

EXTREMAL LAPLACIAN-ENERGY-LIKE INVARIANT OF GRAPHS WITH GIVEN MATCHING NUMBER*

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Abstract. Let G be a graph of order n with Laplacian spectrum $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. The Laplacian-energy-like invariant of graph G , LEL for short, is defined as: $LEL(G) = \sum_{k=1}^{n-1} \sqrt{\mu_k}$. In this note, the extremal (maximal and minimal) LEL among all the connected graphs with given matching number is determined. The corresponding extremal graphs are completely characterized with respect to LEL. Moreover a relationship between LEL and the independence number is presented in this note.

Key words. Laplacian matrix, Laplacian-energy-like, Matching number.

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1. Introduction. Let $G = (V, E)$ be a simple undirected graph with vertex set $V(G) = \{v_1, v_2, v_3, \dots, v_n\}$ and edge set $E(G)$. Also let d_i be the degree of the vertex v_i for $i = 1, 2, \dots, n$. Assume that $A(G)$ is the $(0, 1)$ -adjacency matrix of G and $D(G)$ is the diagonal matrix of vertex degrees. The Laplacian matrix of G is $L(G) = D(G) - A(G)$. The Laplacian polynomial $P(G, \lambda)$ of G is the characteristic polynomial of its Laplacian matrix, $P(G, \lambda) = \det(\lambda I_n - L(G)) = \sum_{k=0}^n (-1)^k c_k \lambda^{n-k}$. The Laplacian matrix $L(G)$ has nonnegative eigenvalues $n \geq \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ [3]. Denote by $S(G) = \{\mu_1, \mu_2, \dots, \mu_n\}$ the spectrum of $L(G)$, i.e., the Laplacian spectrum of G . If the eigenvalue μ_i appears $l_i > 1$ times in $S(G)$, we write them as $\mu_i^{(l_i)}$ for the sake of convenience.

All graphs considered in this paper are finite and simple. For two nonadjacent vertices v_i and v_j , we use $G + e$ to denote the graph obtained by inserting a new edge $e = v_i v_j$ in G . Similarly, for $e \in E(G)$ of graph G , let $G - e$ be the subgraph of G obtained by deleting the edge e from $E(G)$. The complement of graph G is always

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denoted by $\overline{G}$. For two vertex disjoint graphs G_1 and G_2 , we denote by $G_1 \cup G_2$ the graph which consists of two connected components G_1 and G_2 . The *join* of G_1 and G_2 , denoted by $G_1 \vee G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{u_i v_j : u_i \in V(G_1), v_j \in V(G_2)\}$. Given a graph G , a subset $S(G)$ of $V(G)$ is called an *independent set* of G if the subgraph it induces has no edges. The *independence number* of G , denoted by $\alpha(G)$, is defined to be the number of vertices in a largest independent set of G . A subset S of $V(G)$ is called a *dominating set* of G if for every vertex $v \in V - S$, there exists a vertex $u \in S$ such that v is adjacent to u . The *domination number* of graph G , denoted by $\gamma(G)$, is defined as the minimum cardinality of dominating sets of G . Two edges e_1 and e_2 are independent if they do not have a common vertex. A matching of G is a subset of mutually independent edges of G . For a graph G , the *matching number* $\beta(G)$ is the maximum cardinality among the independent sets of edges in G . The *components* of a graph G are its maximal connected subgraphs. Components of odd (even) order are called odd (even) components. For other undefined notation and terminology from graph theory, the readers are referred to [2].

Recently a graph invariant was introduced by Liu and Liu [20]:

$$LEL(G) = \sum_{k=1}^{n-1} \sqrt{\mu_k}.$$

Moreover, in [20], it was shown that $LEL(G)$ has properties similar to those of graph molecular energy, defined by Gutman [7] (for more details on the chemical aspects and mathematical properties of graph energy, the readers are referred to [8]). The LEL has attracted the attention of more and more researchers. In [25], Stevanović *et al.* proved that, for a set of polycyclic aromatic hydrocarbons, LEL is as good as Randić index (a connectivity index) and better than Wiener index (a distance based index) for indicating their chemical properties (such as melting point MP, boiling point BP, and so on). In [9], Gutman *et al.* proved that, for bipartite graph G , LEL coincide with its incidence energy $IE(G)$, as defined in [18]. And some nice results have been obtained, such as for trees ([13, 14, 15, 16, 17, 27]), unicyclic graphs ([24]), bicyclic graphs ([11]) and so on. Recently Zhu [29] characterized the maximal LEL of all graphs with connectivity number and with chromatic number, respectively. Liu, Liu and Tan [21] determined the nine greatest LEL of all connected graphs of order n . For more recent results on the LEL, see [4, 19].

Let $\mathcal{G}_{n,\beta}$ be the set of connected graphs of order n and with matching number β . Zhou and Trinajstić [28] determined the extremal Kirchhoff index of graphs in $\mathcal{G}_{n,\beta}$. Feng *et al.* [5, 6] characterized the extremal graph with respect to spectral radius and Zagreb indices, Harary index and hyper-Wiener index, respectively, among all graphs from $\mathcal{G}_{n,\beta}$. Inspired by their two results, in this paper we determine the extremal LEL among all graphs in $\mathcal{G}_{n,\beta}$. Moreover, some related results including

independence number and domination number have been presented with respect to LEL.

2. Some lemmas. Note that the Laplacian eigenvalues of an edge-deleted graph $G - e$ are interlaced to those of G (see [10]), and $\sum_{i=1}^{n-1} \mu_i(G) - \sum_{i=1}^{n-1} \mu_i(G - e) = 2$, the following lemma can be easily obtained.

LEMMA 2.1. [19, 29] *Let G be a graph with $e \in E(G)$ and two nonadjacent vertices v_i and v_j in $V(G)$. Then we have*

- (1) $LEL(G - e) < LEL(G)$;
- (2) $LEL(G) < LEL(G + e')$ where $e' = v_i v_j$.

LEMMA 2.2. [22] *Let G be a graph of order n with $S(G) = \{\mu_1, \mu_2, \dots, \mu_{n-1}, 0\}$. Then $S(\overline{G}) = \{n - \mu_1, n - \mu_2, \dots, n - \mu_{n-1}, 0\}$.*

LEMMA 2.3. *Let $f(x)$ be a function with $x > 0$ such that $f''(x) > 0$. Then, for $1 < x_1 \leq x_2$, we have $f(1) + f(x_1 + x_2 - 1) > f(x_1) + f(x_2)$.*

Proof. Clearly, it suffices to prove that $f(x_1 + x_2 - 1) - f(x_1) - f(x_2) + f(1) > 0$.

Now we first define a new function: $g(x_1) = f(x_1 + x_2 - 1) - f(x_1) - f(x_2) + f(1)$ where $1 < x_1 \leq x_2$ and x_2 is a constant. Taking the first derivative of $g(x_1)$, noticing that $f''(x) > 0$, i.e., the function $f'(x)$ is strictly increasing, we have

$$g'(x_1) = f'(x_1 + x_2 - 1) - f'(x_1) > 0.$$

Therefore, $g(x_1) > g(1)$. It is equivalent that $f(x_1 + x_2 - 1) - f(x_1) - f(x_2) + f(1) > 0$, which completes the proof of this lemma. $\square$

The following is a well-known result from [1, 26] on the matching number of a graph.

LEMMA 2.4. (The Tutte-Berge Formula) *Suppose that G is a graph of order n with matching number β . Then $n - 2\beta = \max\{o(G - S) - |S| : S \subseteq V(G)\}$ where $o(G)$ denotes the number of odd components in graph G .*

3. Main results. Now we begin the determination of graphs in $\mathcal{G}_{n,\beta}$ that have maximal LEL. If $\beta = 1$, the set $\mathcal{G}_{n,\beta}$ contains K_3 or S_n where S_n denotes a star with n vertices. And the extremal graph in $\mathcal{G}_{4,2}$ with maximal LEL is the complete graph

K_4 . So in the following we always assume that $\beta > 2$ and $n \geq 5$.

LEMMA 3.1. *Let G be a graph from $\mathcal{G}_{n,\beta}$ and with maximal LEL. Then we have*

$$G = K_s \vee \left(\bigcup_{i=1}^q K_{n_i} \right),$$

where $n_1, n_2, \dots, n_q$ are all odd positive integers with $s = q + 2\beta - n$ and $\sum_{i=1}^q n_i = n - s$.

Proof. By Lemma 2.4, we find that there exists a subset S with s vertices in $V(G)$ such that $G - S$ has $q = n + s - 2\beta$ odd components $G_1, G_2, \dots, G_q$ which are of orders $n_1, n_2, \dots, n_q$, respectively. Without loss of generality, we can assume that $n_1 \leq n_2 \leq \dots \leq n_q$. Obviously, $n \geq s + q = n + 2s - 2\beta$. Therefore, we have $s \leq \beta$.

If $G - S$ contains even components, we denote by C the union of these even components. Now we add some edges so that $G[C \cup G_q]$ is changed into a complete graph. And this obtained graph is denoted by G' . Since $n - 2\beta(G') \geq o(G' - S) - |S| = o(G - S) - |S| = n - 2\beta(G)$ and $\beta(G') \geq \beta(G)$, we have $\beta(G') = \beta(G)$, i.e., $G' \in \mathcal{G}_{n,\beta}$. From Lemma 2.1 (2), we have $LEL(G') > LEL(G)$, this is a contradiction to the choice of G .

Thus, we find that $G - S$ does not have even components. Then $\bigcup_{i=1}^q V(G_i) = V(G) - S$. Now we claim that $G[S]$ and G_i are all complete graphs for $i = 1, 2, \dots, q$ and G contains all edges joining each vertex in S and each vertex in G_i where $i = 1, \dots, q$. If not, we can add repeatedly some edges in order to make a graph G' satisfying these above conditions. By a similar reasoning as above, G' also belongs to $\mathcal{G}_{n,\beta}$. Moreover, $LEL(G') > LEL(G)$ from Lemma 2.1 (2). A contradiction occurs to this maximality of $LEL(G)$, which completes the proof of this lemma. $\square$

THEOREM 3.2. *Let b be the largest root of cubic equation*

$$(2\sqrt{2} + 1)y^3 - (2 + \sqrt{n})y^2 - (n + 2\sqrt{2} - 1)y + n + \sqrt{n} = 0.$$

For any graph $G \in \mathcal{G}_{n,\beta}$ with $\beta > 2$ and $n \geq 5$, we have

- (1) *If $\beta = \lfloor \frac{n}{2} \rfloor$, then $LEL(G) \leq (n - 1)\sqrt{n}$ with equality if and only if $G \cong K_n$;*
- (2) *If $b^2 < \beta < \lfloor \frac{n}{2} \rfloor$, then $LEL(G) \leq \sqrt{n} + 2(\beta - 1)\sqrt{2\beta} + n - 2\beta - \beta\sqrt{n}$ with equality if and only if $G \cong K_1 \vee (\overline{K_{n-2\beta}} \cup K_{2\beta-1})$;*
- (3) *If $\beta = b^2$, then $LEL(G) \leq \beta\sqrt{n} + (n - \beta - 1)\sqrt{\beta}$ with equality if and only if $G \cong K_1 \vee (\overline{K_{n-2\beta}} \cup K_{2\beta-1})$ or $G \cong K_\beta \vee \overline{K_{n-\beta}}$.*
- (4) *If $2 < \beta < b^2$, then $LEL(G) \leq \beta\sqrt{n} + (n - \beta - 1)\sqrt{\beta}$ with equality if and only if $G \cong K_\beta \vee \overline{K_{n-\beta}}$.*

Proof. Let G be a graph from $\mathcal{G}_{n,\beta}$ with maximal LEL. By Lemma 3.1, we find that $G = K_s \vee (\bigcup_{i=1}^q K_{n_i})$ with $1 \leq n_1 \leq n_2 \leq \cdots \leq n_q$.

If $s = 0$, then $G - S = G$. Thus, $n - 2\beta = q \leq 1$. If $q = 0$, then $n = 2\beta$. And when $q = 1$, we have $\beta = \frac{n-1}{2}$. In the above two cases, by Lemma 2.1 (2), we have $G = K_n$ with $LEL(G) = (n-1)\sqrt{n}$, which completes the proof of (1). So in the following we assume that $s \geq 1$.

It is well known that $S(K_t) = \{t^{(t-1)}, 0\}$. Then we have

$$S(\bigcup_{i=1}^q K_{n_i}) = \{n_1^{(n_1-1)}, n_2^{(n_2-1)}, \dots, n_q^{(n_q-1)}, 0^{(q)}\}.$$

Considering that $\sum_{i=1}^q n_i = n - s$, by Lemma 2.2, we get

$$\begin{aligned} \overline{S(\bigcup_{i=1}^q K_{n_i})} &= \{(n-s-n_1)^{(n_1-1)}, (n-s-n_2)^{(n_2-1)}, \dots, (n-s-n_q)^{(n_q-1)}, (n-s)^{(q-1)}, 0\} \\ S(\overline{G}) &= S(\overline{K_s} \cup \bigcup_{i=1}^q K_{n_i}) \\ &= \{(n-s-n_1)^{(n_1-1)}, (n-s-n_2)^{(n_2-1)}, \dots, (n-s-n_q)^{(n_q-1)}, (n-s)^{(q-1)}, 0^{(s+1)}\}. \end{aligned}$$

In view of Lemma 2.2, again, we have

$$S(G) = \{n^{(s)}, (s+n_1)^{(n_1-1)}, (s+n_2)^{(n_2-1)}, \dots, (s+n_q)^{(n_q-1)}, s^{(q-1)}, 0\}.$$

By definition, we have

$$LEL(G) = s\sqrt{n} + (n_1-1)\sqrt{s+n_1} + (n_2-1)\sqrt{s+n_2} + \cdots + (n_q-1)\sqrt{s+n_q} + (q-1)\sqrt{s}.$$

Now we define a function $f(x) = (x-1)\sqrt{s+x}$ where $x > 0$ and s is positive constant. By a simple calculation, we have $f''(x) = \frac{4s+3x-1}{4(s+x)\sqrt{s+x}} > 0$. If $3 \leq n_i \leq n_j$, from Lemma 2.3, we have

$$0 + (n_i + n_j - 1)\sqrt{n_i + n_j + s} > (n_i - 1)\sqrt{n_i + s} + (n_j - 1)\sqrt{n_j + s}.$$

Thus, considering the formula $LEL(G)$, replacing the pair (n_i, n_j) by another pair $(1, n_i + n_j - 1)$, we can get a new graph still in $\mathcal{G}_{n,\beta}$ but having a larger LEL.

Repeating the above process, we find that $LEL(G)$ reaches its maximum when $n_1 = n_2 = \cdots = n_{q-1} = 1$ and $n_q = n - s - q + 1$. Note that $n - q + s = 2\beta$. Thus, in this case, we have

$$LEL(G) = s\sqrt{n} + 2(\beta - s)\sqrt{2\beta - s + 1} + (n + s - 2\beta - 1)\sqrt{s}.$$

Now we consider the function

$$h(x) = x\sqrt{n} + 2(\beta - x)\sqrt{2\beta - x + 1} + (n + x - 2\beta - 1)\sqrt{x}$$

with $1 \leq x \leq \beta \leq \lfloor \frac{n}{2} \rfloor$. Next we determine the maximum value of $h(x)$.

Setting $\delta = \beta - x$ and a function $f(x, \delta) = x\sqrt{n} + 2\delta\sqrt{x + 2\delta + 1} + (n - x - 2\delta - 1)\sqrt{x}$ with $1 \leq \delta \leq \beta$, then we have

$$\frac{df(x, \delta)}{d\delta} = 2\sqrt{x + 2\delta + 1} + \frac{2\delta}{\sqrt{x + 2\delta + 1}} - 2\sqrt{x} > 0,$$

and

$$\frac{df(x, \delta)}{dx} = \sqrt{n} + \frac{\delta}{\sqrt{x + 2\delta + 1}} - \sqrt{x} + \frac{n - x - 2\delta - 1}{2\sqrt{x}} > 0.$$

Thus, $f(x, \delta) \leq f(\beta, \delta) = f(\beta, 0) = \beta\sqrt{n} + (n - \beta - 1)\sqrt{\beta} = h(\beta)$, and

$$f(x, \delta) \leq f(x, \beta - 1) = f(1, \beta - 1) = \sqrt{n} + 2(\beta - 1)\sqrt{2\beta} + n - 2\beta = h(1).$$

Therefore, we claim that $h(x)$ reaches its maximum value at $x = 1$ or $x = \beta$. Note that

$$h(1) - h(\beta) = \sqrt{n} + 2(\beta - 1)\sqrt{2\beta} + n - 2\beta - \beta\sqrt{n} - (n - \beta - 1)\sqrt{\beta}.$$

Let $\sqrt{\beta} = y$. It follows that

$$\begin{aligned} h(1) - h(\beta) &= \sqrt{n} + 2(y^2 - 1)\sqrt{2}y + n - 2y^2 - \sqrt{n}y^2 - (n - y^2 - 1)y \\ &= (2\sqrt{2} + 1)y^3 - (2 + \sqrt{n})y^2 - (n + 2\sqrt{2} - 1)y + n + \sqrt{n}. \end{aligned}$$

Let $F(y) = (2\sqrt{2} + 1)y^3 - (2 + \sqrt{n})y^2 - (n + 2\sqrt{2} - 1)y + n + \sqrt{n}$. It is obvious that

$$F(-\sqrt{n}) = -2\sqrt{2}(n - 1)\sqrt{n} - (\sqrt{n} + 1)n < 0,$$

$$F(-1) = 2n - 4 > 0,$$

$$F(\sqrt{2}) = 3\sqrt{2} - \sqrt{n} - (\sqrt{2} - 1)n \leq 3\sqrt{2} - \sqrt{5} - 5(\sqrt{2} - 1) < 0 \text{ for } n \geq 5,$$

$$\begin{aligned} F(\sqrt{\frac{n}{2}}) &= (2\sqrt{2} + 1)\frac{n}{2}\sqrt{\frac{n}{2}} - (2 + \sqrt{n})\frac{n}{2} - (n + 2\sqrt{2} - 1)\sqrt{\frac{n}{2}} + n + \sqrt{n} \\ &= (\frac{2\sqrt{2}+1}{2}\sqrt{\frac{n}{2}} - \frac{\sqrt{n}}{2})n - (n + \sqrt{2} - 1)\sqrt{\frac{n}{2}} \\ &= (\frac{\sqrt{2}-1}{2}n - \sqrt{2} + 1)\sqrt{\frac{n}{2}} > 0. \end{aligned}$$

Therefore, the roots of $F(x) = 0$ lie in the intervals $(-\sqrt{n}, -1)$, $(-1, \sqrt{2})$ and $(\sqrt{2}, \sqrt{\frac{n}{2}})$. Thus, $F(x)$ has exactly one root in $(\sqrt{2}, \sqrt{\frac{n}{2}})$. Assume that this root is b . Clearly, we have $F(1) > F(\beta)$ if $\beta > b^2$ and $F(1) < F(\beta)$ if $\beta < b^2$.

By now we can claim that, in the case $2 \leq \beta < \lfloor \frac{n}{2} \rfloor$, the maximum value of $h(x)$ is attained at $x = 1$ when $\beta > b^2$ and at $x = \beta$ when $\beta < b^2$ and at $x = 1$ or $x = \beta$ if $\beta = b^2$. Bearing in mind that $G = K_s \vee ((n+s-2\beta-1)K_1 \cup K_{2\beta-2s+1}) \cong K_\beta \vee \overline{K_{n-\beta}}$ for $x = \beta$ and $G = K_1 \vee ((n-2\beta)K_1 \cup K_{2\beta-1}) \cong K_1 \vee (\overline{K_{n-2\beta}} \cup K_{2\beta-1})$ for $x = 1$, we complete the proof of this theorem. $\square$

In [23], Stevanović presented a nice connection between the Laplacian-energy-like invariant and the Laplacian coefficients.

LEMMA 3.3. *Let G and H be two graphs of order n . If $c_k(G) \leq c_k(H)$ for $k = 1, 2, \dots, n-1$, then $LEL(G) \leq LEL(H)$. Furthermore, if a strict inequality $c_k(G) < c_k(H)$ holds for some $1 \leq k \leq n-1$, then $LEL(G) < LEL(H)$.*

LEMMA 3.4. [12] *Let G be a connected graph with n vertices which consists of a subgraph H (with at least two vertices) and $n - |H|$ distinct pendent edges (not in H) attached to a vertex v in H . Then*

$$P(G, \lambda) = (\lambda - 1)^{n-|H|} P(H) - (n - |H|) \lambda (\lambda - 1)^{n-|H|-1} P(L_v(H)),$$

where $L_v(H)$ denotes the principal submatrix of $L(H)$ obtained by deleting the row and column corresponding to the vertex v .

Denote by $A(n, \beta)$ a tree obtained from star $S_{n-\beta+1}$ by a pendent edge to each $\beta - 1$ pendent vertices of $S_{n-\beta+1}$. $A(n, \beta)$ is called a *spur* (see [15]). Clearly, the matching number of $A(n, \beta)$ is β .

LEMMA 3.5. [15] *Among all the trees of order n and with matching number $1 \leq \beta \leq \frac{n}{2}$, the tree $A(n, \beta)$ has minimal Laplacian coefficient c_k for every $k = 0, 1, 2, \dots, n$.*

THEOREM 3.6. *Let p, q and r be three roots of cubic equation*

$$\lambda^3 - (n - \beta + 4)\lambda^2 + (3n - 3\beta + 4)\lambda - n = 0.$$

Among all graphs in $\mathcal{G}_{n,\beta}$, the tree $A(n, \beta)$ has minimal LEL with $LEL(A(n, \beta)) = n - 2\beta + \sqrt{5}(\beta - 2) + \sqrt{p} + \sqrt{q} + \sqrt{r}$.

Proof. Lemma 2.1 (1) implies that the extremal graph from $\mathcal{G}_{n,\beta}$ with minimal LEL must be a tree. Combining Lemmas 3.3 and 3.4, the result in this theorem follows immediately except the value of $LEL(A(n, \beta))$. Next we will calculate the value of $LEL(A(n, \beta))$.

Let A_{2k-1} be a tree obtained by attaching a pendent edge to each pendent vertex of star S_k and A'_{2k-1} be the tree obtained by deleting a pendent vertex of A_{2k-1} . By Lemma 3.4, we have

$$(3.1) \quad P(A'_{2\beta-1}, \lambda) = (\lambda - 1)P(A_{2\beta-3}, \lambda) - \lambda(\lambda^2 - 3\lambda + 1)^{\beta-2}$$

and

$$\begin{aligned} P(A_{2\beta-1}, \lambda) &= (\lambda - 1)P(A'_{2\beta-1}, \lambda) - \lambda \left[P(A_{2\beta-3}, \lambda) - (\lambda^2 - 3\lambda + 1)^{\beta-2} \right] \\ &= (\lambda^2 - 3\lambda + 1)P(A_{2\beta-3}, \lambda) - \lambda(\lambda - 2)(\lambda^2 - 3\lambda + 1)^{\beta-2} \quad \text{by (3.1)} \\ &= (\lambda^2 - 3\lambda + 1)^2 P(A_{2\beta-5}, \lambda) - 2\lambda(\lambda - 2)(\lambda^2 - 3\lambda + 1)^{\beta-2} \\ &\vdots \\ &= (\lambda^2 - 3\lambda + 1)^{\beta-2} P(A_3, \lambda) - (\beta - 2)\lambda(\lambda - 2)(\lambda^2 - 3\lambda + 1)^{\beta-2} \\ &= (\lambda^2 - 3\lambda + 1)^{\beta-2} \lambda \left[\lambda^2 - (\beta + 2)\lambda + 2\beta - 1 \right] \quad \text{as } P(A_3, \lambda) = \lambda(\lambda - 1)(\lambda - 3). \end{aligned}$$

By Lemma 3.4 and using above result, we get

$$\begin{aligned} P(A(n, \beta), \lambda) &= (\lambda - 1)^{n-2\beta+1} P(A_{2\beta-1}, \lambda) - (n - 2\beta + 1)\lambda(\lambda - 1)^{n-2\beta}(\lambda^2 - 3\lambda + 1)^{\beta-1} \\ &= (\lambda - 1)^{n-2\beta} \left[(\lambda - 1)P(A_{2\beta-1}, \lambda) - (n - 2\beta + 1)\lambda(\lambda^2 - 3\lambda + 1)^{\beta-1} \right] \\ &= (\lambda - 1)^{n-2\beta} (\lambda^2 - 3\lambda + 1)^{\beta-2} \lambda \left[\lambda^3 - (n - \beta + 4)\lambda^2 + (3n - 3\beta + 4)\lambda - n \right]. \end{aligned}$$

Note that the equation $\lambda^2 - 3\lambda + 1 = 0$ has two roots $\frac{3+\sqrt{5}}{2} = (\frac{\sqrt{5}+1}{2})^2$ and $\frac{3-\sqrt{5}}{2} = (\frac{\sqrt{5}-1}{2})^2$. Thus, our proof for this theorem is completed. $\square$

Before continuing our study, we recall a classical lemma in which a connection between matching number and independence number is presented.

LEMMA 3.7. [2] *For any bipartite graph G of order n , we have $\alpha(G) + \beta(G) = n$.*

THEOREM 3.8. *Let G be a graph of order n with independence number α . Then we have*

- (1) $LEL(G) \leq (n - \alpha)\sqrt{n} + (\alpha - 1)\sqrt{n - \alpha}$ with equality if and only if $G \cong K_{n-\alpha} \vee \overline{K_\alpha}$;
- (2) $LEL(G) \geq 2\alpha - n + \sqrt{5}(n - \alpha - 2) + \sqrt{p} + \sqrt{q} + \sqrt{r}$ with equality if and only if $G \cong A_{n, n-\alpha}$, where p, q, r are the three roots of $\lambda^3 - (\alpha + 4)\lambda^2 + (3\alpha + 4)\lambda - n = 0$.

Proof. From Lemma 2.1 (2), we find that, after adding a new edge into a graph G , the obtained graph has a larger LEL. Considering the computing result (4) in Theorem 3.2, we can obtain immediately the result in (1).

By Lemma 2.1 (1), we claim that, among all graphs of order n and with independence number α , the minimal LEL must be attained at a tree. In view of Theorem 3.6 and Lemma 3.7, replacing β in Theorem 3.6 by $n - \alpha$, we obtain immediately the result in (2) of this theorem. $\square$

It is well known that the complement of the maximum independent set of graph G is just the minimum dominating set in it. Similarly we can easily obtain the following result.

THEOREM 3.9. *Let G be a graph of order n with domination number γ . Then we have*

- (1) $LEL(G) \leq \gamma\sqrt{n} + (n - \gamma - 1)\sqrt{\gamma}$ with equality if and only if $G \cong K_\gamma \vee \overline{K_{n-\gamma}}$;
- (2) $LEL(G) \geq n - 2\gamma + \sqrt{5}(\gamma - 2) + \sqrt{p} + \sqrt{q} + \sqrt{r}$ with equality if and only if $G \cong A_{n,\gamma}$, where p, q, r are the three roots of $\lambda^3 - (n - \gamma + 4)\lambda^2 + (3n - 3\gamma + 4)\lambda - n = 0$.

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REFERENCES

- [1] C. Berge. Sur le couplage maximum d'un graphe. *C.R. Acad. Sci. Paris*, 247:258–259, 1958.
- [2] J.A. Bondy and U.S.R. Murty. *Graph Theory with Applications*. American Elsevier Publishing Co., Inc., New York, 1976.
- [3] D. Cvetković, M. Doob, and H. Sachs. *Spectra of Graphs-Theory and Application*, third edition. Johann Ambrosius Barth, Heidelberg, 1995.
- [4] K.C. Das, K. Xu, and I. Gutman. Comparison between Kirchhoff index and the Laplacian-energy-like invariant. *Linear Algebra Appl.*, 436:3661–3671, 2012.
- [5] L. Feng and A. Ilić. Zagreb, Harary and hyper-Wiener indices of graphs with a given matching number. *Appl. Math. Lett.*, 23:943–948, 2010.
- [6] L. Feng, G. Yu, and X.D. Zhang. Spectral radius of graphs with given matching number. *Linear Algebra Appl.*, 422:133–138, 2007.
- [7] I. Gutman. The energy of a graph. *Ber. Math.-Statist. Sect. Forsch. Graz*, 103:1–22, 1978.
- [8] I. Gutman. The energy of a graph: Old and new results. *Algebraic Combinatorics and Applications*, Springer-Verlag, Berlin, 196–211, 2001.
- [9] I. Gutman, D. Kiani, and M. Mirzakhah. On incidence energy of graphs. *MATCH Commun. Math. Comput. Chem.*, 62:573–580, 2009.
- [10] W. Haemers. Interlacing eigenvalues and graphs. *Linear Algebra Appl.*, 226-228:593–616, 1995.
- [11] C.X. He and H.Y. Shan. On the Laplacian coefficients of bicyclic graphs. *Discrete Math.*, 310:3404–3412, 2010.
- [12] C.X. He, J.Y. Shao, and J.L. He. On the Laplacian spectral radii of bicyclic graphs. *Discrete Math.*, 308:5981–5995, 2008.
- [13] S. He and S. Li. Ordering of trees with fixed matching number by the Laplacian coefficients. *Linear Algebra Appl.*, 435:1171–1186, 2011.

- [14] A. Ilić. On the ordering of trees by the Laplacian coefficients. *Linear Algebra Appl.*, 431:2203–2212, 2009.
- [15] A. Ilić. Trees with minimal Laplacian coefficients. *Comput. Math. Appl.*, 59:2776–2783, 2010.
- [16] A. Ilić, D. Krtinić, and M. Ilić. On Laplacian like energy of trees. *MATCH Commun. Math. Comput. Chem.*, 64:111–122, 2010.
- [17] A. Ilić and M. Ilić. Laplacian coefficients of trees with given number of leaves or vertices of degree two. *Linear Algebra Appl.*, 431:2195–2202, 2009.
- [18] M.R. Jooyandeh, D. Kiani, and M. Mirzakhah. Incidence energy of a graph. *MATCH Commun. Math. Comput. Chem.*, 62:561–572, 2009.
- [19] B. Liu, Y. Huang, and Z. You. A survey on the Laplacian-energy-like invariant. *MATCH Commun. Math. Comput. Chem.*, 66:713–730, 2011.
- [20] J. Liu and B. Liu. A Laplacian-energy-like invariant of a graph. *MATCH Commun. Math. Comput. Chem.*, 59:355–372, 2008.
- [21] M. Liu, B. Liu, and X. Tan. The first to ninth greatest LEL invariants of connected graphs. *Util. Math.*, in press.
- [22] R. Merris. Laplacian matrices of graphs: A survey. *Linear Algebra Appl.*, 197/198:143–176, 1994.
- [23] D. Stevanović. Laplacian-like energy of trees, *MATCH Commun. Math. Comput. Chem.*, 61:407–417, 2009.
- [24] D. Stevanović and A. Ilić. On the Laplacian coefficients of unicyclic graphs. *Linear Algebra Appl.*, 430:2290–2300, 2009.
- [25] D. Stevanović, A. Ilić, C. Onisor, and M. Diudea. LEL-a newly designed molecular descriptor. *Acta Chim. Slov.*, 56:410–417, 2009.
- [26] W.T. Tutte. The factorization of linear graphs. *J. London Math. Soc.*, 22:107–111, 1947.
- [27] X.D. Zhang, X.P. Lv, and Y.H. Chen. Ordering trees by the Laplacian coefficients. *Linear Algebra Appl.*, 431:2414–2424, 2009.
- [28] B. Zhou and N. Trinajstić. The Kirchhoff index and the matching number. *Int. J. of Quantum Chem.*, 109:2978–2981, 2009.
- [29] B.X. Zhu. The Laplacian-energy like of graphs. *Appl. Math. Lett.*, 24:1604–1607, 2011.