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ON CONSTRUCTION OF A COMPLEX FINITE JACOBI MATRIX FROM TWO SPECTRA*

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Abstract. This paper concerns with the inverse spectral problem for two spectra of finite order complex Jacobi matrices (tri-diagonal symmetric matrices with complex entries). The problem is to reconstruct the matrix using two sets of eigenvalues, one for the original Jacobi matrix and one for the matrix obtained by replacing the last diagonal element of the Jacobi matrix by some other number. The uniqueness and existence results for solution of the inverse problem are established and an explicit procedure of reconstruction of the matrix from the two spectra is given.

Key words. Jacobi matrix, Difference equation, Eigenvalue, Normalizing numbers, Inverse spectral problem.

AMS subject classifications. 65F18.

1. Introduction. Let J be an $N \times N$ Jacobi matrix of the form

$$J = \begin{bmatrix} b_0 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_0 & b_1 & a_1 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & b_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{N-3} & a_{N-3} & 0 \\ 0 & 0 & 0 & \cdots & a_{N-3} & b_{N-2} & a_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-2} & b_{N-1} \end{bmatrix}, \tag{1.1}$$

where for each n, a_n and b_n are arbitrary complex numbers such that a_n is different from zero:

$$a_n, b_n \in \mathbb{C}, \quad a_n \neq 0. \tag{1.2}$$

A distinguishing feature of the Jacobi matrix (1.1) from other matrices is that the eigenvalue problem $Jy = \lambda y$ for a column vector $y = \{y_n\}_{n=0}^{N-1}$ is equivalent to the second order linear difference equation

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \tag{1.3}$$

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$$n \in \{0, 1, \ldots, N-1\}, \quad a_{-1} = a_{N-1} = 1,$$

for $\{y_n\}_{n=-1}^{N}$, with the boundary conditions

$$y_{-1} = y_N = 0. \tag{1.4}$$

This allows, using techniques from the theory of three-term linear difference equations [4], to develop a thorough analysis of the eigenvalue problem $Jy = \lambda y$.

If J has distinct eigenvalues $\lambda_1, \ldots, \lambda_p$ with multiplicities $m_1, \ldots, m_p$ and for some

$$\widetilde{b}_{N-1} \neq b_{N-1}, \tag{1.5}$$

the matrix

$$\widetilde{J} = \begin{bmatrix} b_0 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_0 & b_1 & a_1 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & b_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & b_{N-3} & a_{N-3} & 0 \\ 0 & 0 & 0 & \cdots & a_{N-3} & b_{N-2} & a_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-2} & \widetilde{b}_{N-1} \end{bmatrix} \tag{1.6}$$

(in which all a_n and b_n are the same as in J, except b_{N-1} which is replaced by $\widetilde{b}_{N-1}$) has distinct eigenvalues $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_q$ with multiplicities $n_1, \ldots, n_q$, then the collections

$$\{\lambda_k, m_k (k = 1, \ldots, p)\} \quad \text{and} \quad \{\widetilde{\lambda}_i, n_i (i = 1, \ldots, q)\} \tag{1.7}$$

are called the *two spectra* (or the *two-spectra*) of J.

The inverse problem about two spectra consists in reconstruction of the matrix J from its two spectra. This problem consists of the following parts:

(i) To elucidate the uniqueness problem consisting of whether the matrix J and the number $\widetilde{b}_{N-1}$ are determined uniquely by the two spectra of J.

(ii) To find necessary and sufficient conditions for two collections of numbers in (1.7) to be the two spectra for some matrix of the form (1.1) with entries from class (1.2).

(iii) To indicate an algorithm for the construction of the matrix J and the number $\widetilde{b}_{N-1}$ from the two spectra of J.

For real finite Jacobi matrices, this problem was completely solved by the author in [2]. Note that in case of real entries the finite Jacobi matrix is selfadjoint and its eigenvalues are real and distinct. In the complex case, however, the Jacobi matrix is, in general, no longer selfadjoint and its eigenvalues may be complex and multiple.

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In the present paper we show, by reducing the inverse problem about two spectra to the inverse problem about spectral data consisting of the eigenvalues and normalizing numbers of the matrix, that the complex Jacobi matrix and the number $\widetilde{b}_{N-1}$ are determined from two spectra given in (1.7) uniquely up to signs of the off-diagonal elements of the matrix. We indicate also necessary and sufficient conditions for two collections of numbers of the form given in (1.7) to be two spectra of a Jacobi matrix J of the form (1.1) with entries belonging to the class (1.2), as well as a procedure of reconstruction of J and $\widetilde{b}_{N-1}$ from the two spectra.

For given collections in (1.7), assuming that $\lambda_k \neq \widetilde{\lambda}_i$ for all possible values of k and i, and that

$$\sum_{k=1}^{p} m_k \lambda_k \neq \sum_{i=1}^{q} n_i \widetilde{\lambda}_i, \tag{1.8}$$

$$\sum_{k=1}^{p} \frac{1}{(m_k-1)!} \lim_{\lambda\to\lambda_k} \frac{d^{m_k-1}}{d\lambda^{m_k-1}} \frac{1}{\prod\limits_{l=1,l\neq k}^{p} (\lambda-\lambda_l)^{m_l} \prod\limits_{i=1}^{q} (\lambda-\widetilde{\lambda}_i)^{n_i}} \neq 0, \tag{1.9}$$

we define the number a by

$$\frac{1}{a} = \sum_{k=1}^{p} \frac{1}{(m_k-1)!} \lim_{\lambda\to\lambda_k} \frac{d^{m_k-1}}{d\lambda^{m_k-1}} \frac{1}{\prod\limits_{l=1,l\neq k}^{p} (\lambda-\lambda_l)^{m_l} \prod\limits_{i=1}^{q} (\lambda-\widetilde{\lambda}_i)^{n_i}} \tag{1.10}$$

and construct the numbers

$$\beta_{kj} = \frac{a}{(m_k-j)!} \lim_{\lambda\to\lambda_k} \frac{d^{m_k-j}}{d\lambda^{m_k-j}} \frac{1}{\prod\limits_{l=1,l\neq k}^{p} (\lambda-\lambda_l)^{m_l} \prod\limits_{i=1}^{q} (\lambda-\widetilde{\lambda}_i)^{n_i}} \tag{1.11}$$

$$(j=1,\ldots,m_k;\, k=1,\ldots,p).$$

Then we set

$$s_l = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \binom{l}{j-1} \beta_{kj} \lambda_k^{l-j+1}, \quad l=0,1,2,\ldots, \tag{1.12}$$

where $\binom{l}{j-1}$ is a binomial coefficient and we put $\binom{l}{j-1}=0$ if $j-1>l$. Using these numbers we introduce the determinants

$$D_n = \begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n} \end{vmatrix}, \quad n=0,1,2,\ldots \tag{1.13}$$

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The main result of this paper is the following theorem.

THEOREM 1.1. *Let two collections of numbers in (1.7) be given, where* $\lambda_1, \ldots, \lambda_p$ *are distinct complex numbers with* $p \in \{1, \ldots, N\}$ *and* $m_1, \ldots, m_p$ *are positive integers such that* $m_1 + \cdots + m_p = N$*; the* $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_q$ *are distinct complex numbers with* $q \in \{1, \ldots, N\}$ *and* $n_1, \ldots, n_q$ *are positive integers such that* $n_1 + \cdots + n_q = N$*. In order for these collections to be two spectra for a Jacobi matrix* J *of the form (1.1) with entries belonging to the class (1.2), it is necessary and sufficient that the following two conditions are satisfied:*

(i) $\lambda_k \neq \widetilde{\lambda}_i$ *for all* $k \in \{1, \ldots, p\}, i \in \{1, \ldots, q\}$*, and (1.8), (1.9) hold;*
(ii) $D_n \neq 0$*, for* $n \in \{1, 2, \ldots, N-1\}$*, where* D_n *is the determinant defined by (1.13), (1.12), (1.11), (1.10).*

Under the conditions (i) and (ii) the entries a_n *and* b_n *of the matrix* J *for which the collections in (1.7) are two spectra, are recovered by the formulae*

$$a_n = \frac{\pm\sqrt{D_{n-1}D_{n+1}}}{D_n}, \quad n \in \{0, 1, \ldots, N-2\}, \quad D_{-1} = 1, \tag{1.14}$$

$$b_n = \frac{\Delta_n}{D_n} - \frac{\Delta_{n-1}}{D_{n-1}}, \quad n \in \{0, 1, \ldots, N-1\}, \quad \Delta_{-1} = 0, \quad \Delta_0 = s_1, \tag{1.15}$$

where D_n *is defined by (1.13), (1.12), (1.11), (1.10), and* Δ_n *is the determinant obtained from the determinant* D_n *by replacing in* D_n *the last column by the column with the components* $s_{n+1}, s_{n+2}, \ldots, s_{2n+1}$*. Further, the element* $\widetilde{b}_{N-1}$ *of the matrix* $\widetilde{J}$ *corresponding to the matrix* J *and defined by (1.6) is determined by the formula*

$$\widetilde{b}_{N-1} = b_{N-1} + \sum_{i=1}^{q} n_i \widetilde{\lambda}_i - \sum_{k=1}^{p} m_k \lambda_k. \tag{1.16}$$

It follows from the above solution of the inverse problem about two spectra that the matrix (1.1) is not uniquely restored from the two spectra. This is linked with the fact that the a_n are determined from (1.14) uniquely up to a sign. To ensure that the inverse problem is uniquely solvable, we have to specify additionally a sequence of signs + and −. Namely, let $\{\sigma_0, \sigma_1, \ldots, \sigma_{N-2}\}$ be a given finite sequence, where for each $n \in \{0, 1, \ldots, N-2\}$ the σ_n is + or −. We have 2^{N-1} such different sequences. Now to determine a_n uniquely from (1.14) for $n \in \{0, 1, \ldots, N-2\}$ we can choose the sign σ_n when extracting the square root. In this way, we get precisely 2^{N-1} distinct Jacobi matrices possessing the same two spectra. The inverse problem is solved uniquely from the data consisting of the two spectra and a sequence $\{\sigma_0, \sigma_1, \ldots, \sigma_{N-2}\}$ of signs + and −. Thus, we can say that the inverse problem with respect to the two

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spectra is solved uniquely up to signs of the off-diagonal elements of the recovered Jacobi matrix.

The paper is organized as follows. Section 2 is auxiliary and presents a solution of the inverse spectral problem for complex finite Jacobi matrices in terms of the eigenvalues and normalizing numbers. Lastly, Section 3 presents the solution of the inverse problem for complex finite Jacobi matrices in terms of the two spectra.

2. Inverse problem with respect to eigenvalues and normalizing numbers. In the sequel, we will use the following well-known useful lemma. We bring it here for easy reference.

Lemma 2.1. *Let $A(\lambda)$ and $B(\lambda)$ be polynomials with complex coefficients and $\deg A < \deg B = N$. Suppose that $B(\lambda) = b(\lambda - z_1)^{m_1} \cdots (\lambda - z_p)^{m_p}$, where $z_1, \ldots, z_p$ are distinct complex numbers, b is a nonzero complex number, and $m_1, \ldots, m_p$ are positive integers such that $m_1 + \cdots + m_p = N$. Then there exist uniquely determined complex numbers $a_{kj}(j = 1, \ldots, m_k; k = 1, \ldots, p)$ such that*

$$\frac{A(\lambda)}{B(\lambda)} = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \frac{a_{kj}}{(\lambda - z_k)^j} \tag{2.1}$$

for all values of λ different from $z_1, \ldots, z_p$. The numbers a_{kj} are given by the equation

$$a_{kj} = \frac{1}{(m_k - j)!} \lim_{\lambda \to z_k} \frac{d^{m_k - j}}{d\lambda^{m_k - j}} \left[(\lambda - z_k)^{m_k} \frac{A(\lambda)}{B(\lambda)} \right], \tag{2.2}$$

$$j = 1, \ldots, m_k; \quad k = 1, \ldots, p.$$

Proof. For each $k \in \{1, \ldots, p\}$, we have

$$\frac{A(\lambda)}{B(\lambda)} = \frac{C_k(\lambda)}{(\lambda - z_k)^{m_k}}, \tag{2.3}$$

where the function

$$C_k(\lambda) = (\lambda - z_k)^{m_k} \frac{A(\lambda)}{B(\lambda)}$$

$$= \frac{A(\lambda)}{b(\lambda - z_1)^{m_1} \cdots (\lambda - z_{k-1})^{m_{k-1}} (\lambda - z_{k+1})^{m_{k+1}} \cdots (\lambda - z_p)^{m_p}}$$

is regular (analytic) at z_k. We can expand $C_k(\lambda)$ into a Taylor series about the point z_k,

$$C_k(\lambda) = \sum_{s=0}^{\infty} d_{ks} (\lambda - z_k)^s, \tag{2.4}$$

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where

$$d_{ks} = \frac{C_k^{(s)}(z_k)}{s!}, \quad s = 0, 1, 2, \ldots$$

Substituting (2.4) in (2.3) we get that near z_k,

$$\frac{A(\lambda)}{B(\lambda)} = \sum_{s=0}^{m_k-1} \frac{d_{ks}}{(\lambda - z_k)^{m_k-s}} + (a \;\; Taylor \;\; series \;\; about \;\; z_k). \tag{2.5}$$

Consider the function

$$\Phi(\lambda) = \frac{A(\lambda)}{B(\lambda)} - \sum_{k=1}^{p} \sum_{s=0}^{m_k-1} \frac{d_{ks}}{(\lambda - z_k)^{m_k-s}}. \tag{2.6}$$

By (2.5) and (2.6), near z_l,

$$\Phi(\lambda) = (a \;\; Taylor \;\; series \;\; about \;\; z_l) - \sum_{k=1, k\neq l}^{p} \sum_{s=0}^{m_k-1} \frac{d_{ks}}{(\lambda - z_k)^{m_k-s}}$$

and this is analytic at z_l ($l = 1, \ldots, p$). Therefore, the function $\Phi(\lambda)$ is analytic everywhere, that is, $\Phi(\lambda)$ is an entire function. Next, since $\deg A < \deg B$,

$$\Phi(\lambda) \to 0 \;\; as \;\; |\lambda| \to \infty.$$

Thus, the entire function $\Phi(\lambda)$ is bounded and tends to zero as $|\lambda| \to \infty$. By the well-known Liouville theorem, we conclude that $\Phi(\lambda) \equiv 0$. Thus, we have

$$\frac{A(\lambda)}{B(\lambda)} = \sum_{k=1}^{p} \sum_{s=0}^{m_k-1} \frac{d_{ks}}{(\lambda - z_k)^{m_k-s}} = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \frac{d_{k,m_k-j}}{(\lambda - z_k)^j}$$

and

$$d_{k,m_k-j} = \frac{C_k^{(m_k-j)}(z_k)}{(m_k-j)!} = \frac{1}{(m_k-j)!} \lim_{\lambda\to z_k} \frac{d^{m_k-j}}{d\lambda^{m_k-j}} \left[(\lambda - z_k)^{m_k} \frac{A(\lambda)}{B(\lambda)} \right].$$

These prove (2.1) and (2.2). Note that decomposition (2.1) is unique as for the a_{kj} in this decomposition Eq. (2.2) necessarily holds. □

Now it is easy to get the following important for us consequence of this lemma.

Lemma 2.2. *Let $A(\lambda)$ be a polynomial in λ with complex coefficients and* $\deg A \leq N$*, where N is a positive integer. Next, let for some $p \in \{1, \ldots, N\}, z_1, \ldots, z_p$ be distinct complex numbers and $m_1, \ldots, m_p$ be positive integers such that $m_1 + \cdots + m_p = N$. Further, suppose that*

$$A^{(j)}(z_k) = 0 \;\; for \;\; j = 0, 1, \ldots, m_k - 1; k = 1, \ldots, p. \tag{2.7}$$

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Then there is a constant c such that

$$A(\lambda) = c(\lambda - z_1)^{m_1} \cdots (\lambda - z_p)^{m_p}, \tag{2.8}$$

where $c = 0$ if $\deg A < N$ and $c \neq 0$ if $\deg A = N$.

Proof. Let us set

$$B(\lambda) = (\lambda - z_1)^{m_1} \cdots (\lambda - z_p)^{m_p}. \tag{2.9}$$

If $\deg A < N$, then applying Lemma 2.1 to $A(\lambda)/B(\lambda)$ we can write (2.1), (2.2). From condition (2.7) and Eq. (2.2) it follows that

$$a_{kj} = 0 \quad (j = 1, \ldots, m_k; k = 1, \ldots, p).$$

Then (2.1) gives $A(\lambda)/B(\lambda) = 0$ for all values of λ different from $z_1, \ldots, z_p$, that is, $A(\lambda) \equiv 0$. Thus, (2.8) holds with $c = 0$ if $\deg A < N$.

Let now $\deg A = N$. Then dividing $A(\lambda)$ by $B(\lambda)$ with a reminder term, we can write

$$A(\lambda) = cB(\lambda) + R(\lambda), \tag{2.10}$$

where c is a nonzero complex number and $R(\lambda)$ is a polynomial of $\deg R < N$. It follows from (2.7), (2.9), and (2.10) that

$$R^{(j)}(z_k) = 0 \quad \textit{for} \quad j = 0, 1, \ldots, m_k - 1; k = 1, \ldots, p.$$

Then by the first part of the lemma we get that $R(\lambda) \equiv 0$ and (2.10) takes the form (2.8). □

Next in this section, we follow the author's paper [1]. Given a Jacobi matrix J of the form (1.1) with the entries (1.2), consider the eigenvalue problem $Jy = \lambda y$ for a column vector $y = \{y_n\}_{n=0}^{N-1}$, that is equivalent to the problem (1.3), (1.4). Denote by $\{P_n(\lambda)\}_{n=-1}^{N}$ and $\{Q_n(\lambda)\}_{n=-1}^{N}$ the solutions of Eq. (1.3) satisfying the initial conditions

$$P_{-1}(\lambda) = 0, \quad P_0(\lambda) = 1; \tag{2.11}$$

$$Q_{-1}(\lambda) = -1, \quad Q_0(\lambda) = 0. \tag{2.12}$$

For each $n \geq 0$, $P_n(\lambda)$ is a polynomial of degree n and is called a polynomial of first kind and $Q_n(\lambda)$ is a polynomial of degree $n - 1$ and is known as a polynomial of second kind. These polynomials can be found recurrently from Eq. (1.3) using initial

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conditions (2.11) and (2.12). The leading terms of the polynomials $P_n(\lambda)$ and $Q_n(\lambda)$ have the forms

$$P_n(\lambda) = \frac{\lambda^n}{a_0 a_1 \cdots a_{n-1}} + \cdots, \quad Q_n(\lambda) = \frac{\lambda^{n-1}}{a_0 a_1 \cdots a_{n-1}} + \cdots.$$

The equality

$$\det(J - \lambda I) = (-1)^N a_0 a_1 \cdots a_{N-2} P_N(\lambda) \tag{2.13}$$

holds (see [1, 2]) so that the eigenvalues of the matrix J coincide with the zeros of the polynomial $P_N(\lambda)$.

Further, the identity

$$P_{N-1}(\lambda) Q_N(\lambda) - P_N(\lambda) Q_{N-1}(\lambda) = 1 \tag{2.14}$$

holds (see [1, Lemma 4]).

Let $R(\lambda) = (J - \lambda I)^{-1}$ be the resolvent of the matrix J (by I we denote the identity matrix of needed dimension) and e_0 be the N-dimensional column vector with the components $1, 0, \ldots, 0$. The rational function

$$w(\lambda) = -\langle R(\lambda) e_0, e_0 \rangle = \left\langle (\lambda I - J)^{-1} e_0, e_0 \right\rangle, \tag{2.15}$$

introduced earlier in [3], we call the *resolvent function* of the matrix J, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{C}^N$. This function is known also as the Weyl-Titchmarsh function of J.

The entries $R_{nm}(\lambda)$ of the matrix $R(\lambda) = (J - \lambda I)^{-1}$ (resolvent of J) are of the form

$$R_{nm}(\lambda) = \begin{cases} P_n(\lambda)[Q_m(\lambda) + M(\lambda) P_m(\lambda)], & 0 \le n \le m \le N-1, \\ P_m(\lambda)[Q_n(\lambda) + M(\lambda) P_n(\lambda)], & 0 \le m \le n \le N-1, \end{cases} \tag{2.16}$$

(see [1, 2]) where

$$M(\lambda) = -\frac{Q_N(\lambda)}{P_N(\lambda)}. \tag{2.17}$$

According to (2.15), (2.16), (2.17) and using initial conditions (2.11), (2.12), we get

$$w(\lambda) = -R_{00}(\lambda) = -M(\lambda) = \frac{Q_N(\lambda)}{P_N(\lambda)}. \tag{2.18}$$

By (2.13), we have

$$P_N(\lambda) = c(\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_p)^{m_p},$$

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where c is a nonzero constant and $\lambda_1, \dots, \lambda_p$ are distinct eigenvalues of J with multiplicities $m_1, \dots, m_p$. Therefore, we can decompose the rational function $w(\lambda)$ expressed by (2.18) into partial fractions (Lemma 2.1) to get

$$w(\lambda) = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \frac{\beta'_{kj}}{(\lambda - \lambda_k)^j}, \tag{2.19}$$

where

$$\beta'_{kj} = \frac{1}{(m_k - j)!} \lim_{\lambda \to \lambda_k} \frac{d^{m_k - j}}{d\lambda^{m_k - j}} \left[(\lambda - \lambda_k)^{m_k} \frac{Q_N(\lambda)}{P_N(\lambda)} \right] \tag{2.20}$$

$$(j = 1, \dots, m_k; \quad k = 1, \dots, p)$$

are called the *normalizing numbers* of the matrix J.

The collection of the eigenvalues and normalizing numbers

$$\{\lambda_k, \beta'_{kj} (j = 1, \dots, m_k; k = 1, \dots, p)\}, \tag{2.21}$$

of the matrix J of the form (1.1), (1.2) is called the *spectral data* of this matrix.

Thus, the spectral data consist of the eigenvalues and associated normalizing numbers derived by decomposing the resolvent function (Weyl-Titchmarsh function) $w(\lambda)$ into partial fractions using the eigenvalues.

Determination of the spectral data of a given Jacobi matrix is called the *direct spectral problem* for this matrix.

The *inverse spectral problem* consists in reconstruction of the matrix J from its spectral data. This problem was solved by the author in [1] and we will present here the final result.

Let us set

$$s'_l = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \binom{l}{j-1} \beta'_{kj} \lambda_k^{l-j+1}, \quad l = 0, 1, 2, \dots, \tag{2.22}$$

where $\binom{l}{j-1}$ is a binomial coefficient and we put $\binom{l}{j-1} = 0$ if $j - 1 > l$. Next, using these numbers s'_l we introduce the determinants

$$D'_n = \begin{vmatrix} s'_0 & s'_1 & \cdots & s'_n \\ s'_1 & s'_2 & \cdots & s'_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s'_n & s'_{n+1} & \cdots & s'_{2n} \end{vmatrix}, \quad n = 0, 1, 2, \dots \tag{2.23}$$

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Let us bring an important property of the determinants D'_n in the form of a lemma.

LEMMA 2.3. *Given any collection (2.21), for the determinants D'_n defined by (2.23), (2.22), we have $D'_n = 0$ for $n \geq N$, where $N = m_1 + \cdots + m_p$.*

Proof. Given a collection (2.21), define a linear functional Ω on the linear space of all polynomials in λ with complex coefficients as follows: if $G(\lambda)$ is a polynomial then the value $\langle \Omega, G(\lambda) \rangle$ of the functional Ω on the element (polynomial) G is

$$\langle \Omega, G(\lambda) \rangle = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \beta_{kj} \frac{G^{(j-1)}(\lambda_k)}{(j-1)!}. \tag{2.24}$$

Let $m \geq 0$ be a fixed integer and set

$$T(\lambda) = \lambda^m (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_p)^{m_p} \tag{2.25}$$

$$= t_m \lambda^m + t_{m+1} \lambda^{m+1} + \cdots + t_{m+N-1} \lambda^{m+N-1} + \lambda^{m+N}.$$

Then, due to (2.24),

$$\left\langle \Omega, \lambda^l T(\lambda) \right\rangle = 0, \quad l = 0, 1, 2, \ldots \tag{2.26}$$

Consider (2.26) for $l = 0, 1, 2, \ldots, N+m$, and substitute (2.25) in it for $T(\lambda)$. Taking into account that

$$\left\langle \Omega, \lambda^l \right\rangle = s'_l, \quad l = 0, 1, 2, \ldots,$$

where s'_l is defined by (2.22), we get

$$t_m s_{l+m} + t_{m+1} s_{l+m+1} + \cdots + t_{m+N-1} s_{l+m+N-1} + s_{l+m+N} = 0,$$

$$l = 0, 1, 2, \ldots, N+m.$$

Therefore, $(0, \ldots, 0, t_m, t_{m+1}, \ldots, t_{m+N-1}, 1)$ is a nontrivial solution of the homogeneous system of linear algebraic equations

$$x_0 s_l + x_1 s_{l+1} + \cdots + x_m s_{l+m} + x_{m+1} s_{l+m+1} + \cdots + x_{m+N-1} s_{l+m+N-1}$$

$$+ x_{m+N} s_{l+m+N} = 0, \quad l = 0, 1, 2, \ldots, N+m,$$

with the unknowns $x_0, x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+N-1}, x_{m+N}$. Therefore, the determinant of this system, which coincides with D'_{N+m}, must be zero. ∎

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The solution of the above inverse problem with respect to eigenvalues and normalizing numbers is given by the following theorem (see Theorem 6 in [1]).

THEOREM 2.4. *Let an arbitrary collection (2.21) of numbers be given, where* $1 \leq p \leq N, m_1, \ldots, m_p$ *are positive integers with* $m_1 + \cdots + m_p = N, \lambda_1, \ldots, \lambda_p$ *are distinct complex numbers. In order for this collection to be the spectral data for a Jacobi matrix* J *of the form (1.1) with entries belonging to the class (1.2), it is necessary and sufficient that the following two conditions are satisfied:*

(i) $\sum_{k=1}^{p} \beta'_{k1} = 1$;
(ii) $D'_n \neq 0,$ *for* $n \in \{1, 2, \ldots, N-1\}$, *where* D'_n *is the determinant defined by (2.23), (2.22).*

Under the conditions (i) and (ii) the entries a_n *and* b_n *of the matrix* J *for which the collection (2.21) is spectral data, are recovered by the formulae*

$$a_n = \frac{\pm\sqrt{D'_{n-1}D'_{n+1}}}{D'_n}, \quad n \in \{0, 1, \ldots, N-2\}, \quad D'_{-1} = 1, \tag{2.27}$$

$$b_n = \frac{\Delta'_n}{D'_n} - \frac{\Delta'_{n-1}}{D'_{n-1}}, \quad n \in \{0, 1, \ldots, N-1\}, \quad \Delta'_{-1} = 0, \quad \Delta'_0 = s'_1, \tag{2.28}$$

where D'_n *is defined by (2.23), (2.22), and* Δ'_n *is the determinant obtained from the determinant* D'_n *by replacing in* D'_n *the last column by the column with the components* $s'_{n+1}, s'_{n+2}, \ldots, s'_{2n+1}$.

Note that the condition (ii) of Theorem 6 in [1] contains an extra condition which requires that $D'_N = 0$. However, by Lemma 2.3 of the present paper, that condition is fulfilled automatically and is therefore redundant. Concerning the formulas (2.27) and (2.28) for a_n and b_n, respectively, see formulas (3.35) and (3.36) in [1].

It follows from (2.27) and (2.28) that the inverse problem with respect to the spectral data is solved uniquely up to signs of the off-diagonal elements of the recovered Jacobi matrix.

Let us remark that in the case of arbitrary real distinct numbers $\lambda_1, \ldots, \lambda_N$ and positive numbers $\beta'_1, \ldots, \beta'_N$ the condition (ii) of Theorem 2.4 is satisfied automatically and in this case, we have $D'_n > 0$, for $n \in \{1, 2, \ldots, N-1\}$; see [2, Lemma 7]. However, in the complex case the condition (ii) of Theorem 2.4 need not be satisfied automatically. Indeed, let $N = 3$ and as the collection (2.21) we take

$$\{\lambda_1, \lambda_2, \lambda_3, \beta'_1, \beta'_2, \beta'_3\},$$

where $\lambda_1, \lambda_2, \lambda_3, \beta'_1, \beta'_2, \beta'_3$ are arbitrary complex numbers such that

$$\lambda_1 \neq \lambda_2, \quad \lambda_1 \neq \lambda_3, \quad \lambda_2 \neq \lambda_3,$$

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$$\beta_1' \neq 0, \quad \beta_2' \neq 0, \quad \beta_3' \neq 0, \quad \beta_1' + \beta_2' + \beta_3' = 1.$$

We have

$$s_l' = \beta_1'\lambda_1^l + \beta_2'\lambda_2^l + \beta_3'\lambda_3^l, \quad l = 0, 1, 2, \ldots,$$

and it is not difficult to show that

$$D_1' = \begin{vmatrix} s_0' & s_1' \\ s_1' & s_2' \end{vmatrix}$$

$$= \beta_1'\beta_2'(\lambda_1 - \lambda_2)^2 + \beta_1'\beta_3'(\lambda_1 - \lambda_3)^2 + \beta_2'\beta_3'(\lambda_2 - \lambda_3)^2,$$

$$D_2' = \begin{vmatrix} s_0' & s_1' & s_2' \\ s_1' & s_2' & s_3' \\ s_2' & s_3' & s_4' \end{vmatrix} = \beta_1'\beta_2'\beta_3'(\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_3)^2,$$

$$\Delta_0' = s_1' = \beta_1'\lambda_1 + \beta_2'\lambda_2 + \beta_3'\lambda_3,$$

$$\Delta_1' = \begin{vmatrix} s_0' & s_2' \\ s_1' & s_3' \end{vmatrix} = \beta_1'\beta_2'(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2$$

$$+\beta_1'\beta_3'(\lambda_1 + \lambda_3)(\lambda_1 - \lambda_3)^2 + \beta_2'\beta_3'(\lambda_2 + \lambda_3)(\lambda_2 - \lambda_3)^2,$$

$$\Delta_2' = \begin{vmatrix} s_0' & s_1' & s_3' \\ s_1' & s_2' & s_4' \\ s_2' & s_3' & s_5' \end{vmatrix} = \beta_1'\beta_2'\beta_3' \begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 \end{vmatrix}.$$

We see that the condition $D_1' \neq 0$ is not satisfied automatically, and therefore, one must require $D_1' \neq 0$ as a condition. For example, if

$$\beta_1' = \beta_2' = \beta_3' = \frac{1}{3}, \quad \lambda_1 = \frac{1 \pm i\sqrt{3}}{2}, \quad \lambda_2 = 1, \quad \lambda_3 = 0,$$

then we get $D_1' = 0$.

3. Construction from two spectra. Let J be an $N \times N$ Jacobi matrix of the form (1.1) with entries satisfying (1.2). Define $\widetilde{J}$ to be the Jacobi matrix given by (1.6), where the number $\widetilde{b}_{N-1}$ satisfies (1.5). Denote by $\lambda_1, \ldots, \lambda_p$ all the distinct eigenvalues of the matrix J and by $m_1, \ldots, m_p$ their multiplicities, respectively, as the

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roots of the characteristic polynomial $\det(J-\lambda I)$ so that $1 \le p \le N, m_1+\cdots+m_p = N$, and

$$\det(\lambda I - J) = (\lambda-\lambda_1)^{m_1}\cdots(\lambda-\lambda_p)^{m_p}. \tag{3.1}$$

Further, denote by $\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_q$ all the distinct eigenvalues of the matrix $\widetilde{J}$ and by $n_1,\ldots,n_q$ their multiplicities, respectively, as the roots of the characteristic polynomial $\det(\widetilde{J}-\lambda I)$ so that $1 \le q \le N, n_1+\cdots+n_q = N$, and

$$\det(\lambda I - \widetilde{J}) = (\lambda-\widetilde{\lambda}_1)^{n_1}\cdots(\lambda-\widetilde{\lambda}_q)^{n_q}. \tag{3.2}$$

Recall that we call the collections $\{\lambda_k, m_k (k=1,\ldots,p)\}$ and $\{\widetilde{\lambda}_i, n_i (i=1,\ldots,q)\}$ the *two spectra* of the matrix J.

Our goal in this section is to prove Theorem 1.1 which solves the inverse problem for two spectra, consisting in the reconstruction of the matrix J and the number $\widetilde{b}_{N-1}$ from the two spectra of J. We will reduce the inverse problem for two spectra to the inverse problem for eigenvalues and normalizing numbers solved above in Section 2.

First let us study some necessary properties of the two spectra of the Jacobi matrix J.

Let $P_n(\lambda)$ and $Q_n(\lambda)$ be the polynomials of the first and second kind for the matrix J. The similar polynomials for the matrix $\widetilde{J}$ we denote by $\widetilde{P}_n(\lambda)$ and $\widetilde{Q}_n(\lambda)$. By (2.13), we have

$$\det(J-\lambda I) = (-1)^N a_0 a_1 \cdots a_{N-2} P_N(\lambda), \tag{3.3}$$

$$\det\left(\widetilde{J}-\lambda I\right) = (-1)^N a_0 a_1 \cdots a_{N-2} \widetilde{P}_N(\lambda), \tag{3.4}$$

so that the eigenvalues $\lambda_1,\ldots,\lambda_p$ and $\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_q$ of the matrices J and $\widetilde{J}$ and their multiplicities coincide with the zeros and their multiplicities of the polynomials $P_N(\lambda)$ and $\widetilde{P}_N(\lambda)$, respectively.

Lemma 3.1. *For the resolvent function $w(\lambda)$ of the matrix J, defined by (2.15), the following formula holds:*

$$w(\lambda) = \frac{\widetilde{Q}_N(\lambda)}{\widetilde{P}_N(\lambda)} + \frac{b_{N-1}-\widetilde{b}_{N-1}}{P_N(\lambda)\widetilde{P}_N(\lambda)}. \tag{3.5}$$

Proof. Comparing the matrices J and $\widetilde{J}$ defined by (1.1) and (1.6), respectively, we can see that

$$\widetilde{P}_n(\lambda) = P_n(\lambda), \quad n \in \{-1,0,1,\ldots,N-1\},$$

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$$a_{N-2}P_{N-2}(\lambda) + b_{N-1}P_{N-1}(\lambda) + P_N(\lambda) = \lambda P_{N-1}(\lambda), \tag{3.6}$$

$$a_{N-2}P_{N-2}(\lambda) + \widetilde{b}_{N-1}P_{N-1}(\lambda) + \widetilde{P}_N(\lambda) = \lambda P_{N-1}(\lambda), \tag{3.7}$$

and

$$\widetilde{Q}_n(\lambda) = Q_n(\lambda), \quad n \in \{-1, 0, 1, \ldots, N-1\},$$

$$a_{N-2}Q_{N-2}(\lambda) + b_{N-1}Q_{N-1}(\lambda) + Q_N(\lambda) = \lambda Q_{N-1}(\lambda), \tag{3.8}$$

$$a_{N-2}Q_{N-2}(\lambda) + \widetilde{b}_{N-1}Q_{N-1}(\lambda) + \widetilde{Q}_N(\lambda) = \lambda Q_{N-1}(\lambda). \tag{3.9}$$

Subtracting (3.6) and (3.7), and also (3.8) and (3.9), we get

$$\widetilde{P}_N(\lambda) - P_N(\lambda) = (b_{N-1} - \widetilde{b}_{N-1})P_{N-1}(\lambda),$$

$$\widetilde{Q}_N(\lambda) - Q_N(\lambda) = (b_{N-1} - \widetilde{b}_{N-1})Q_{N-1}(\lambda).$$

Hence

$$P_{N-1}(\lambda) = \frac{\widetilde{P}_N(\lambda) - P_N(\lambda)}{b_{N-1} - \widetilde{b}_{N-1}}, \quad Q_{N-1}(\lambda) = \frac{\widetilde{Q}_N(\lambda) - Q_N(\lambda)}{b_{N-1} - \widetilde{b}_{N-1}}. \tag{3.10}$$

Replacing $P_{N-1}(\lambda)$ and $Q_{N-1}(\lambda)$ in the identity (2.14) by their expressions from (3.10), we get

$$\widetilde{P}_N(\lambda)Q_N(\lambda) - P_N(\lambda)\widetilde{Q}_N(\lambda) = b_{N-1} - \widetilde{b}_{N-1}. \tag{3.11}$$

Dividing both sides by $P_N(\lambda)\widetilde{P}_N(\lambda)$ gives

$$\frac{Q_N(\lambda)}{P_N(\lambda)} - \frac{\widetilde{Q}_N(\lambda)}{\widetilde{P}_N(\lambda)} = \frac{b_{N-1} - \widetilde{b}_{N-1}}{P_N(\lambda)\widetilde{P}_N(\lambda)}.$$

Therefore, by formula (2.18) for the resolvent function $w(\lambda)$, we obtain (3.5). □

Lemma 3.2. *The matrices J and $\widetilde{J}$ have no common eigenvalues, that is, $\lambda_k \neq \widetilde{\lambda}_i$ for all values of k and i.*

Proof. Suppose that λ is an eigenvalue of the matrices J and $\widetilde{J}$. Then by (3.3) and (3.4), we have $P_N(\lambda) = \widetilde{P}_N(\lambda) = 0$. But this is impossible by (3.11) and the condition $\widetilde{b}_{N-1} \neq b_{N-1}$. □

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The following lemma allows us to calculate the difference $\widetilde{b}_{N-1} - b_{N-1}$ in terms of the two spectra.

LEMMA 3.3. *The equality (trace formula)*

$$\sum_{k=1}^{p} m_k \lambda_k - \sum_{i=1}^{q} n_i \widetilde{\lambda}_i = b_{N-1} - \widetilde{b}_{N-1} \tag{3.12}$$

holds.

Proof. For any matrix $A = [a_{jk}]_{j,k=1}^{N}$ the spectral trace of A coincides with the matrix trace of A: if $\mu_1, \ldots, \mu_p$ are the distinct eigenvalues of A of multiplicities $m_1, \ldots, m_p$ as the roots of the characteristic polynomial $\det(A - \lambda I)$, then

$$\sum_{k=1}^{p} m_k \mu_k = \sum_{k=1}^{N} a_{kk}.$$

Indeed, this follows from

$$\det(\lambda I - A) = (\lambda - \mu_1)^{m_1} \cdots (\lambda - \mu_p)^{m_p}$$

by comparison of the coefficients of λ^{N-1} on the two sides. Therefore, we can write

$$\sum_{k=1}^{p} m_k \lambda_k = b_0 + b_1 + \cdots + b_{N-2} + b_{N-1},$$

$$\sum_{i=1}^{q} n_i \widetilde{\lambda}_i = b_0 + b_1 + \cdots + b_{N-2} + \widetilde{b}_{N-1}.$$

Subtracting the last two equalities side by side we arrive at (3.12). ◻

The following statement follows from Lemma 3.3.

COROLLARY 3.4. *Under the condition (1.5),*

$$\sum_{k=1}^{p} m_k \lambda_k - \sum_{i=1}^{q} n_i \widetilde{\lambda}_i \neq 0.$$

The following lemma gives a formula for calculating the normalizing numbers $\beta'_{kj} (j = 1, \ldots, m_k; k = 1, \ldots, p)$ in terms of the two spectra.

LEMMA 3.5. *Under the condition (1.5), the statement in (1.9) is valid and if we define the number a by (1.10), then for each $k \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, m_k\}$ the*

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formula

$$\beta'_{kj} = \frac{a}{(m_k-j)!} \lim_{\lambda\to\lambda_k} \frac{d^{m_k-j}}{d\lambda^{m_k-j}} \frac{1}{\prod\limits_{l=1,l\neq k}^{p} (\lambda-\lambda_l)^{m_l} \prod\limits_{i=1}^{q} (\lambda-\widetilde{\lambda}_i)^{n_i}} \tag{3.13}$$

holds.

Proof. Substituting (2.19) in the left-hand side of (3.5) established in Lemma 3.1, we can write

$$\sum_{k=1}^{p}\sum_{j=1}^{m_k} \frac{\beta'_{kj}}{(\lambda-\lambda_k)^j} = \frac{\widetilde{Q}_N(\lambda)}{\widetilde{P}_N(\lambda)} + \frac{b_{N-1}-\widetilde{b}_{N-1}}{P_N(\lambda)\widetilde{P}_N(\lambda)}.$$

Hence, we get, using (3.3), (3.4),

$$\begin{aligned}\beta'_{kj} &= \frac{1}{(m_k-j)!} \lim_{\lambda\to\lambda_k} \frac{d^{m_k-j}}{d\lambda^{m_k-j}} \left\{(\lambda-\lambda_k)^{m_k}\left[\frac{\widetilde{Q}_N(\lambda)}{\widetilde{P}_N(\lambda)} + \frac{b_{N-1}-\widetilde{b}_{N-1}}{P_N(\lambda)\widetilde{P}_N(\lambda)}\right]\right\} \\ &= \frac{b_{N-1}-\widetilde{b}_{N-1}}{(m_k-j)!} \lim_{\lambda\to\lambda_k} \frac{d^{m_k-j}}{d\lambda^{m_k-j}} \left[(\lambda-\lambda_k)^{m_k}\frac{1}{P_N(\lambda)\widetilde{P}_N(\lambda)}\right] \\ &= \frac{c(b_{N-1}-\widetilde{b}_{N-1})}{(m_k-j)!} \lim_{\lambda\to\lambda_k} \frac{d^{m_k-j}}{d\lambda^{m_k-j}} \left[(\lambda-\lambda_k)^{m_k}\frac{1}{\det(\lambda I-J)\det\left(\lambda I-\widetilde{J}\right)}\right],\end{aligned} \tag{3.14}$$

where

$$c = (a_0a_1\cdots a_{N-2})^2.$$

We have used the fact that since $\widetilde{P}_N(\lambda_k)\neq 0$ by Lemma 3.2,

$$\lim_{\lambda\to\lambda_k} \frac{d^{m_k-j}}{d\lambda^{m_k-j}} \left\{(\lambda-\lambda_k)^{m_k}\frac{\widetilde{Q}_N(\lambda)}{\widetilde{P}_N(\lambda)}\right\} = 0 \quad (j=1,\ldots,m_k).$$

Substituting (3.1) and (3.2) in (3.14), we get

$$\beta'_{kj} = \frac{a'}{(m_k-j)!} \lim_{\lambda\to\lambda_k} \frac{d^{m_k-j}}{d\lambda^{m_k-j}} \frac{1}{\prod\limits_{l=1,l\neq k}^{p} (\lambda-\lambda_l)^{m_l} \prod\limits_{i=1}^{q} (\lambda-\widetilde{\lambda}_i)^{n_i}}, \tag{3.15}$$

where

$$a' = c(b_{N-1}-\widetilde{b}_{N-1}).$$

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Next, putting $j = 1$ in Eq. (3.15) and then summing this equation over $k = 1, \ldots, p$ and taking into account the condition (i) of Theorem 2.4, we get

$$1 = a' \sum_{k=1}^{p} \frac{1}{(m_k - 1)!} \lim_{\lambda \to \lambda_k} \frac{d^{m_k - 1}}{d\lambda^{m_k - 1}} \frac{1}{\prod\limits_{l=1, l \neq k}^{p} (\lambda - \lambda_l)^{m_l} \prod\limits_{i=1}^{q} (\lambda - \widetilde{\lambda}_i)^{n_i}}.$$

Hence, (1.9) and $a' = a$ follow and (3.15) coincides with (3.13). ∎

The following statement follows from Lemma 3.5.

Corollary 3.6. *We have*

$$\beta'_{kj} = \beta_{kj} \quad \textit{and hence} \quad s'_l = s_l, \quad D'_n = D_n,$$

where β'_{kj}, s'_l, D'_n *are defined by (2.20), (2.22), (2.23) and* β_{kj}, s_l, D_n *by (1.11), (1.12), (1.13).*

Let us now prove Theorem 1.1 stated in the Introduction.

Proof. The necessity of the conditions (i) and (ii) of Theorem 1.1 follows from Lemmas 3.2, 3.5, Corollary 3.4, and Theorem 2.4 by Corollary 3.6. To prove sufficiency suppose that two collections of numbers in (1.7) are given which satisfy the conditions of Theorem 1.1. We construct $\beta_{kj} (j = 1, \ldots, m_k; k = 1, \ldots, p)$ by (1.11), (1.10). It follows directly that

$$\sum_{k=1}^{p} \beta_{k1} = 1.$$

Thus, the collection $\{\lambda_k, \beta_{kj} (j = 1, \ldots, m_k; k = 1, \ldots, p)\}$ satisfies conditions (i) and (ii) of Theorem 2.4 (with β'_{kj} replaced by β_{kj}), and hence, there exists a Jacobi matrix J of the form (1.1) with entries from the class (1.2) such that λ_k are the eigenvalues of the multiplicity m_k and β_{kj} are the corresponding normalizing numbers for J. Having the matrix J, in particular, its entry b_{N-1}, we construct the number $\widetilde{b}_{N-1}$ by

$$\widetilde{b}_{N-1} = b_{N-1} + \sum_{i=1}^{q} n_i \widetilde{\lambda}_i - \sum_{k=1}^{p} m_k \lambda_k \tag{3.16}$$

and then the matrix $\widetilde{J}$ by (1.6) according to the matrix J and (3.16). Note that by the condition (1.8), we have

$$\widetilde{b}_{N-1} \neq b_{N-1}. \tag{3.17}$$

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It remains to show that $\widetilde{\lambda}_i$ are the eigenvalues of the constructed matrix $\widetilde{J}$, of multiplicity n_i. To do this we denote the eigenvalues of $\widetilde{J}$ by $\widetilde{\mu}_1, \ldots, \widetilde{\mu}_s$ and their multiplicities by $\widetilde{n}_1, \ldots, \widetilde{n}_s$. We have to show that $s = q, \widetilde{\mu}_i = \widetilde{\lambda}_i, \widetilde{n}_i = n_i (i = 1, \ldots, q)$. Let us set

$$f(\lambda) = \prod_{l=1}^{p} (\lambda - \lambda_l)^{m_l}, \quad g(\lambda) = \prod_{i=1}^{q} (\lambda - \widetilde{\lambda}_i)^{n_i}, \quad h(\lambda) = \prod_{i=1}^{s} (\lambda - \widetilde{\mu}_i)^{\widetilde{n}_i}.$$

By the direct problem, we have (Lemma 3.5)

$$\beta_{kj} = \frac{\widetilde{a}}{(m_k - j)!} \lim_{\lambda \to \lambda_k} \frac{d^{m_k - j}}{d\lambda^{m_k - j}} \left[(\lambda - \lambda_k)^{m_k} \frac{1}{f(\lambda) h(\lambda)} \right], \tag{3.18}$$

where

$$\frac{1}{\widetilde{a}} = \sum_{k=1}^{p} \frac{1}{(m_k - 1)!} \lim_{\lambda \to \lambda_k} \frac{d^{m_k - 1}}{d\lambda^{m_k - 1}} \frac{1}{\prod\limits_{l=1, l \neq k}^{p} (\lambda - \lambda_l)^{m_l} \prod\limits_{i=1}^{s} (\lambda - \widetilde{\mu}_i)^{\widetilde{n}_i}}.$$

On the other hand, by our construction of β_{kj}, we have (1.11), (1.10) which can be written in the form

$$\beta_{kj} = \frac{a}{(m_k - j)!} \lim_{\lambda \to \lambda_k} \frac{d^{m_k - j}}{d\lambda^{m_k - j}} \left[(\lambda - \lambda_k)^{m_k} \frac{1}{f(\lambda) g(\lambda)} \right], \tag{3.19}$$

where

$$\frac{1}{a} = \sum_{k=1}^{p} \frac{1}{(m_k - 1)!} \lim_{\lambda \to \lambda_k} \frac{d^{m_k - 1}}{d\lambda^{m_k - 1}} \frac{1}{\prod\limits_{l=1, l \neq k}^{p} (\lambda - \lambda_l)^{m_l} \prod\limits_{i=1}^{q} (\lambda - \widetilde{\lambda}_i)^{n_i}}.$$

Subtracting (3.18) and (3.19) side by side we get

$$\lim_{\lambda \to \lambda_k} \frac{d^{m_k - j}}{d\lambda^{m_k - j}} \frac{A(\lambda)}{F_k(\lambda)} = 0 \quad (j = 1, \ldots, m_k; k = 1, \ldots, p), \tag{3.20}$$

where

$$A(\lambda) = \widetilde{a} g(\lambda) - a h(\lambda) = \widetilde{a} \prod_{i=1}^{q} (\lambda - \widetilde{\lambda}_i)^{n_i} - a \prod_{i=1}^{s} (\lambda - \widetilde{\mu}_i)^{\widetilde{n}_i}, \tag{3.21}$$

$$F_k(\lambda) = \frac{f(\lambda) g(\lambda) h(\lambda)}{(\lambda - \lambda_k)^{m_k}} = \prod_{l=1, l \neq k}^{p} (\lambda - \lambda_l)^{m_l} \prod_{i=1}^{q} (\lambda - \widetilde{\lambda}_i)^{n_i} \prod_{i=1}^{s} (\lambda - \widetilde{\mu}_i)^{\widetilde{n}_i}. \tag{3.22}$$

Note that by (3.21), $A(\lambda)$ is a polynomial of degree $\leq N$ because $n_1 + \cdots + n_q = N$ and $\widetilde{n}_1 + \cdots + \widetilde{n}_s = N$; besides, by (3.22), $F_k(\lambda_k) \neq 0 (k = 1, \ldots, p)$ observing that $\widetilde{\mu}_i$'s are distinct from λ_k by Lemma 3.2.

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Taking $j = m_k$ in (3.20), we get

$$\frac{A(\lambda_k)}{F_k(\lambda_k)} = 0, \quad that \quad is, \quad A(\lambda_k) = 0.$$

Next, taking $j = m_k - 1$ in (3.20), we get

$$\frac{A'(\lambda_k)F_k(\lambda_k) - A(\lambda_k)F_k'(\lambda_k)}{F_k^2(\lambda_k)} = 0,$$

and hence, $A'(\lambda_k) = 0$. Continuing in this way, we find that

$$A^{(j)}(\lambda_k) = 0 \quad for \quad all \quad j = 0, 1, \dots, m_k - 1; k = 1, \dots, p.$$

Therefore, applying Lemma 2.2, we get that

$$A(\lambda) = c(\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_p)^{m_p},$$

that is,

$$\widetilde{a} \prod_{i=1}^{q} (\lambda - \widetilde{\lambda}_i)^{n_i} - a \prod_{i=1}^{s} (\lambda - \widetilde{\mu}_i)^{\widetilde{n}_i} = c \prod_{k=1}^{p} (\lambda - \lambda_k)^{m_k}, \tag{3.23}$$

where c is a constant. Hence

$$(\widetilde{a} - a)\lambda^N - \left(\widetilde{a} \sum_{i=1}^{q} n_i \widetilde{\lambda}_i - a \sum_{i=1}^{s} \widetilde{n}_i \widetilde{\mu}_i \right) \lambda^{N-1} + \cdots$$

$$= c\lambda^N - \left(c \sum_{k=1}^{p} m_k \lambda_k \right) \lambda^{N-1} + \cdots$$

and we get

$$\widetilde{a} - a = c, \tag{3.24}$$

$$\widetilde{a} \sum_{i=1}^{q} n_i \widetilde{\lambda}_i - a \sum_{i=1}^{s} \widetilde{n}_i \widetilde{\mu}_i = c \sum_{k=1}^{p} m_k \lambda_k. \tag{3.25}$$

Next, from (3.16), we have

$$\sum_{i=1}^{q} n_i \widetilde{\lambda}_i - \sum_{k=1}^{p} m_k \lambda_k = \widetilde{b}_{N-1} - b_{N-1},$$

and applying Lemma 3.3 to the matrices J and $\widetilde{J}$ we can write

$$\sum_{i=1}^{s} \widetilde{n}_i \widetilde{\mu}_i - \sum_{k=1}^{p} m_k \lambda_k = \widetilde{b}_{N-1} - b_{N-1}.$$

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From the last two equations we get

$$\sum_{i=1}^{q} n_i \widetilde{\lambda}_i = \sum_{i=1}^{s} \widetilde{n}_i \widetilde{\mu}_i. \tag{3.26}$$

Taking (3.26) into account in (3.25) we obtain

$$(\widetilde{a} - a) \sum_{i=1}^{s} \widetilde{n}_i \widetilde{\mu}_i = c \sum_{k=1}^{p} m_k \lambda_k.$$

Hence, by (3.24),

$$c \left(\sum_{k=1}^{p} m_k \lambda_k - \sum_{i=1}^{s} \widetilde{n}_i \widetilde{\mu}_i \right) = 0, \quad that \quad is, \quad c(b_{N-1} - \widetilde{b}_{N-1}) = 0.$$

Therefore, $c = 0$ by (3.17), and hence, $\widetilde{a} = a$ by (3.24). Now (3.23) gives

$$\prod_{i=1}^{q} (\lambda - \widetilde{\lambda}_i)^{n_i} = \prod_{i=1}^{s} (\lambda - \widetilde{\mu}_i)^{\widetilde{n}_i}$$

for all values of λ. It follows that $q = s, \widetilde{\lambda}_i = \widetilde{\mu}_i, n_i = \widetilde{n}_i (i = 1, \ldots, q)$ with a possible reorder of the $\widetilde{\mu}_i$'s.

Formulae (1.14), (1.15) follow from (2.27), (2.28) by Corollary 3.6 and formula (1.16) follows from Lemma 3.3. □

Formulae (1.14), (1.15), and (1.16) show that the two spectra in (1.7) determine the Jacobi matrix J of the form (1.1) in the class (1.2) and the number $\widetilde{b}_{N-1}$ in $\mathbb{C}$ in the matrix $\widetilde{J}$ defined by (1.6) uniquely up to signs of the off-diagonal elements of J.

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