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TECHNIQUES FOR IDENTIFYING INERTIALLY ARBITRARY PATTERNS*

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Abstract. Two techniques used to show a matrix pattern is spectrally arbitrary are the nilpotent-Jacobian method and more recently the nilpotent-centralizer method. This paper presents generalizations of both techniques, which are then used to show that certain non-spectrally-arbitrary patterns are inertially arbitrary. A flaw in a method used in three previous publications on inertially arbitrary patterns is discussed. By using the techniques developed here, it is shown that all of the patterns in the three papers affected by the flaw are nevertheless inertially arbitrary.

Key words. Inertially arbitrary, Nilpotent-centralizer method, Nilpotent-Jacobian method, Refined inertia, Sign pattern, Zero-nonzero pattern.

AMS subject classifications. 15A18, 15B35.

1. Definitions. A *sign pattern* is a matrix with entries in $\{+,-,0\}$, where $+$ (resp., $-$) represents a positive (resp., negative) real number. A *zero-nonzero pattern* is a matrix with entries in $\{*,0\}$, where $*$ represents a nonzero real number. We use the term *pattern* when statements hold for both sign patterns and zero-nonzero patterns. The *sign* of a real number a, denoted by $sgn(a)$, is defined as

$$sgn(a) = \begin{cases} + & \text{if } a > 0, \\ - & \text{if } a < 0, \text{ and} \\ 0 & \text{if } a = 0. \end{cases}$$

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The *qualitative class* of a pattern $\mathcal{A} = [\alpha_{ij}]$, denoted by $Q(\mathcal{A})$, is the set of all real matrices $A = [a_{ij}]$ such that:

(i) if $\alpha_{ij} = *$, then $a_{ij} \neq 0$,
(ii) if $\alpha_{ij} = +$, then $a_{ij} > 0$,
(iii) if $\alpha_{ij} = -$, then $a_{ij} < 0$, and
(iv) if $\alpha_{ij} = 0$, then $a_{ij} = 0$.

If $A \in Q(\mathcal{A})$, then A is a *realization* of $\mathcal{A}$. Furthermore, if $A \in Q(\mathcal{A})$ is an $n \times n$ nilpotent matrix, that is, A has characteristic polynomial $c_A(z)$ equal to z^n, then A is a *nilpotent realization* of $\mathcal{A}$.

The *inertia* of a square matrix A, denoted by $i(A)$, is the ordered triple of nonnegative integers $(a_+,\, a_-,\, a_0)$ where a_+ (resp., a_- and a_0) is the number of eigenvalues of A with positive (resp., negative and zero) real part. The *inertia* of a square pattern $\mathcal{A}$ is the set $i(\mathcal{A}) = \{i(A) : A \in Q(\mathcal{A})\}$. A pattern $\mathcal{A}$ of order n is *inertially arbitrary* if $i(\mathcal{A})$ contains every ordered triple of nonnegative integers (a_+, a_-, a_0) with $a_+ + a_- + a_0 = n$.

The *refined inertia* of a real square matrix A, introduced in [14] and denoted by $ri(A)$, is the ordered 4-tuple of nonnegative integers $(a_+, a_-, a_z, 2a_p)$ where a_+ (resp., a_-) is the number of eigenvalues of A with positive (resp., negative) real part, and a_z (resp., $2a_p$) is the number of zero (resp., nonzero pure imaginary) eigenvalues of A. In this context, the inertia of a matrix is then $i(A) = (a_+, a_-, a_z + 2a_p)$. The *refined inertia* of a square pattern $\mathcal{A}$ is the set $ri(\mathcal{A}) = \{ri(A) : A \in Q(\mathcal{A})\}$. A pattern $\mathcal{A}$ of order n is *refined inertially arbitrary* if $ri(\mathcal{A})$ contains every ordered 4-tuple of nonnegative integers $(a_+, a_-, a_z, 2a_p)$ with $a_+ + a_- + a_z + 2a_p = n$.

The *spectrum* of a square matrix A, denoted by $\sigma(A)$, is the multiset of eigenvalues of A. The *spectrum* of a square pattern $\mathcal{A}$ is the set $\sigma(\mathcal{A}) = \{\sigma(A) : A \in Q(\mathcal{A})\}$. A pattern $\mathcal{A}$ of order n is *spectrally arbitrary* if every multiset of n complex numbers, closed under complex conjugation, is in the spectrum of $\mathcal{A}$. Equivalently, $\mathcal{A}$ is spectrally arbitrary if, for every monic real polynomial $r(z)$ of degree n, there exists a matrix $A \in Q(\mathcal{A})$ such that the characteristic polynomial of A is $c_A(z) = r(z)$.

For patterns $\mathcal{A}$ and $\mathcal{B}$ of order n, if $\mathcal{A}$ is obtained from $\mathcal{B}$ by replacing some (or possibly none) of the nonzero symbols in $\mathcal{B}$ with 0, then $\mathcal{A}$ is a *subpattern* of $\mathcal{B}$ and $\mathcal{B}$ is a *superpattern* of $\mathcal{A}$. Two patterns are *equivalent* if one can be obtained from the other via some combination of transposition, negation, diagonal similarity, and permutation similarity. If two patterns $\mathcal{A}$ and $\mathcal{B}$ are equivalent, then $\mathcal{A}$ is inertially arbitrary (resp., spectrally arbitrary, refined inertially arbitrary) if and only if $\mathcal{B}$ is inertially arbitrary (resp., spectrally arbitrary, refined inertially arbitrary).

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2. Introduction. Spectrally and inertially arbitrary sign patterns were introduced in [8] while refined inertially arbitrary patterns were introduced in [14]. In [8], the nilpotent-Jacobian method is developed to show a pattern $\mathcal{A}$ is spectrally arbitrary by first finding a nilpotent matrix $A \in Q(\mathcal{A})$ and then verifying that a certain Jacobian matrix is nonsingular so that the Implicit Function Theorem can be applied. In [9], the generalized $n \times m$ Jacobian matrix for a matrix A of order n with m nonzero entries is discussed and the nilpotent-centralizer method is developed using an algebraic property of a certain nilpotent matrix to show a pattern is spectrally arbitrary. In the proofs of both the nilpotent-Jacobian method and the nilpotent-centralizer method, a nilpotent matrix is critical. We generalize both of these methods by replacing the nilpotent matrices by other realizations. In Sections 3 and 4, we demonstrate the consequences of this replacement for the nilpotent-Jacobian method and nilpotent-centralizer method, respectively. We also demonstrate how both of these generalizations can be applied to show a pattern is inertially arbitrary.

In [15, Lemma 5], [5, Lemma 3.2], and [6, Lemma 5], a modified nilpotent-Jacobian method is employed to show that certain patterns are inertially arbitrary. However, the arguments presented for these three lemmas are flawed. In the Appendix, we describe the flaw. In Section 5, we demonstrate that the patterns in question, appearing in the three papers [15, 5, 6], are nevertheless inertially arbitrary.

3. Generalizations of the nilpotent-Jacobian method. In this section, we generalize the nilpotent-Jacobian method so that it may be used to obtain sets of spectra, inertias, and characteristic polynomials for a pattern. We first give a name to a property that a matrix (not a pattern) can satisfy. Throughout this paper we restrict our attention to real matrices.

Let $A = [a_{ij}]$ be a matrix of order n. If A satisfies the following conditions, then we say that A *allows a nonzero Jacobian*:

(i) A has $m \geq n$ nonzero entries.

(ii) Among those m nonzero entries, there are n nonzero entries, say $a_{i_1 j_1}, \ldots, a_{i_n j_n}$, such that if X is the matrix obtained from A by replacing the entries $a_{i_1 j_1}, \ldots, a_{i_n j_n}$ by real variables $x_1, \ldots, x_n$ and $c_X(z) = z^n + p_1 z^{n-1} + p_2 z^{n-2} + \cdots + p_{n-1} z + p_n$ is the characteristic polynomial of X, then the Jacobian matrix of order n with (i, j) entry equal to $\frac{\partial p_i}{\partial x_j}(x_1, \ldots, x_n)$ is nonsingular at $(x_1, \ldots, x_n) = (a_{i_1 j_1}, \ldots, a_{i_n j_n})$.

Using this definition, we restate the nilpotent-Jacobian method.

THEOREM 3.1 (The nilpotent-Jacobian method). *Let A be a nilpotent realization of a square pattern $\mathcal{A}$ such that A allows a nonzero Jacobian. Then every superpattern of $\mathcal{A}$ is spectrally arbitrary.*

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Next, we consider the consequences of replacing the nilpotent matrix in the nilpotent-Jacobian method by an arbitrary realization that allows a nonzero Jacobian.

THEOREM 3.2. *Let $\mathcal{A}$ be a pattern of order n and suppose there is a matrix $A \in Q(\mathcal{A})$ that allows a nonzero Jacobian. If the characteristic polynomial of A is*

$$c_A(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n,$$

then for any $b_1, \ldots, b_n \in \mathbb{R}$ sufficiently close to $a_1, \ldots, a_n$, respectively, each superpattern $\hat{\mathcal{A}}$ of pattern $\mathcal{A}$ has a realization $B \in Q(\hat{\mathcal{A}})$ whose characteristic polynomial is

$$c_B(z) = z^n + b_1 z^{n-1} + b_2 z^{n-2} + \cdots + b_{n-1} z + b_n.$$

Proof. The result follows by using A in place of a nilpotent matrix in a proof of the nilpotent-Jacobian method. For completeness, we provide a full proof.

Let A be a realization of $\mathcal{A}$ that allows a nonzero Jacobian. Suppose $a_{i_1 j_1}, \ldots, a_{i_n j_n}$ are n nonzero entries of A that give rise to a nonsingular Jacobian matrix. Let X be the matrix obtained from A by replacing the entries $a_{i_1 j_1}, \ldots, a_{i_n j_n}$ by real variables $x_1, \ldots, x_n$, and let the characteristic polynomial of X be given by

$$c_X(z) = z^n + p_1 z^{n-1} + p_2 z^{n-2} + \cdots + p_{n-1} z + p_n,$$

where $p_i = p_i(x_1, \ldots, x_n)$ is differentiable in each x_k. We seek real values of $x_1, \ldots, x_n$ with $sgn(x_k) = sgn(a_{i_k j_k})$ such that

$$\begin{aligned} p_1(x_1, \ldots, x_n) - b_1 &= 0, \\ p_2(x_1, \ldots, x_n) - b_2 &= 0, \\ &\vdots \\ p_n(x_1, \ldots, x_n) - b_n &= 0. \end{aligned} \tag{3.1}$$

If $(b_1, \ldots, b_n) = (a_1, \ldots, a_n)$, then a solution to the system of equations in (3.1) is $(x_1, \ldots, x_n) = (a_{i_1 j_1}, \ldots, a_{i_n j_n})$. This is because the coefficient of z^{n-k} in the characteristic polynomial of A is equal to $p_k(a_{i_1 j_1}, \ldots, a_{i_n j_n}) = a_k$. Define functions $f_1, \ldots, f_n$ of $x_1, \ldots, x_n, b_1, \ldots, b_n$ to be the left sides of the equations in (3.1), in order. As $p_k(x_1, \ldots, x_n)$ is the coefficient of z^{n-k} of the characteristic polynomial of X, it must be a polynomial expression in the entries of the matrix X, and hence, each f_k has continuous partial derivatives with respect to all $2n$ variables. Since A allows a nonzero Jacobian and $\frac{\partial f_i}{\partial x_j} = \frac{\partial p_i}{\partial x_j}$, the Jacobian matrix of order n with (i, j) entry equal to $\frac{\partial f_i}{\partial x_j}$ is nonsingular at $(x_1, \ldots, x_n, b_1, \ldots, b_n) = (a_{i_1 j_1}, \ldots, a_{i_n j_n}, a_1, \ldots, a_n)$.

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By the Implicit Function Theorem, for $(b_1, \ldots, b_n)$ sufficiently close to $(a_1, \ldots, a_n)$, there are unique continuous functions $x_1, \ldots, x_n$ of $b_1, \ldots, b_n$ that maintain $f_1 = f_2 = \cdots = f_n = 0$. For $(b_1, \ldots, b_n)$ sufficiently close to $(a_1, \ldots, a_n)$, we have $sgn(x_k) = sgn(a_{i_k j_k})$ for each k, thereby yielding a realization of $\mathcal{A}$ with the desired characteristic polynomial. By using a similar argument as done in [8], the result also holds for any superpattern of $\mathcal{A}$. $\square$

Since the coefficients of the characteristic polynomial of a matrix are continuous functions of its eigenvalues, Theorem 3.2 can be used to provide information on the spectrum of a square pattern $\mathcal{A}$.

Corollary 3.3. *Let $\mathcal{A}$ be a pattern of order n and suppose there is a matrix $A \in Q(\mathcal{A})$ that allows a nonzero Jacobian. If A has spectrum*

$$\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\},$$

then for any multiset $\{\mu_1, \mu_2, \ldots, \mu_n\}$ of n complex numbers, closed under complex conjugation, with each μ_k sufficiently close to λ_k, each superpattern $\hat{\mathcal{A}}$ of pattern $\mathcal{A}$ has a realization $B \in Q(\hat{\mathcal{A}})$ whose spectrum is

$$\sigma(B) = \{\mu_1, \mu_2, \ldots, \mu_n\}.$$

Corollary 3.3 may be used to provide a list of refined inertias a pattern allows.

Lemma 3.4. *Let $\mathcal{A}$ be a pattern of order n and $\hat{\mathcal{A}}$ a superpattern of $\mathcal{A}$. Suppose $\mathcal{A}$ has a realization A that allows a nonzero Jacobian, and let $ri(A) = (a_+, a_-, a_z, 2a_p)$.*

(i) If $a_z \geq 1$, then $(a_+ + 1, a_-, a_z - 1, 2a_p)$, $(a_+, a_- + 1, a_z - 1, 2a_p) \in ri(\hat{\mathcal{A}})$.
(ii) If $a_z \geq 2$, then $(a_+, a_-, a_z - 2, 2(a_p + 1)) \in ri(\hat{\mathcal{A}})$.
(iii) If $a_p \geq 1$, then $(a_+ + 2, a_-, a_z, 2(a_p - 1))$, $(a_+, a_- + 2, a_z, 2(a_p - 1)) \in ri(\hat{\mathcal{A}})$.

Furthermore, for each modified refined inertia, there is a realization of $\hat{\mathcal{A}}$ with this refined inertia that allows a nonzero Jacobian.

Proof. Since $ri(A) = (a_+, a_-, a_z, 2a_p)$, the spectrum of A can be partitioned as

$$\sigma(A) = \{\lambda_1, \ldots, \lambda_{a_+}\} \cup \{\mu_1, \ldots, \mu_{a_-}\} \cup \bigcup_{i=1}^{a_z} \{0\} \cup \{\pm\beta_1 i, \ldots, \pm\beta_{a_p} i\},$$

where each $\lambda_k \in \mathbb{C}$ has positive real part, each $\mu_k \in \mathbb{C}$ has negative real part, and each $\beta_k \in \mathbb{R}$ is positive. Note that $\sigma(A)$ is closed under complex conjugation as A is a real matrix. Since A allows a nonzero Jacobian, by Corollary 3.3, for every superpattern $\hat{\mathcal{A}}$ of $\mathcal{A}$ there is a realization $B \in Q(\hat{\mathcal{A}})$ with arbitrary spectrum (closed under complex conjugation) sufficiently close to $\sigma(A)$.

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If $a_z \geq 1$, then a zero eigenvalue of A can be perturbed to give a positive (resp., negative) eigenvalue, and hence, for a sufficiently small perturbation there will be a realization $B \in Q(\hat{\mathcal{A}})$ with refined inertia $(a_+ + 1, a_-, a_z - 1, 2a_p)$ (resp., $(a_+, a_- + 1, a_z - 1, 2a_p)$).

The other refined inertias can be obtained in a similar manner. Two real zero eigenvalues can be perturbed to give a pair of pure imaginary eigenvalues $\pm\beta i$. Finally, a pair of pure imaginary eigenvalues can be perturbed to give two eigenvalues with positive real parts or two eigenvalues with negative real parts.

By continuity, for each modified refined inertia, there is a realization of $\hat{\mathcal{A}}$ with this refined inertia that allows a nonzero Jacobian. □

As demonstrated in the next example, the condition that A allows a nonzero Jacobian is necessary in the statement of Lemma 3.4.

Example 3.5. Consider the sign pattern

$$\mathcal{A} = \begin{bmatrix} 0 & + & 0 & 0 & 0 \\ - & 0 & + & 0 & 0 \\ 0 & - & - & + & 0 \\ 0 & 0 & - & 0 & + \\ 0 & 0 & 0 & - & 0 \end{bmatrix}.$$

If $A \in Q(\mathcal{A})$ is the realization where each nonzero entry has magnitude equal to 1, then $ri(A) = (0, 3, 0, 2)$. We show in Example 4.8 that A does not allow a nonzero Jacobian. Since it is known (see [13]) that $\mathcal{A}$ allows only eigenvalues with nonpositive real part, $\mathcal{A}$ does not allow the refined inertia $(2, 3, 0, 0)$.

By applying Lemma 3.4 recursively to an initial refined inertia $(a_+, a_-, a_z, 2a_p)$ with $a_z \geq 2$, a kind of "North-East Lemma" for patterns is obtained: see [1, Lemma 1.1] for a North-East Lemma for an inverse eigenvalue problem on graphs.

Corollary 3.6. *Let $\mathcal{A}$ be a pattern of order n and $\hat{\mathcal{A}}$ a superpattern of $\mathcal{A}$. Suppose $\mathcal{A}$ has a realization A such that $ri(A) = (a_+,\, a_-,\, a_z,\, 2a_p)$ with $a_z \geq 2$. If A allows a nonzero Jacobian, then for every $n_+ \geq a_+$, $n_- \geq a_-$ and $n_0 \geq 0$ with $n_+ + n_- + n_0 = n$, it follows that $(n_+, n_-, n_0) \in i(\hat{\mathcal{A}})$.*

The next result, which was first proved in [3], follows from Corollary 3.6 with $a_+ = a_- = 0$ and gives a technique that may be used to show a pattern is inertially arbitrary.

Theorem 3.7. [3, Theorem 2.13] *Let $\mathcal{A}$ be a pattern of order n and suppose $\mathcal{A}$ has a realization A that has refined inertia $ri(A) = (0, 0, a_z, 2a_p)$ for some $a_z \geq 2$. If A allows a nonzero Jacobian, then every superpattern of $\mathcal{A}$ is inertially arbitrary.*

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Note that if none of the realizations A of an inertially arbitrary pattern $\mathcal{A}$ that have $ri(A) = (0, 0, a_z, 2a_p)$ for some $a_z \geq 2$ allows a nonzero Jacobian, then Theorem 3.7 cannot be used to show that $\mathcal{A}$ is inertially arbitrary.

Example 3.8. Following [3, Example 2.16], observe that the patterns

$$\mathcal{N}_4 = \begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ - & - & 0 & 0 \\ 0 & 0 & - & - \end{bmatrix} \quad \text{and} \quad \mathcal{N}_4^* = \begin{bmatrix} * & * & 0 & 0 \\ 0 & 0 & * & * \\ * & * & 0 & 0 \\ 0 & 0 & * & * \end{bmatrix}$$

do not allow the refined inertia $(0, 0, 0, 4)$, but they are shown to be inertially arbitrary in [4] and [7]. Suppose matrix A is a realization of $\mathcal{N}_4$ or $\mathcal{N}_4^*$, where $ri(A)$ is equal to either $(0, 0, 2, 2)$ or $(0, 0, 4, 0)$. If A allows a nonzero Jacobian, then Lemma 3.4 implies that $(0, 0, 0, 4) \in ri(\mathcal{A})$, which is a contradiction. Hence, none of the realizations of $\mathcal{N}_4$ or $\mathcal{N}_4^*$ with refined inertia $(0, 0, 2, 2)$ or $(0, 0, 4, 0)$ allow a nonzero Jacobian. Therefore, Theorem 3.7 cannot be applied to show $\mathcal{N}_4$ and $\mathcal{N}_4^*$ are inertially arbitrary.

In cases where the refined inertia $ri(A) = (0, 0, a_z, 2a_p)$ with $a_z \geq 2$ cannot be used to show that a square pattern is inertially arbitrary, the following consequence of Corollary 3.6 may be applicable.

Theorem 3.9. *Let $\mathcal{A}$ be a pattern of order n. Suppose $A, B \in Q(\mathcal{A})$ have refined inertias $ri(A) = (1, 0, a_z, 2a_p)$ and $ri(B) = (0, 1, b_z, 2b_p)$ such that $a_z, b_z \geq 2$. If both A and B allow a nonzero Jacobian and $(0, 0, n) \in i(\hat{\mathcal{A}})$ for some superpattern $\hat{\mathcal{A}}$ of $\mathcal{A}$, then $\hat{\mathcal{A}}$ is inertially arbitrary.*

In the following sections, we demonstrate how Theorems 3.7 and 3.9 can be used to show a pattern is inertially arbitrary.

4. Generalizations of the nilpotent-centralizer method. In this section, we extend the nilpotent-centralizer method, recently developed in [9] for showing that a pattern is spectrally arbitrary, to give other techniques to show that a pattern is inertially arbitrary.

Recall that the *index* of a nilpotent matrix A is the smallest positive integer k such that $A^k = O$. It was observed in [2] that it is necessary that a nilpotent matrix A of order n has index n for A to allow a nonzero Jacobian. Consequently, it is also a necessary condition for a nilpotent matrix to have index n if it is to be successfully employed in the nilpotent-centralizer method. Note that if a nilpotent matrix has index n, then it is *nonderogatory*, that is, its minimum polynomial is equal to its characteristic polynomial. We discover in this section that a nonderogatory matrix is a key necessary condition to extend these techniques to identify inertially arbitrary patterns. In particular, Corollary 4.5 states that if the minimum polynomial of A

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is not the same as its characteristic polynomial, then A does not allow a nonzero Jacobian.

Let A be a matrix of order n with $m \geq n$ nonzero entries, say $a_{i_1 j_1}, \ldots, a_{i_m j_m}$. Suppose X is the matrix obtained from A by replacing $a_{i_k j_k}$ by real variables x_k for $1 \leq k \leq m$. Let $c_X(z) = z^n + p_1 z^{n-1} + \cdots + p_{n-1} z + p_n$, where $p_i \in \mathbb{R}[x_1, \ldots, x_m]$ for $1 \leq i \leq n$. Define $g : \mathbb{R}^m \to \mathbb{R}^n$ by

$$g(x_1, x_2, \ldots, x_m) = (p_1(x_1, x_2, \ldots, x_m), p_2(x_1, x_2, \ldots, x_m), \ldots, p_n(x_1, x_2, \ldots, x_m)).$$

We call g *the polynomial map of* A, and denote the Jacobian $\mathrm{Jac}(g)$ of g evaluated at $x_k = a_{i_k, j_k}$ $(1 \leq k \leq m)$ by $\mathrm{Jac}(g)|_A$.

First, we reinterpret the property that a matrix allows a nonzero Jacobian.

Observation 4.1. Let A be a matrix of order n and g be the polynomial map of A. Then A allows a nonzero Jacobian if and only if $\mathrm{Jac}(g)|_A$ has rank n.

The following lemma from [9] describes the Jacobian matrix of the polynomial map g of a square matrix A in terms of the adjugate of $zI - A$. For a matrix M, the notation M_{ij} denotes its (i, j) entry.

Lemma 4.2. [9, Lemma 3.1] *Let A be a matrix of order n, $Y = [y_{ij}]$ be a matrix with the same zero-nonzero pattern as A and whose nonzero entries are distinct indeterminates, and $c_Y(z) = z^n + p_1 z^{n-1} + \cdots + p_{n-1} z + p_n$ be the characteristic polynomial of Y with $p_i \in \mathbb{R}[y_{11}, y_{12}, \ldots, y_{nn}]$. Then for $y_{ij} \neq 0$,*

$$-\frac{\partial p_k}{\partial y_{ij}}\bigg|_{Y=A} = \text{the coefficient of } z^{n-k} \text{ in } [\mathrm{adj}(zI - A)]_{ji}. \tag{4.1}$$

When A is a nilpotent matrix, Garnett and Shader [9, Lemma 3.3] show that

$$\mathrm{adj}(zI - A) = \sum_{k=0}^{n-1} A^k z^{n-1-k}$$

and derive in [9, Theorem 3.4] that line (4.1) simplifies to

$$\frac{\partial p_k}{\partial y_{ij}}\bigg|_{Y=A} = -(A^{k-1})_{ji}.$$

Now, we consider the case when A is not necessarily nilpotent. For any square matrix B, $B\,\mathrm{adj}(B) = \det(B)I$ and thus

$$(zI - A)\,\mathrm{adj}(zI - A) = c_A(z)I, \tag{4.2}$$

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where $c_A(z)$ is the characteristic polynomial of A. Thus, as was done by Leverrier [12, p. 167], comparing coefficients of the powers of z in (4.2) gives

$$\mathrm{adj}(zI - A) = \sum_{k=0}^{n-1} h_k(A) z^{n-1-k} \tag{4.3}$$

for some real polynomials $h_k(z)$ of degree k, $1 \leq k \leq n-1$, with $h_0(A) = I$. Thus, using Lemma 4.2, the following result generalizes Theorem 3.4 in [9].

THEOREM 4.3. *Let A be a matrix of order n, $Y = [y_{ij}]$ be a matrix with the same zero-nonzero pattern as A and whose nonzero entries are distinct indeterminates, and $c_Y(z) = z^n + p_1 z^{n-1} + \cdots + p_{n-1} z + p_n$ be the characteristic polynomial of Y with $p_i \in \mathbb{R}[y_{11}, y_{12}, \ldots, y_{nn}]$. Then for $y_{ij} \neq 0$,*

$$\left.\frac{\partial p_k}{\partial y_{ij}}\right|_{Y=A} = -\left[h_{k-1}(A)\right]_{ji},$$

where $h_k(z)$ is defined in (4.3).

The *Hadamard product* of matrices A and B of the same order is the entrywise product $A \circ B$ with $[A \circ B]_{ij} = a_{ij} b_{ij}$. Mimicking Theorem 3.5 in [9], we have the following result without the nilpotent condition on A.

THEOREM 4.4. *Let A be a matrix of order n and g be its polynomial map. Then $\mathrm{Jac}(g)|_A$ has rank less than n if and only if there exists a nonzero polynomial $q(z) \in \mathbb{R}[z]$ of degree at most $n-1$ such that $q(A) \circ A^T = O$.*

Proof. Suppose A is a matrix of order n with polynomial map g. If $\mathrm{Jac}(g)|_A$ has rank less than n, then by Theorem 4.3, there exist scalars $c_0, c_1, \ldots, c_{n-1}$ not all zero such that $(c_0 I + c_1 h_1(A) + \cdots + c_{n-1} h_{n-1}(A)) \circ A^T = O$. But this implies that there exists a nonzero polynomial $q(z) \in \mathbb{R}[z]$ of degree at most $n-1$ (since each $h_k(z)$ has degree at most $n-1$) such that $q(A) \circ A^T = O$.

Conversely, if $\mathrm{Jac}(f)|_A$ has rank n, then by Theorem 4.3, the set $W = \{I \circ A^T, h_1(A) \circ A^T, \ldots, h_{n-1}(A) \circ A^T\}$ is a linearly independent set. But $\mathrm{span}(W) \subset \mathrm{span}(W')$ for $W' = \{I \circ A^T, A \circ A^T, A^2 \circ A^T, \ldots, A^{n-1} \circ A^T\}$. Thus, W' is a linearly independent set and $q(A) \circ A^T \neq O$ for any nonzero polynomial $q(z) \in \mathbb{R}[z]$ of degree at most $n-1$. ∎

COROLLARY 4.5. *If A is a derogatory matrix of order n, then A does not allow a nonzero Jacobian.*

Proof. If A is derogatory and $q(z)$ is the minimum polynomial of A, then $\deg(q(z)) \leq n-1$ and $q(A) = O$. The result follows from Theorem 4.4 and Observation 4.1. ∎

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The *centralizer* of a matrix A is the set of all matrices that commute with A. Analogous to Lemma 3.6 in [9], we have the following lemma.

LEMMA 4.6. *Let A be a nonderogatory matrix of order n. There exists a nonzero polynomial $q(z) \in \mathbb{R}[z]$ of degree at most $n-1$ such that $q(A) \circ A^T = O$ if and only if there is a nonzero matrix B in the centralizer of A such that $B \circ A^T = O$.*

Proof. Let A be a nonderogatory matrix of order n.

Suppose there is a nonzero polynomial $q(z) \in \mathbb{R}[z]$ of degree at most $n-1$ with $q(A) \circ A^T = O$. Since A is nonderogatory, the matrices $I, A, A^2, \ldots, A^{n-1}$ are linearly independent. This implies $q(A) \neq 0$, since $q(z) \neq 0$. Furthermore, $q(A)$ commutes with A. Thus, we may take $B = q(A)$.

Suppose there is a nonzero matrix B in the centralizer of A such that $B \circ A^T = O$. Since A is nonderogatory, by [11, Theorem 3.2.4.2], $B = q(A)$ for some nonzero polynomial $q(z) \in \mathbb{R}[z]$ of degree at most $n-1$. ☐

Observation 4.1, Theorem 4.4, and Lemma 4.6 give rise to the following result.

THEOREM 4.7. *Let A be a nonderogatory matrix. Then A allows a nonzero Jacobian if and only if the zero matrix is the only matrix B in the centralizer of A such that $B \circ A^T = O$.*

In the following example, we use Theorem 4.7 to show that the matrix mentioned in Example 3.5 does not allow a nonzero Jacobian.

EXAMPLE 4.8. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 & -1 & 1 \\ 0 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Then A is nonderogatory, $B \circ A^T = O$, and $BA = AB$. Therefore, by Theorem 4.7, A does not allow a nonzero Jacobian.

Using Theorems 3.7 and 4.7, we obtain the following result that gives a centralizer technique applicable to inertially arbitrary patterns.

THEOREM 4.9. *Let $\mathcal{A}$ be a pattern of order n and suppose $\mathcal{A}$ has a realization A that has refined inertia $ri(A) = (0, 0, a_z, 2a_p)$ with $a_z \geq 2$. If A is nonderogatory and the only matrix B in the centralizer of A satisfying $B \circ A^T = O$ is the zero matrix, then every superpattern of $\mathcal{A}$ is inertially arbitrary.*

Note that if $a_z = n$ in Theorem 4.9, then every superpattern of $\mathcal{A}$ is in fact spectrally arbitrary [9]. However if $a_z < n$, this is not in general true; see, for

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example, $\mathcal{W}_n$ in Section 4.1 and $\mathcal{G}_n$ with $n = 2k+1$ in Section 4.3, which do not allow refined inertia $(0, 0, n, 0)$, and thus are not spectrally arbitrary.

A *proper lower Hessenberg* pattern $\mathcal{A} = [\alpha_{ij}]$ of order n has $\alpha_{i,i+1} \neq 0$ and $\alpha_{ij} = 0$ for $1 \leq i \leq n-1$ and $j \geq i+2$. A particular example is a *full tridiagonal* pattern, which has all defined entries α_{kk}, $\alpha_{k,k+1}$, and $\alpha_{k,k-1}$ nonzero, $1 \leq k \leq n$. In the next two theorems, we apply Theorem 4.9 using the following remark.

Remark 4.10. [10, Theorem 7.4.4] Every proper lower Hessenberg matrix is nonderogatory.

Theorem 4.11. *Let $\mathcal{A}$ be a full tridiagonal sign pattern. If there exists a matrix $A \in Q(\mathcal{A})$ having refined inertia $(0, 0, a_z, 2a_p)$ with $a_z \geq 2$, then all superpatterns of $\mathcal{A}$ are inertially arbitrary.*

Proof. If there exists a matrix $A \in Q(\mathcal{A})$ having refined inertia $(0, 0, a_z, 2a_p)$ with $a_z \geq 2$, then A is nonderogatory by Remark 4.10. Consider a matrix B that commutes with A and has the property that $B \circ A^T = O$. Following the argument in [9, Lemma 4.6], the matrix B must be the zero matrix. Therefore, by Theorem 4.9, $\mathcal{A}$ and all its superpatterns are inertially arbitrary. ∎

Theorem 4.12. *Let $\mathcal{A}$ be a proper lower Hessenberg sign pattern with the first column having all entries nonzero. If $A \in Q(\mathcal{A})$ has refined inertia $(0, 0, a_z, 2a_p)$ with $a_z \geq 2$, then all superpatterns of $\mathcal{A}$ are inertially arbitrary.*

Proof. By Remark 4.10, any matrix $A \in Q(\mathcal{A})$ is nonderogatory. Suppose there exists a matrix $A \in Q(\mathcal{A})$ having refined inertia $(0, 0, a_z, 2a_p)$ with $a_z \geq 2$. Consider a matrix B such that $BA = AB$ and $B \circ A^T = O$. We use induction on the rows of B to prove that $B = O$. The first row of B is all zeros, since the first column of A is nonzero. Therefore, the first row of BA is all zeros, but the first row of AB is a nonzero multiple of the second row of B, hence the second row of B is all zeros. Now suppose the first $r-1$ rows of B are zero. Then row $r-1$ of BA is zero. Row $r-1$ of AB is a linear combination of rows 1 to r of B with a nonzero coefficient for row r of B. Now since rows 1 to $r-1$ in B are zero, it follows that row r of B is zero. Therefore, by induction $B = O$ and by Theorem 4.9, $\mathcal{A}$ and all of its superpatterns are inertially arbitrary. ∎

Example 4.13. Consider the following sign pattern $\mathcal{A}$ and a realization A:

$$\mathcal{A} = \begin{bmatrix} + & + & 0 & 0 & 0 & 0 \\ - & 0 & + & 0 & 0 & 0 \\ - & 0 & 0 & + & 0 & 0 \\ - & 0 & 0 & 0 & + & 0 \\ + & 0 & 0 & 0 & 0 & + \\ - & 0 & 0 & 0 & - & - \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & 1 & 0 & 0 & 0 & 0 \\ -b & 0 & 1 & 0 & 0 & 0 \\ -c & 0 & 0 & 1 & 0 & 0 \\ -d & 0 & 0 & 0 & 1 & 0 \\ e & 0 & 0 & 0 & 0 & 1 \\ -f & 0 & 0 & 0 & -g & -h \end{bmatrix}$$

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with $a, \dots, h$ positive. Note that without loss of generality any $A \in Q(\mathcal{A})$ can be scaled as above [16, Lemma 2]. The coefficient of x^2 in the characteristic polynomial of A is $bg + ch + d > 0$ and so $\mathcal{A}$ is not spectrally arbitrary. If $a = b = h = \frac{1}{2}$, $c = \frac{17}{8}$, $d = \frac{9}{16}$, $e = \frac{83}{8}$, $f = \frac{161}{64}$, and $g = \frac{19}{4}$, then the eigenvalues of A are $\{0, 0, \pm i, \pm 2i\}$. Thus, by Theorem 4.12, $\mathcal{A}$ and its superpatterns are inertially arbitrary.

Example 4.13 addresses a remark made in [3]. In particular, this example shows that there exists an inertially arbitrary sign pattern that does not allow a positive and a negative principal minor of every order, correcting a statement made in [8].

5. Corrections to the literature. In this section we revisit the patterns presented in [15, 5, 6] that were claimed to be inertially arbitrary using a flawed technique. We now prove that these patterns are indeed inertially arbitrary.

5.1. The patterns $\mathcal{W}_n$ and $\mathcal{W}_n^*$ in [6]. In [6], it is stated that the sign pattern $\mathcal{W}_n$ is inertially arbitrary for all $n \geq 6$ and its zero-nonzero pattern $\mathcal{W}_n^*$ is inertially arbitrary for all $n \geq 5$. Although the proof of Lemma 5 in [6] is flawed, the following result has recently been proved.

PROPOSITION 5.1. [3, Example 2.15] *For $n \geq 6$, every superpattern of $\mathcal{W}_n$ is inertially arbitrary.*

We now consider the zero-nonzero pattern $\mathcal{W}_n^*$.

PROPOSITION 5.2. *For $n \geq 5$, every superpattern of $\mathcal{W}_n^*$ is inertially arbitrary.*

Proof. By Proposition 5.1, every superpattern of $\mathcal{W}_n^*$ is inertially arbitrary for all $n \geq 6$. It remains to show that every superpattern of $\mathcal{W}_5^*$ is inertially arbitrary. Note that [6, Theorem 9] gives two matrices in $Q(\mathcal{W}_5^*)$ with inertias $(1, 1, 3)$ and $(1, 3, 1)$. Furthermore, both of these matrices allow a nonzero Jacobian (using the same choice of variables as in [3, Example 2.15]). By Theorem 3.2, every superpattern of $\mathcal{W}_5^*$ allows the inertias $(1, 1, 3)$, $(1, 3, 1)$ and also $(3, 1, 1)$ by negation.

Note that the matrix

$$W = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 6 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 6 & 0 & 0 & -1 \\ 0 & 6 & 0 & 0 & -1 \end{bmatrix} \in Q(\mathcal{W}_5^*)$$

has refined inertia $ri(W) = (0, 0, 1, 4)$. Furthermore, if g is the polynomial map of W, then $\mathrm{rank}(\mathrm{Jac}(g)|_W) = 5$. By recursive application of Lemma 3.4 to the refined inertia $(0, 0, 1, 4)$, the zero-nonzero pattern $\mathcal{W}_5^*$ and its superpatterns can attain every

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inertia (w_+, w_-, w_0) where the product w_+w_- is even and $w_+ + w_- + w_0 = 5$. Therefore, every superpattern of $\mathcal{W}_5^*$ is inertially arbitrary. ☐

Note that the nilpotent-Jacobian method cannot be applied to show that a sign pattern with less than $2n$ nonzero entries is inertially arbitrary; see [16, Theorem 6]. However, in Proposition 5.1, Theorem 3.7 is successfully applied to sign patterns $\mathcal{W}_n$ $(n \geq 6)$ with $2n-1$ nonzero entries to show that the sign patterns are inertially arbitrary.

The sign pattern $\mathcal{M}_n$ was introduced in [6] and was declared to be inertially arbitrary since it allows the same characteristic polynomials as $\mathcal{W}_n$ (see [6, Theorem 10]). By Propositions 5.1 and 5.2, Theorem 10 in [6] can be strengthened to include superpatterns of $\mathcal{M}_n$ and $\mathcal{M}_n^*$.

5.2. The sign patterns $\mathcal{N}_{2,1}$ and $\mathcal{N}_{2,2}$ in [5]. In [5, Lemma 3.2], the method used to show that the sign patterns

$$\mathcal{N}_{2,1} = \begin{bmatrix} + & + & + & 0 \\ - & - & - & 0 \\ 0 & 0 & 0 & + \\ - & - & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{N}_{2,2} = \begin{bmatrix} + & + & + & 0 \\ - & - & - & 0 \\ 0 & 0 & 0 & - \\ - & - & 0 & 0 \end{bmatrix}$$

allow specific inertias is flawed and hence, Proposition 3.4 in [5] is also affected by the flaw. In the following we provide an alternative proof of Proposition 3.4 in [5] using Theorem 3.9.

Proposition 5.3. *The sign patterns $\mathcal{N}_{2,1}$ and $\mathcal{N}_{2,2}$ are inertially arbitrary.*

Proof. First, note that setting every nonzero entry in both sign patterns $\mathcal{N}_{2,1}$ or $\mathcal{N}_{2,2}$ to have magnitude 1 gives matrices with characteristic polynomial z^4 (see [5, Lemma 3.2]). Thus, both sign patterns allow inertia $(0, 0, 4)$.

We next provide realizations A and B with refined inertias $ri(A) = (1, 0, 3, 0)$ and $ri(B) = (0, 1, 3, 0)$ and show that both A and B allow a nonzero Jacobian. Let

$$X = \begin{bmatrix} x_1 & x_2 & 1 & 0 \\ -2 & -1 & -x_3 & 0 \\ 0 & 0 & 0 & v \\ -1 & -x_4 & 0 & 0 \end{bmatrix}.$$

We analyze the sign patterns $\mathcal{N}_{2,1}$ and $\mathcal{N}_{2,2}$ simultaneously by fixing either $v = 1$ or $v = -1$, accordingly. Let A be the matrix obtained from X by setting $(x_1, x_2, x_3, x_4) = (2, 1, 1, 1)$. Then $v = 1$ (resp., $v = -1$) gives $ri(A) = (1, 0, 3, 0)$ and $A \in Q(\mathcal{N}_{2,1})$ (resp., $A \in Q(\mathcal{N}_{2,2})$). It can be verified that the Jacobian matrix evaluated at $(x_1, x_2, x_3, x_4) = (2, 1, 1, 1)$ is nonsingular.

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Let

$$Y = \begin{bmatrix} y_1 & y_2 & 1 & 0 \\ -2 & -2 & -y_3 & 0 \\ 0 & 0 & 0 & v \\ -1 & -y_4 & 0 & 0 \end{bmatrix}.$$

As above, we analyze the sign patterns $\mathcal{N}_{2,1}$ and $\mathcal{N}_{2,2}$ simultaneously by fixing either $v = 1$ or $v = -1$, accordingly. Let B be the matrix obtained from Y by setting $(y_1, y_2, y_3, y_4) = (1, 1, 1, 1)$. Then $v = 1$ (resp., $v = -1$) gives $ri(B) = (0, 1, 3, 0)$ and $B \in Q(\mathcal{N}_{2,1})$ (resp., $B \in Q(\mathcal{N}_{2,2})$). It can be verified that the Jacobian matrix evaluated at $(y_1, y_2, y_3, y_4) = (1, 1, 1, 1)$ is nonsingular.

Theorem 3.9 then shows both $\mathcal{N}_{2,1}$ and $\mathcal{N}_{2,2}$ are inertially arbitrary. $\square$

5.3. The sign pattern $\mathcal{G}_{2k+1}$ in [15]. For $k \geq 2$, let

$$A_{2k+1} = \left[\begin{array}{cccccccc} -a_1 & -a_2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 1 & a & 0 & 0 & & & \vdots \\ 0 & 0 & 0 & -a_3 & -1 & 0 & & \\ 0 & -1 & 0 & 0 & -a_4 & \ddots & \ddots & \vdots \\ -1 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & -a_{2k-1} & -1 \\ 0 & -1 & 0 & 0 & & & \ddots & -a_{2k} \\ -1 & 0 & 0 & 0 & \cdots & & 0 & 0 \end{array}\right] \in Q\left(\mathcal{G}_{2k+1}\right),$$

with $a > 0$ and $a_i > 0$ for $1 \leq i \leq 2k$, as in [15]. The sign pattern $\mathcal{G}_{2k+1}$ was claimed to be inertially arbitrary for $k \geq 2$, but the proof of Lemma 5 in [15] uses a flawed argument. Using Theorem 3.7, we provide a new argument to show that $\mathcal{G}_{2k+1}$ is inertially arbitrary.

Let $\widetilde{A}_{2k+1}$ denote the matrix with $a = 1$ and $a_i = 1$, $1 \leq i \leq 2k$. We show that $ri(\widetilde{A}_{2k+1}) = (0, 0, 2k-1, 2)$ and $\widetilde{A}_{2k+1}$ allows a nonzero Jacobian. To do so, we follow the methods presented in [15, pp. 270-275]. We first verify that the sign pattern $\mathcal{G}_5$ is inertially arbitrary using Theorem 4.9. Let M be a matrix of order n, and let α, β be nonempty subsets of $\{1, 2, \ldots, n\}$. Then $M(\alpha, \beta)$ (resp., $M[\alpha, \beta]$) denotes the submatrix of M obtained by removing (resp., retaining) rows indexed by α and columns indexed by β. When $\alpha = \beta$, we use $M(\alpha)$ and $M[\alpha]$, respectively.

Proposition 5.4. *Every superpattern of $\mathcal{G}_5$ is inertially arbitrary.*

Proof. Let $A = \widetilde{A}_5$. Matrix A has refined inertia $(0, 0, 3, 2)$ and A is nonderogatory. Take B such that $B \circ A^T = O$. Consider A and B partitioned so that the $(1, 1)$

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block $A_{11} = A[\{123\}]$ and the $(2,2)$ block $A_{22} = A[\{45\}]$. Partition matrix B in the same manner. Then $B_{11}A_{11} = O$ and $B_{21}A_{12} = O$. The $(1,1)$ block of $AB = BA$ then gives $b_{14} = b_{25} = b_{13} = 0$ and $b_{23} = -b_{33}$, so $A_{21}B_{12} = 0$. The $(2,2)$ block then gives $b_{55} = b_{44}$. The $(1,2)$ block gives $b_{34} = b_{35} = b_{23} = 0$, so $b_{33} = 0$. Hence, $b_{44} = b_{55} = 0$ and $b_{45} = 0$. Lastly, the $(2,1)$ block gives $b_{52} = b_{51} = 0$ implying that $b_{42} = b_{41} = 0$, and finally, that $B = O$. Therefore, by Theorem 4.9, every superpattern of $\mathcal{G}_5$ is inertially arbitrary. ◻

Let $c_{2k+1}(z)$ be the characteristic polynomial of A_{2k+1}. The cofactor expansion of $\det(zI - A_{2k+1})$ along the last row gives

$$\det[(zI - A_{2k+1})(\{2k+1\},\{1\})]$$

$$= \det\begin{bmatrix} a_2 & 1 \\ z-1 & -a \end{bmatrix} \det\begin{bmatrix} a_3 & 1 \\ z & a_4 \end{bmatrix} \cdots \det\begin{bmatrix} a_{2k-1} & 1 \\ z & a_{2k} \end{bmatrix}$$

and $\det[(zI - A_{2k+1})(2k+1)]$ equals

$$zc_{2k-1}(z) + \det\begin{bmatrix} z+a_1 & 1 \\ -2 & -a \end{bmatrix} \det\begin{bmatrix} a_3 & 1 \\ z & a_4 \end{bmatrix} \cdots \det\begin{bmatrix} a_{2k-3} & 1 \\ z & a_{2k-2} \end{bmatrix} \det[a_{2k-1}].$$

Thus,

$$\begin{aligned} c_{2k+1}(z) &= (-a_2a - z + 1)(a_3a_4 - z)\cdots(a_{2k-1}a_{2k} - z) + z^2c_{2k-1} \\ &\quad + z(a_{2k-1})(2 - a(z+a_1))(a_3a_4 - z)\cdots(a_{2k-3}a_{2k-2} - z). \end{aligned} \tag{5.1}$$

Define

$$\begin{aligned} g_{2k+1}(z) &= (1 - z - a_2a)\prod_{r=2}^{k}(a_{2r-1}a_{2r} - z) \\ &\quad + a_{2k-1}z(2 - a(z+a_1))\prod_{r=2}^{k-1}(a_{2r-1}a_{2r} - z). \end{aligned} \tag{5.2}$$

Since the degree of each summand of $g_{2k+1}(z)$ is k, it follows that $\deg(g_{2k+1}(z)) \leq k$. For $k \geq 3$,

$$c_{2k+1}(z) = z^2c_{2k-1}(z) + g_{2k+1}(z). \tag{5.3}$$

The next result identifies a realization of $\mathcal{G}_{2k+1}$ with refined inertia $(0,0,2k-1,2)$.

Proposition 5.5. [15, Proposition 7] *Let $k \geq 2$. If $a_i = 1$ for $1 \leq i \leq 2k$ and $a = 1$, then*

$$c_{2k+1}(z) = z^{2k+1} + z^{2k-1}.$$

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Proof. When $k = 2$, it can be verified that $c_5(z) = z^5 + z^3$. Assume that $k \geq 3$ and proceed by induction. By (5.3) and the induction hypothesis,

$$c_{2k+1}(z) = z^2(z^{2k-1} + z^{2k-3}) + g_{2k+1}(z).$$

Since $a_i = 1$ for $1 \leq i \leq 2k$ and $a = 1$, it follows from (5.2) that

$$g_{2k+1}(z) = -z(1-z)^{k-1} + z(1-z)^{k-1} = 0,$$

which gives the result. □

For consistency with the notation used in [15], let $f_i(A_{2k+1})$ be the coefficient of $z^{(2k+1)-i}$ in $c_{2k+1}(z)$ for $1 \leq i \leq 2k$. Define $f_i(A_{2k+1}) = 0$ for $i \geq 2k+1$.

We show that the Jacobian matrix has the form

$$J = \left.\frac{(\partial f_1, \ldots, \partial f_{2k+1})}{(\partial a_1, \ldots, \partial a_{2k}, \partial a)}\right|_{(1,\ldots,1)} = \left[\begin{array}{cc|cccc} 1 & 0 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & 0 & \cdots & \cdots & 0 \\ \hline \# & \# & -1 & \# & \cdots & \# \\ \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \# \\ \# & \# & 0 & \cdots & 0 & -1 \end{array}\right], \tag{5.4}$$

where each # represents a real number.

Proposition 5.6. *For any $k \geq 2$,*

$$f_1(A_{2k+1}) = -1 + a_1, \quad f_2(A_{2k+1}) = -a_1 + 2a_2, \quad f_3(A_{2k+1}) = 1 - a_3 a,$$
$$f_{2k}(A_{2k+1}) = (2 - a_1 a)\prod_{r=3}^{2k-1} a_r - \prod_{r=3}^{2k} a_r - (1 - a_2 a)\sum_{s=2}^{k}\prod_{\substack{r=2\\ r\neq s}}^{k} a_{2r-1}a_{2r},$$
$$f_{2k+1}(A_{2k+1}) = (1 - a_2 a)\prod_{r=3}^{2k} a_r.$$

Proof. The formulas for both $f_1(A_{2k+1})$ and $f_2(A_{2k+1})$ follow from the negative of the trace and the sum of the 2×2 principal minors, see, e.g., [15, Proposition 3]. For $f_3(A_{2k+1})$, note that $f_3(A_5) = 1 - a_3 a$ and proceed by induction on k. Finally, $f_{2k}(A_{2k+1})$ follows from (5.1) by looking at the coefficient of z, and $f_{2k+1}(A_{2k+1})$ follows by setting $z = 0$ in (5.1). □

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Using Proposition 5.6, the first three rows and last two rows of the Jacobian matrix can be seen to have the form specified in (5.4) when evaluated at $(a_1, \ldots, a_{2k}, a) = (1, \ldots, 1)$.

PROPOSITION 5.7. *If* $3 \leq j \leq 2k-2$, *then* $\left.\frac{\partial g_{2k+1}(z)}{\partial a_j}\right|_{(1,\ldots,1)} = 0$.

Proof. By (5.2), for $3 \leq j \leq 2k-2$,

$$\frac{\partial g_{2k+1}(z)}{\partial a_j} = (1 - z - a_2 a)\left(\prod_{\substack{r=2 \\ r\neq\lceil j/2\rceil}}^{k} (a_{2r-1}a_{2r} - z)\right) a_s$$
$$+a_{2k-1}z(2 - a(z + a_1))\left(\prod_{\substack{r=2 \\ r\neq\lceil j/2\rceil}}^{k-1} (a_{2r-1}a_{2r} - z)\right) a_s,$$

where $a_s = a_{j+1}$ if j is odd, and $a_s = a_{j-1}$ if j is even. When evaluated at $(a_1, \ldots, a_{2k}, a) = (1, \ldots, 1)$, the zero polynomial is obtained. □

PROPOSITION 5.8. *If* $4 \leq i \leq 2k-1$ *and* $3 \leq j \leq i-1$, *then* $\left.\frac{\partial f_i(A_{2k+1})}{\partial a_j}\right|_{(1,\ldots,1)} = 0$ *and* $\left.\frac{\partial f_i(A_{2k+1})}{\partial a_i}\right|_{(1,\ldots,1)} = -1$.

Proof. For $w = a_j$ or a_i, $\frac{\partial f_i(A_{2k+1})}{\partial w}$ is the sum of $\frac{\partial f_i(A_{2k-1})}{\partial w}$ and the coefficient of z^{2k+1-i} in $\frac{\partial g_{2k+1}(z)}{\partial w}$ by (5.3). The result now follows by induction on k along with Propositions 5.6 and 5.7. □

By Proposition 5.8, the Jacobian matrix has the form specified in (5.4). Therefore, $\widetilde{A}_{2k+1}$ allows a nonzero Jacobian. By Proposition 5.5, $ri(\widetilde{A}_{2k+1}) = (0, 0, 2k-1, 2)$, and hence by Theorem 3.7, $\mathcal{G}_{2k+1}$ is inertially arbitrary. In fact, we have proven the following stronger result.

COROLLARY 5.9. *For* $k \geq 2$, *every superpattern of* $\mathcal{G}_{2k+1}$ *is inertially arbitrary.*

6. Appendix. We illustrate a flaw in a technique used in [15, Lemma 5],[5, Lemma 3.2], and [6, Lemma 5]. Each of the results in question demonstrated that a Jacobian matrix had full rank if one of the coefficients of a related characteristic polynomial was assumed to be nonzero. The remainder of the arguments did not adequately take into account that the coefficient in question was being fixed before the Implicit Function Theorem was employed.

We illustrate the flaw by focusing on Lemma 3.2 in [5] as an example and use the notation from that paper. This Lemma 3.2 makes the claim that for any $r_1, r_2, r_3, r_4 \in \mathbb{R}$ with $r_3 \neq 0$, there exists a matrix $A \in Q(\mathcal{N}_{2,1})$ such that $p_A(x) = x^4 + r_1x^3 + r_2x^2 +$

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$r_3x + r_4$. With the flawed proof in hand, consider the following example. Suppose we want to obtain the polynomial $p(x) = x^4 + x^3 + x^2 + 7x + 1$. In particular, $c^3r_3 = 7$ for the scalar c in the first sentence of the proof. For any choice of r_3, no matter how small, choose c so that $c^3r_3 = 7$. In the middle of the proof, r_3 is fixed to be some positive number (indirectly fixing the scaling factor c). For any such positive fixed value of r_3, the Jacobian is shown to be nonzero. For the fixed value of r_3 there is some ϵ_{r_3}, such that if r_1, r_2 and r_4 are within ϵ_{r_3} of zero, then the polynomial $x^4 + r_1x^3 + r_2x^2 + r_3x + r_4$ can be obtained. To obtain $p(x)$ above, it is necessary to scale by c, but $c = (\frac{7}{r_3})^{\frac{1}{3}}$, because r_3 is fixed. Thus, focusing on the coefficient of x^2 as an example, $c^2r_2 = 1$, that is $r_2 = (\frac{r_3}{7})^{\frac{2}{3}}$. However it is not guaranteed that $(\frac{r_3}{7})^{\frac{2}{3}} \leq \epsilon_{r_3}$. Of course, a smaller value of r_3 could be chosen. But that smaller value of r_3 might result in an even smaller ϵ_{r_3}, and it could still be the case that $r_2 > \epsilon_{r_3}$. Thus, the flawed argument assumes that ϵ_{r_3} is independent of r_3.

Below we provide a corrected statement for Lemma 3.2 in [5] to help illustrate the flaw discussed above.

Lemma 6.1. (Correction to [5, Lemma 3.2]) *For any $r_1, r_2, r_4 \in \mathbb{R}$ there exists a matrix $A \in Q(\mathcal{N}_{2,1})$ and $B \in Q(\mathcal{N}_{2,2})$ such that $p_A(x) = p_B(x) = x^4 + r_1x^3 + r_2x^2 + r_3x + r_4$, for $|r_3|$ sufficiently large with respect to r_1, r_2, r_4.*

Proof. The proof is identical to that in [5] with the last sentence

> "Thus, for any r_1, r_2, r_4 and any $r_3 \neq 0$ sufficiently close to 0, there exist positive values $a_{11}, a_{22}, a_{12}, a_{41}, a_{42}$ such that $p_A(x) = x^4 + r_1x^3 + r_2x^2 + r_3x + r_4$ with $r_3 \neq 0$."

replaced by the following sentences.

> "Thus, for $r_3 \neq 0$ given, and for any r_1, r_2, r_4 sufficiently close to 0, there exist positive values $a_{11}, a_{22}, a_{12}, a_{41}, a_{42}$ such that $p_A(x) = x^4 + r_1x^3 + r_2x^2 + r_3x + r_4$. Suppose we wish to obtain an arbitrary polynomial $p(x) = x^4 + sx^3 + tx^2 + ux + v$ with u sufficiently large with respect to s, t and v. Then choose r_1, r_2 and r_4 sufficiently close to zero with the additional property that $cr_1 = s$, $c^2r_2 = t$ and $c^4r_4 = v$ for some $c \neq 0$. Then $\mathcal{A}$ realizes the characteristic polynomial $p(x) = x^4 + sx^3 + tx^2 + c^3r_3x + v$. In fact, by further scaling down r_1, r_2 and r_4 appropriately, $\mathcal{A}$ can realize any characteristic polynomial $p(x) = x^4 + sx^3 + tx^2 + kx + v$ with $|k| \geq |c^3r_3|$, that is, with the coefficient of x sufficiently large." ∎

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