z) \in \mathcal{S}_+^J(A)$ *only if, for* $z = (|\lambda_1|^2 + |\lambda_2|^2)/2 - 2|\sigma|^2$*,* $|x+iy-\lambda_1| < |x+iy-\lambda_2|$ *and* $(x,y,z) \in \mathcal{S}_-^J(A)$ *only if, for* $z = (|\lambda_1|^2 + |\lambda_2|^2)/2 - 2|\sigma|^2$*,* $|x+iy-\lambda_1| > |x+iy-\lambda_2|$.

(2) For $2|\sigma| > |\lambda_1 - \lambda_2|$*, then* $\mathcal{S}^J(A)$ *is the union of the two sheets of an hyperboloid (without interior) centered at*

$$\left(\frac{\lambda_1 + \lambda_2}{2}, \frac{|\lambda_1|^2 + |\lambda_2|^2}{2} - 2|\sigma|^2 \right).$$

Moreover, $(x,y,z) \in \mathcal{S}_+^J(A)$ *only if* $z > -(\sqrt{|\sigma|^2 - (|\lambda_1 - \lambda_2|/2)^2} - |\sigma|)^2$ *and* $(x,y,z) \in \mathcal{S}_-^J(A)$ *only if* $z < -(\sqrt{|\sigma|^2 - (|\lambda_1 + \lambda_2|/2)^2} - |\sigma|)^2$.

(3) If $|\lambda_1 - \lambda_2| = 2|\sigma|$*, then* $\mathcal{S}^J(A)$ *degenerates into the plane equidistant from* $(\Re\lambda_1, \Im\lambda_1, 0)$ *and* $(\Re\lambda_2, \Im\lambda_2, 0)$.

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(4) For $\sigma = 0$, $\mathcal{S}^J(A)$ degenerates into two half-rays.

Proof. (1) We may assume $\lambda_1 = -\lambda_2 = \lambda > 0$, otherwise, we may replace A by a matrix of the form $\mu(A - \nu I)$ for some $\mu, \nu \in \mathbb{C}$ with $|\mu| = 1$. We consider

$$A = \begin{bmatrix} \lambda & 2\sigma \\ 0 & -\lambda \end{bmatrix}, \quad \lambda, \sigma \geq 0.$$

Suppose $(x, y, z) \in S^J(A)$. Then, there exists a J_2-unitary matrix V such that

$$V^{\#}AV = \begin{bmatrix} e & f \\ g & -e \end{bmatrix}, \quad V^{\#}A^{\#}AV = \begin{bmatrix} |e|^2 - |g|^2 & * \\ * & |e|^2 - |f|^2 \end{bmatrix},$$

where $e = x + iy$. Thus,

$$z = |e|^2 - |g|^2 = x^2 + y^2 - |g|^2, \tag{2.3}$$

the norm of the generating vector being positive. The case of a generating vector with negative norm is similarly treated. It is clear that

$$\mathrm{Tr}\,(V^{\#}A^{\#}AV) = 2|e|^2 - |f|^2 - |g|^2 = 2\lambda^2 - 4|\sigma|^2. \tag{2.4}$$

We also obtain $\det\,(V^{\#}AV) = -e^2 - fg = -\lambda^2$, so that

$$|f|^2|g|^2 = (\lambda^2 - e^2)(\lambda^2 - \bar{e}^2). \tag{2.5}$$

We show that, for $0 < \sigma < \lambda$, $\mathcal{S}^J_+(A)$ and $\mathcal{S}^J_-(A)$ are the right and left sheets of the hyperboloid (with nontransverse axis parallel to the X-axis):

$$\frac{x^2}{\lambda^2 - \sigma^2} - \frac{y^2}{\sigma^2} - \frac{(z - \lambda^2 + 2\sigma^2)^2}{4\sigma^2(\lambda^2 - \sigma^2)} = 1, \tag{2.6}$$

given by $x \geq \sqrt{\lambda^2 - \sigma^2}$ and $x \leq \sqrt{\lambda^2 - \sigma^2}$, respectively. We sketch the proof of the previous assertion. We rewrite (2.4) and (2.5) as

$$|f|^2 + |g|^2 = 2(x^2 + y^2 + 2\sigma^2 - \lambda^2),$$
$$|f|^2|g|^2 = 16(\sigma^2 x^2 + (\sigma^2 - \lambda^2)y^2 + (\sigma^2 - \lambda^2)\sigma^2),$$

so that

$$|g|^2 = x^2 + y^2 - \lambda^2 + 2\sigma^2 \pm 2\sqrt{\sigma^2 x^2 + (\sigma^2 - \lambda^2)y^2 + (\sigma^2 - \lambda^2)\sigma^2}.$$

Having in mind (2.3), we obtain

$$0 = z - \lambda^2 + 2\sigma^2 \pm 2\sqrt{\sigma^2 x^2 + (\sigma^2 - \lambda^2)y^2 + (\sigma^2 - \lambda^2)\sigma^2},$$

and (2.6) easily follows. The vertex of the right sheet of the hyperboloid is the point $(x, y, z) = (\sqrt{\lambda^2 - \sigma^2}, 0, \lambda^2 - 2\sigma^2)$ and x, y is on the right sheet if $|x + iy - \lambda| <$

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$|x+iy+\lambda|$ for $z=\lambda^2-2\sigma^2$. The vertex of the left sheet of the hyperboloid is the point $(x,y,z)=(-\sqrt{\lambda^2-\sigma^2},0,\lambda^2-2\sigma^2)$ and x,y is on the left sheet if $|x+iy-\lambda|>|x+iy+\lambda|$, for $z=\lambda^2-2\sigma^2$.

(2) For $\sigma>\lambda$, by similar arguments to those leading to (2.6), we can prove that $\mathcal{S}^J(A)$ is the union of the two sheets of the hyperboloid (without interior)

$$-\frac{x^2}{\sigma^2-\lambda^2}-\frac{y^2}{\sigma^2}+\frac{(z-\lambda^2+2\sigma^2)^2}{4\sigma^2(\sigma^2-\lambda^2)}=1. \tag{2.7}$$

We observe that the nontransverse axis of the hyperboloid is parallel to the Z-axis. Having in mind (2.3), we obtain, for $\mathcal{S}^J_+(A)$, the upper sheet of the hyperboloid:

$$0=z-\lambda^2+2\sigma^2-2\sqrt{\sigma^2x^2+(\sigma^2-\lambda^2)y^2+(\sigma^2-\lambda^2)\sigma^2}.$$

Similarly, we find for $\mathcal{S}^J_-(A)$, the lower sheet of the hyperboloid:

$$0=z-\lambda^2+2\sigma^2+2\sqrt{\sigma^2x^2+(\sigma^2-\lambda^2)y^2+(\sigma^2-\lambda^2)\sigma^2}.$$

(3) Now, we investigate the degenerate cases. For $\sigma=0$, the indefinite Davis-Wielandt shell degenerates into two half rays: $-\infty<x\le-\lambda,\ y=0,\ z=\lambda^2$ $(\mathcal{S}^J_-(A))$, or $\lambda\le x<+\infty,\ y=0,\ z=\lambda^2\ \ (\mathcal{S}^J_+(A))$.

(4) For $\sigma=\lambda$, the Davis-Wielandt shell degenerates into the plane $x=0,\ z,y\in\mathbb{R}$. ∎

Suppose that $J=J_2\oplus\cdots\oplus J_2\oplus -I_s\in M_{2r+s}$ $(2r+s\ge3)$, and A is a quadratic operator in the form

$$A=\bigoplus_{j=1}^{r}A_r\oplus\lambda_2I_s,\quad A_j=\begin{bmatrix}\lambda_1 & 2\sigma_j\\ 0 & \lambda_2\end{bmatrix},\quad \sigma_1\ge\cdots\ge\sigma_r\ge0. \tag{2.8}$$

Let $\Omega_+=\operatorname{conv}\left(\bigcup_{j=1}^r\mathcal{S}^{J_2}_+(A_j)\right)$ and $\Omega_-=\operatorname{conv}\left(\bigcup_{j=1}^r\mathcal{S}^{J_2}_-(A_j)\cup\{(\lambda_2,\lambda_2\overline{\lambda_2})\}\right)$, where $\operatorname{conv}(S)$ denotes the convex hull of the set S. Since $\{(\lambda_2,\lambda_2\overline{\lambda_2})\}\subset\mathcal{S}^{\,J}_-(A_r)$, we get $\Omega_-=\operatorname{conv}\left(\bigcup_{j=1}^r\mathcal{S}^{J_2}_-(A_j)\right)$. For any $\omega_+\in\Omega_+,\ \omega_-\in\Omega_-$, consider the rays of the lines defined by ω_+,ω_- with extreme at ω_+ (ω_-) and not containing ω_- (ω_+). For commodity, we denote the union of all these rays by $\operatorname{psconv}(\Omega_+\cup\Omega_-)$.

In the proof of the next theorem, we use the following auxiliary lemma and the concept of *support plane*. Assume that $\mathcal{S}^{\,J}_-(A)\cap\mathcal{S}^{\,J}_+(A)=\emptyset$. A support plane of $\mathcal{S}(A)$ is a plane that touches $\mathcal{S}_+(A)$ (resp. $\mathcal{S}_-(A)$) and does not intersect $\mathcal{S}_-(A)$ (resp. $\mathcal{S}_+(A)$).

Lemma 2.6. *Let us consider 2 hyperbolas h_1,h_2 with the same center O, and let us denote by h_j^+,h_j^- the two branches of the hyperbola h_j. Moreover, let us assume*

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that a line passing through O defines tow half-planes such that one of them contains the branches h_1^+, h_2^+ and the other one contains the branches h_1^-, h_2^-. Then, the set

$$S = \mathrm{conv}\,(h_1^+ \cup h_2^+) \cup \mathrm{conv}\,(h_1^- \cup h_2^-)$$

is pseudoconvex.

Proof. Elementary. □

THEOREM 2.7. *Let $A : \mathcal{H} \to \mathcal{H}$ be a non-scalar quadratic operator satisfying (1.1) and (1.2), with the eigenvalues of X arranged in non-increasing order $\sigma_1 \geq \cdots \geq \sigma_r \geq 0$ and let $J = USU^* = J_2 \oplus \cdots \oplus J_2 \oplus -I_s \in M_{2r+s}$ $(2r+s \geq 3)$. One of the following holds:*

(1) If $2\sigma_1 < |\lambda_1 - \lambda_2|$, then $\mathcal{S}^J(A) = \mathrm{psconv}\mathcal{S}^{J_2}(A_1)$.

(2) (i) If there exists a k $(1 \leq k \leq r-1)$ such that $2\sigma_k > |\lambda_1 - \lambda_2|$ and $2\sigma_{k+1} < |\lambda_1 - \lambda_2|$, then

$$\mathcal{S}^J(A) = \mathrm{conv}\,\left(\mathcal{S}_+^{J_2}(A_1) \cup \mathcal{S}_+^{J_2}(A_{k+1})\right) \cup \mathrm{conv}\,\left(\mathcal{S}_-^{J_2}(A_2) \cup \mathcal{S}_-^{J_2}(A_1)\right).$$

(ii) If $2\sigma_k > |\lambda_1 - \lambda_2|$, $1 \leq k \leq r$, then $\mathcal{S}^J(A)$ consists of the two sheets of an hyperboloid, namely, $\mathcal{S}_+^{J_2}(A_r)$ and $\mathcal{S}_-^{J_2}(A_r)$, with the interior of each sheet.

(3) If $2\sigma_1 = |\lambda_1 - \lambda_2|$, then $\mathcal{S}^J(A)$ degenerates into the plane

$$\{(x,y,z) : |x+iy-\lambda_1| = |x+iy-\lambda_2|,\ z \in \mathbb{R}\}.$$

(4) If $\sigma_1 = 0$, then $\mathcal{S}^J(A)$ degenerates into the half-rays of the line defined by λ_1, λ_2 with endpoints $(\Re\lambda_1, \Im\lambda_1,\ (|\lambda_1|^2 + |\lambda_2|^2)/2)$ and $(\Re\lambda_2, \Im\lambda_2, (|\lambda_1|^2 + |\lambda_2|^2)/2)$ and not containing $(\Re\lambda_2, \Im\lambda_2, (|\lambda_1|^2 + |\lambda_2|^2)/2)$, $(\Re\lambda_1, \Im\lambda_1,\ (|\lambda_1|^2 + |\lambda_2|^2)/2)$, respectively.

Proof. (1) Consider the polynomial in the variables u, v, w, t

$$f(u,v,w,t) = \det\,\left(\frac{u}{2}(A + A^\#) + \frac{v}{2i}(A - A^\#) + w(A^\# A) + tI\right).$$

For A as in (2.8), we obtain $f(u,v,w,t) = \prod_{j=1}^{r+1} f_j(u,v,w,t)$, where

$$f_j(u,v,w,t) = \det\,\left(\frac{u}{2}(A_j + A_j^\#) + \frac{v}{2i}(A_j - A_j^\#) + w(A_j^\# A_j) + tI_2\right), \quad j = 1, \ldots, r$$

and

$$f_{r+1}(u,v,w,t) = \left(\frac{u}{2}(\lambda_2 + \bar{\lambda}_2) + \frac{v}{2i}(\lambda_2 - \bar{\lambda}_2) + w(\bar{\lambda}_2\lambda_2) + t\right)^s.$$

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The projection of $\mathcal{S}^J(A)$ on the direction (u,v,w) with $u^2+v^2+w^2=1$, coincides with

$$W^J\left(\frac{u}{2}(A+A^{\#})+\frac{v}{2i}(A-A^{\#})+w(A^{\#}A)\right).$$

If $2\sigma_1<|\lambda_1-\lambda_2|$, then the sets $\mathcal{S}^{J_2}(A_j),\ j=1,\dots,r$, are hyperboloids by Lemma 2.5 (1). Having in mind that $\sigma_1\geq\cdots\geq\sigma_r$, these hyperboloids are nested:

$$\operatorname{conv} S_{\pm}^{J_2}(A_r)\subseteq\cdots\subseteq\operatorname{conv} S_{\pm}^{J_2}(A_1).$$

So, if $2\sigma_1<|\lambda_1-\lambda_2|$, the plane perpendicular to the direction (u,v,w) supports $\mathcal{S}^{J_2}(A_1)$ and henceforth $\mathcal{S}^J(A)\subseteq \operatorname{psconv}\mathcal{S}^{J_2}(A_1)$.

Next we prove that the reverse inclusion also holds. We show that $\mathcal{S}^J(A)\supseteq \operatorname{conv}\mathcal{S}_+^{J_2}(A_1)$ and $\mathcal{S}^J(A)\supseteq\operatorname{conv}\mathcal{S}_-^{J_2}(A_1)$. Obviously,

$$\operatorname{conv}\mathcal{S}_-^{J_2}(A_1)=\operatorname{conv}\bigcup_{j=1}^{r}\mathcal{S}_-^{J_2}(A_j).$$

Now, $\operatorname{conv}\bigcup_{j=1}^{r}\mathcal{S}_-^{J_2}(A_j)$ is the set of points

$$(x+iy,z)=\frac{\displaystyle\sum_{j=1}^{r}(x_j+iy_j,z_j)p_j}{\displaystyle\sum_{j=1}^{r}p_j},$$

where $(x_j+y_j,z_j)\in\mathcal{S}_-^{J_2}(A_j),\ p_j\geq 0,\ j=1,\dots,r$. There exists a

$$\zeta=[\zeta_1^T,\dots,\zeta_r^T,\zeta'^T]^T,$$

with $\zeta_1,\dots,\zeta_r\in\mathbb{C}^2,\ \zeta'\in\mathbb{C}^s,\ \zeta_j^*J_2\zeta_j<0,\ j=1,\dots,r$, and $\zeta'^*\zeta'=0$, such that

$$x_j+iy_j=\frac{\zeta_j^*J_2A_j\zeta_j}{\zeta_j^*J_2\zeta_j},\quad z_j=\frac{\zeta_j^*A_j^*J_2A_j\zeta_j}{\zeta_j^*J_2\zeta_j},\quad p_j=-\zeta_j^*J_2\zeta_j.$$

Thus, $\mathcal{S}^J(A)\supseteq\operatorname{conv}\mathcal{S}_-^{J_2}(A_1)$. Similarly, we show that $\mathcal{S}^J(A)\supseteq\operatorname{conv}\mathcal{S}_+^{J_2}(A_1)$. We have proved that, if $2\sigma_1<|\lambda_1-\lambda_2|$, then

$$\mathcal{S}^J(A)\supseteq\operatorname{conv}\mathcal{S}_+^{J_2}(A_1)\cup\operatorname{conv}\mathcal{S}_-^{J_2}(A_1)=\operatorname{psconv}\mathcal{S}^{J_2}(A_1).$$

Henceforth, $\mathcal{S}^J(A)=\operatorname{psconv}\mathcal{S}^{J_2}(A_1)$. Having in mind Lemma 2.5 (1), the result follows.

(2) (i) Let $(x+iy,z)\in\mathcal{S}^J(A),\ (x,y,z\in\mathbb{R})$. Then there exists a

$$\zeta=[\zeta_1^T,\dots,\zeta_r^T,\zeta'^T]^T,$$

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with $\zeta^* J\zeta \neq 0$ and $\zeta_1, \ldots, \zeta_r \in \mathbb{C}^2,\ \zeta' \in \mathbb{C}^s$, such that

$$x + iy = \frac{\zeta^* JA\zeta}{\zeta^* J\zeta}, \quad z = \frac{\zeta^* A^* JA\zeta}{\zeta^* J\zeta}.$$

Assume that $\zeta_j^* J_2\zeta_j \neq 0,\ j = 1, \ldots, r,\ \zeta^{'*}\zeta' \neq 0$ (otherwise, perturb ζ and use a continuity argument). Then

$$x + iy = \frac{\displaystyle\sum_{j=1}^{r}(x_j + iy_j)\zeta_j^* J_2\zeta_j - \lambda_2 \zeta'^* \zeta'}{\displaystyle\sum_{j=1}^{r}\zeta_j^* J_2\zeta_j - \zeta'^*\zeta'},$$

$$z = \frac{\displaystyle\sum_{j=1}^{r} z_j\zeta_j^* J_2\zeta_j - \lambda_2\overline{\lambda_2}\zeta'^*\zeta'}{\displaystyle\sum_{j=1}^{r}\zeta_j^* J_2\zeta_j - \zeta'^*\zeta'},$$

where

$$x_j + iy_j = \frac{\zeta_j^* J_2 A_j\zeta_j}{\zeta_j^* J_2\zeta_j}, \quad z_j = \frac{\zeta_j^* A_j^* J_2 A_j\zeta_j}{\zeta_j^* J_2\zeta_j}.$$

We observe that the hyperboloids $\mathcal{S}_+^{J_2}(A_1), \ldots, \mathcal{S}_+^{J_2}(A_k)$, as well as the hyperboloids $\mathcal{S}_+^{J_2}(A_{k+1}), \ldots, \mathcal{S}_+^{J_2}(A_r)$, are nested, that is

$$\operatorname{conv}\mathcal{S}_+^{J_2}(A_1) \subseteq \cdots \subseteq \operatorname{conv}\mathcal{S}_+^{J_2}(A_k),$$

and

$$\operatorname{conv}\mathcal{S}_+^{J_2}(A_{k+1}) \subseteq \cdots \subseteq \operatorname{conv}\mathcal{S}_+^{J_2}(A_r).$$

Also

$$\operatorname{conv}\mathcal{S}_-^{J_2}(A_1) \subseteq \cdots \subseteq \operatorname{conv}\mathcal{S}_-^{J_2}(A_k),$$

and

$$\operatorname{conv}\mathcal{S}_-^{J_2}(A_{k+1}) \subseteq \cdots \subseteq \operatorname{conv}\mathcal{S}_-^{J_2}(A_r).$$

Let

$$x' + iy' = \frac{\displaystyle\sum_{j=k+1}^{r}(x_j + iy_j)\zeta_j^* J_2\zeta_j - \lambda_2\zeta'^*\zeta'}{\displaystyle\sum_{j=k+1}^{r}\zeta_j^* J_2\zeta_j - \zeta'^*\zeta'}, \quad z' = \frac{\displaystyle\sum_{j=k+1}^{r} z_j\zeta_j^* J_2\zeta_j - \lambda_2\overline{\lambda_2}\zeta'^*\zeta'}{\displaystyle\sum_{j=k+1}^{r}\zeta_j^* J_2\zeta_j - \zeta'^*\zeta'},$$

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and

$$x'' + iy'' = \frac{\displaystyle\sum_{j=1}^{k}(x_j + iy_j)\zeta_j^* J_2 \zeta_j}{\displaystyle\sum_{j=1}^{k}\zeta_j^* J_2 \zeta_j}, \quad z'' = \frac{\displaystyle\sum_{j=1}^{k} z_j \zeta_j^* J_2 \zeta_j}{\displaystyle\sum_{j=1}^{k}\zeta_j^* J_2 \zeta_j}.$$

Clearly,

$$(x+iy, z) = (x'+iy', z')\frac{\sum_{j=k+1}^{r}\zeta_j^* J_2 \zeta_j}{\displaystyle\sum_{j=1}^{r}\zeta_j^* J_2 \zeta_j - \zeta'^*\zeta'} + (x''+iy'', z'')\frac{\sum_{j=1}^{k}\zeta_j^* J_2 \zeta_j}{\displaystyle\sum_{j=1}^{r}\zeta_j^* J_2 \zeta_j - \zeta'^*\zeta'}.$$

We have shown that if

$$(x+iy, z) = \left(\frac{\zeta^* J A \zeta}{\zeta^* J \zeta}, \frac{\zeta^* A^* J A \zeta}{\zeta^* J \zeta}\right),$$

then $(x+iy, z) \in \operatorname{psconv}\left(\mathcal{S}^{J_2}(A_1) \cup \mathcal{S}^{J_2}(A_{r+1})\right)$. Conversely, if

$$(x+iy, z) \in \operatorname{psconv}\left(\mathcal{S}^{J_2}(A_1) \cup \mathcal{S}^{J_2}(A_{r+1})\right),$$

then there exists a $\zeta \in \mathbb{C}^{2r+s}$ such that

$$(x+iy, z) = \left(\frac{\zeta^* J A \zeta}{\zeta^* J \zeta}, \frac{\zeta^* A^* J A \zeta}{\zeta^* J \zeta}\right).$$

By Lemma 2.6, (2) (i) holds. Indeed, let Π be a supporting plane of $\mathcal{S}_+^{J_2}(A_1) \cup \mathcal{S}_+^{J_2}(A_{k+1})$, and Π_0 a plane passing through the origin and through a point where Π touches $\mathcal{S}_+^{J_2}(A_1)\cup\mathcal{S}_+^{J_2}(A_{k+1})$. We show that Π does not intersect $\mathcal{S}_-^{J_2}(A_1)\cup\mathcal{S}_-^{J_2}(A_{k+1})$. Clearly, Π_0 intersects $\mathcal{S}_+^{J_2}(A_1)$ and $\mathcal{S}_+^{J_2}(A_{k+1})$ along two branches of hyperbolae, and Π along a line supporting the union of those branches. By Lemma 2.6, this line does not intersect the complementary branches. Thus, the assertion follows.

(2) (ii) By Lemma 2.5 (2), the sets $\mathcal{S}_+^{J_2}(A_j)$ and $\mathcal{S}_-^{J_2}(A_j)$, $j = 1, \dots r$, are the upper and the lower sheets of hyperboloids, respectively. These hyperboloids are nested, that is

$$\operatorname{conv}\mathcal{S}_+^{J_2}(A_1) \subseteq \operatorname{conv}\mathcal{S}_+^{J_2}(A_2) \subseteq \cdots \subseteq \operatorname{conv}\mathcal{S}_+^{J_2}(A_r),$$

and

$$\operatorname{conv}\mathcal{S}_-^{J_2}(A_1) \subseteq \operatorname{conv}\mathcal{S}_-^{J_2}(A_2) \subseteq \cdots \subseteq \operatorname{conv}\mathcal{S}_-^{J_2}(A_r).$$

This may be easily seen having in mind (2.7) and $\sigma_1 \geq \cdots \geq \sigma_r$.

(3) It is an easy consequence of Lemma 2.5 (3).

(4) It is an immediate consequence of Lemma 2.5 (4). ☐

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3. Final remarks. In this note we have studied the indefinite numerical range, rank-k numerical range and the Wielandt-shell of a quadratic operator. A special case when the operator J is of a form compatible with A as in (1.2) was considered. Under this restriction, some results parallel to those obtained for generalized numerical ranges of quadratic operators acting on Hilbert spaces were derived. The lack of generality on the choice of the indefinite forms here assumed is due to the following reasons. Given an indefinite matrix $S \in M_n$, there exists a non-singular matrix $R \in M_n$ such that R^*SR is of the form $J = I_r \oplus (-I_{n-r})$. Hence, for any $A \in M_n$, $W^S(A) = W^J(R^*AR)$. So the investigation of $W^S(A)$ can be reduced to the case $S = J$. However, in the case $\mathcal{H}$ is infinite dimensional, there may not exist any bounded linear operator such that $R^*SR = I_{H_1} \oplus (-I_{H_2})$. Even if such an operator R exists, the operator R^*AR may not be quadratic. Moreover, in the case R^*AR is quadratic, it might not be in the form (1.2). Our choice of the form of the operator J was motivated by simplifying reasons. We also notice that in the case of the indefinite rank-k numerical ranges and Wielandt-shells, only finite dimensional quadratic operators have been considered. It is challenging to continue in a more general setting the research here initiated. Quadratic operators in Hilbert spaces have been generalized in [15]. The indefinite analogue would deserve investigation.

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