



COLIN DE VERDIÈRE PARAMETERS OF CHORDAL GRAPHS*

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Abstract. The Colin de Verdière parameters, μ and ν , are defined to be the maximum nullity of certain real symmetric matrices associated with a given graph. In this work, both of these parameters are calculated for all chordal graphs. For ν the calculation is based solely on maximal cliques, while for μ the calculation depends on split subgraphs. For the case of μ our work extends some recent work on computing μ for split graphs.

Key words. Chordal graphs, Colin de Verdière parameters, Maximum nullity, Positive semi-definite matrices, Schur complements, Split graphs, Tree-width.

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1. Introduction. Let $G = (V, E)$ be an undirected graph with no loops but possibly multiple edges (a *multigraph*) with vertex set $V = \{1, 2, \dots, n\}$. A graph G is called *simple* if it contains no loops nor multiple edges. Define $\mathcal{S}(G)$ as the set of all n -by- n real symmetric matrices $A = [a_{ij}]$, where

- $a_{ij} \neq 0$ whenever $i \neq j$ and i and j are adjacent by a single edge, and
- $a_{ij} = 0$ whenever $i \neq j$ and i and j are not adjacent.

Note that the entry a_{ij} for $i \neq j$ is not restricted if $ij \in E$ is a multiple edge, and that the main diagonal entries are not restricted in general.

Let $\mathcal{P}(G)$ be the set of positive semidefinite matrices in $\mathcal{S}(G)$. If $A \in \mathcal{P}(G)$, then each diagonal entry a_{ii} is nonnegative and if there is a nonzero off-diagonal entry in the i th row of A then $a_{ii} > 0$. By $M(G)$ and $M_+(G)$ we mean the largest possible nullity of any matrix in $\mathcal{S}(G)$ and in $\mathcal{P}(G)$, respectively.

For a multigraph G , a matrix $M \in \mathcal{S}(G)$ satisfies the *Strong Arnold Property* (SAP) with respect to G if there does not exist a nonzero symmetric matrix B such that $MB = A(G) \circ B = I \circ B = 0$, where \circ is the entry-wise (Hadamard) product, and $A(G)$ is the adjacency matrix of G (whose (i, j) -entry is the number of edges between vertices i and j). Note that the use of the adjacency matrix in the definition

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of the SAP means that such a matrix B is required to have a zero entry in position (i, j) whenever there is at least one edge between the vertices i and j .

For a multigraph G , $\nu(G)$ is the maximum nullity of matrices in $\mathcal{P}(G)$ that have the SAP with respect to G , and $\mu(G)$ (see [6], where μ is defined for simple graphs) is the maximum nullity among all real symmetric matrices $M = [m_{ij}]$ that satisfy

- (M1) for all $i \neq j$, $m_{ij} \leq 0$, $m_{ij} < 0$ if i and j are adjacent in G by a single edge, and $m_{ij} = 0$ if i and j are not adjacent;
- (M2) M has exactly one negative eigenvalue and that eigenvalue has multiplicity one;
- (M3) M has the SAP with respect to G .

The parameters μ and ν have been studied extensively over the past twenty years and have arisen in the context of vertex connectivity [19, 20] and in connection with cliques and chordal graphs [8, 16]. Colin de Verdière's original definition of ν was for simple graphs [7], and this definition was extended to a multigraph setting (and to complex matrices) in [12, 15].

For graphs that are not connected, μ and ν can be found by taking a maximum over connected components (unless the graph has no edges in the case of μ). We therefore assume all graphs are connected unless stated otherwise. Both parameters μ and ν are minor monotone for simple graphs [6, 7], and the minor monotonicity of ν for multigraphs was shown in [15].

For chordal graphs it is known (see [8, 12, 13, 15]) that both μ and ν can take on only one of two values (depending on the tree-width or maximum clique size). Utilizing an operation called orthogonal removal, which generalizes to vertices of any degree the ΔY and $Y\Delta$ transforms (see [17]), we determine precisely when a chordal simple graph can take on each of the values allowed for both μ and ν . For the case of μ , this extends and makes use of precise results for split graphs from the recent work in [8]. A by-product of our analysis is a precise result for ν of chordal multigraphs. In addition, the conditions required in all cases are completely combinatorial, in that they are solely based on maximal cliques or on split subgraphs. Also, the conclusions depend only on tree-width or clique-size, and hence do not actually require the computation of nullities or verification of the SAP. Furthermore, to the best of our knowledge, using orthogonal removal for this purpose is novel (although Schur complements were used in [16] in conjunction with computing μ), and appears to have some utility for chordal graphs and perhaps beyond chordal graphs.

We have defined μ above for multigraphs for the purpose of completeness. However, our analysis of μ for chordal graphs (see Section 5) mainly involves simple graphs, unlike the corresponding one for ν (see Section 4). To our knowledge, μ has

not been previously considered for multigraphs, and therefore, properties of μ in this more general setting may be of interest for further investigation.

2. Preliminaries. A subgraph of G induced by $V' \subseteq V$, denoted $G[V']$, has vertex set V' and edge set consisting of those edges of G where both vertices are elements of V' . If $V' = V(G) \setminus \{v\}$, we denote $G[V']$ by $G - v$. The *neighborhood* of a vertex v of a graph G , denoted $N(v)$, is the set of vertices of G adjacent to v . The *closed neighborhood* of a vertex v , $N[v]$, is $N(v) \cup \{v\}$.

The *simple-neighborhood* of a vertex v in a multigraph G , denoted by $N_1(v)$, is defined to be those vertices $u \in N(v)$ such that u is adjacent to v by a single edge. In a simple graph, $N_1(v) = N(v)$ for every vertex v . A vertex v is called *singly-isolated* in a multigraph G if $N_1(v)$ is an empty set.

By a *complete graph*, we mean a graph where all vertices are pairwise adjacent, and we denote a simple complete graph on n vertices by K_n . A *multicomplete graph* is a complete graph with no single edges, and such a graph on n vertices will be denoted by $K_n^=$. A *clique* is a complete subgraph of a multigraph. For a multigraph G , we let $\omega(G)$ denote the number of vertices of a largest clique in G . A vertex v is called *simplicial* if $G[N(v)]$ is a clique.

A graph G is called *chordal* if G contains no induced cycles of length four or more. A useful view of chordal graphs is that they have a tree-like structure in which their maximal cliques play the role of vertices. Suppose G_1 and G_2 are graphs each of which contains the clique K_p . If we identify the copy of K_p in G_1 with that in G_2 , then the resulting graph G is called a *clique sum* of G_1 and G_2 (along the clique K_p). If G_1 is the clique K_q and G_2 is any chordal graph containing the clique K_p , $p < q$, then the clique sum of G_1 and G_2 along K_p is also chordal. In fact, chordal graphs are just the sequential clique sums of arbitrary cliques [3, 9]. The parameter μ was considered for the clique sum of two general graphs G_1 and G_2 in [16], where it was shown to be closely related to $\max\{\mu(G_1), \mu(G_2)\}$.

A special subclass of chordal graphs are those known as k -trees. A k -tree is constructed sequentially by starting with a complete graph on $k + 1$ vertices and connecting each new vertex to the vertices of an existing clique on k vertices. Observe that every graph is a subgraph of a k -tree, for some value of k at most the number of vertices of G . For a graph G , the *tree-width* of G , denoted by $\text{tw}(G)$, is the minimum k such that G is a subgraph of a k -tree [4].

2.1. Vector representations. Given a set of vectors $\vec{V} = \{\vec{v}_1, \dots, \vec{v}_n\}$ in \mathbb{R}^m , let $X = [\vec{v}_1 \ \dots \ \vec{v}_n]$ be the matrix whose columns are vectors from \vec{V} . Then $X^T X$ is a positive semidefinite matrix, called the *Gram matrix* of \vec{V} . Given a class of matrices $\mathcal{C}(G)$ associated to a multigraph G , we say \vec{V} is a *vector representation*

for G if the Gram matrix of \vec{V} belongs to $\mathcal{C}(G)$ [22]. For example, matrices in $\mathcal{P}(G)$ correspond to *orthogonal* vector representations where $\langle \vec{v}_i, \vec{v}_j \rangle \neq 0$ if i and j are joined by a single edge and $\langle \vec{v}_i, \vec{v}_j \rangle = 0$ if i and j are not adjacent [19]. Since any n -by- n real positive semidefinite matrix A may be factored as $Y^T Y$ for some n -by- n real matrix Y with $\text{rank } A = \text{rank } Y$, each positive semidefinite matrix is the Gram matrix of a suitable set of vectors.

Symmetric matrices that satisfy M1–M3 are also related to certain vector representations.

LEMMA 2.1 ([17]). *Let M satisfy M1–M3 for a multigraph G on n vertices. Then there exists an n -by- n matrix $A = [a_{ij}]$ that satisfies*

- (A1) *for all $i \neq j$, $a_{ij} < 1$ if i and j are adjacent by a single edge in G and $a_{ij} = 1$ otherwise;*
- (A2) *A is positive semidefinite;*
- (A3) *$A - J$ has the SAP with respect to G ,*

such that the nullity of A is one more than the nullity of M , where J is the matrix of all ones.

For a connected multigraph, only the definition of condition A1 is modified from [17]. The vector representations connected with these matrices require inner products equal to one if vertices are not adjacent and less than one if vertices are adjacent.

3. Schur complements. For an n -by- n matrix A and $\alpha, \beta \subseteq \{1, 2, \dots, n\}$, the submatrix of A lying in rows indexed by α and columns indexed by β will be denoted by $A[\alpha, \beta]$. If $\alpha = \beta$, then the principal submatrix of A lying in rows and columns indexed by α is denoted by $A[\alpha]$. If $A[\alpha]$ is nonsingular, then the *Schur complement of $A[\alpha]$ in A* is the matrix $A[\alpha^c] - A[\alpha^c, \alpha](A[\alpha])^{-1}A[\alpha, \alpha^c]$, where α^c represents the complement of the set α relative to $\{1, 2, \dots, n\}$.

LEMMA 3.1 (cf. [23]). *Suppose $A = [a_{ij}]$ is a real symmetric matrix, a_{ii} is positive for some i , and S is the Schur complement of a_{ii} in A . If A is positive semidefinite, then S is also positive semidefinite. If A satisfies M2, then so does S . Finally, $\text{rank}(S) = \text{rank}(A) - 1$.*

Proof. Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix},$$

in which A_{11} is k -by- k and nonsingular. Then

$$\begin{bmatrix} I & 0 \\ -A_{12}^T A_{11}^{-1} & I \end{bmatrix} A \begin{bmatrix} I & -A_{11}^{-1} A_{12} \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix}, \quad (3.1)$$

Moreover, since A is symmetric, it follows from (3.1) and Sylvester's Law of Inertia [18] that the number of positive, negative and zero eigenvalues of A is the same as the number of positive, negative, and zero eigenvalues, respectively, of

$$\begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix}.$$

Using permutation similarity if required, assume $i = k = 1$, so that $A_{11} = a_{11} > 0$, and thus, S has the same number of negative and zero eigenvalues as A . \square

If A is an n -by- n real symmetric matrix with $a_{nn} \neq 0$, then the (i, j) -entry, s_{ij} , in the Schur complement, S , of a_{nn} in A is given by

$$s_{ij} = \frac{1}{a_{nn}} \det \begin{bmatrix} a_{ij} & a_{in} \\ a_{nj} & a_{nn} \end{bmatrix}. \quad (3.2)$$

Given the graph G , we incorporate the graph of S and view it as a graph (or multigraph) derived from the graph of A (this was referred to as orthogonal removal in [5]). Given three vertices u, v, w , let $e(u, v, w)$ be the product of the number of edges between u and v and the number of edges between w and v . Define two operations as follows: For $G \ominus v$, in the induced subgraph $G - v$ of G , between any $u, w \in N(v)$, insert $e(u, v, w)$ edges [21].

For $G \ominus_\mu v$, first observe that beyond two, the number of edges between two vertices is immaterial in determining the matrices that satisfy conditions M1–M3. For \ominus_μ , we will not be as specific in the following definition to help avoid a need for redefinition later: if either $u, w \in N(v)$ are adjacent by a single edge in G or $e(u, v, w) = 1$, then let u and w be adjacent by a single edge in $G \ominus_\mu v$; otherwise, u and w should be adjacent by multiple edges in $G \ominus_\mu v$.

Note that if G is a simple graph and v is a simplicial vertex, then $G \ominus_\mu v = G - v$. Using Lemma 3.1 and (3.2), we have the following:

LEMMA 3.2. *Suppose G is a multigraph, v is a vertex of G , $A = [a_{ij}]$ is a real symmetric matrix, $a_{vv} > 0$, and S is the Schur complement of a_{vv} in A . If $A \in \mathcal{P}(G)$, then $S \in \mathcal{P}(G \ominus v)$. If A satisfies M1 for G , then S satisfies M1 for $G \ominus_\mu v$.*

Finally, we are able to observe the relationship between Schur complements and the SAP.

LEMMA 3.3. *Let G be a multigraph, v a vertex of G , $M = [m_{ij}] \in \mathcal{S}(G)$, $m_{vv} > 0$, and S the Schur complement of m_{vv} in M . If M has the SAP with respect to G , then S has the SAP with respect to $G \ominus v$ and $G \ominus_\mu v$. If v is simplicial and S has the SAP with respect to either $G \ominus v$ or $G \ominus_\mu v$, then M has the SAP with respect to G .*

Proof. First, note that $G \ominus_\mu v$ is a subgraph of $G \ominus v$ and they only differ on $N(v)$ which induces a clique in both, so that the SAP with respect to $G \ominus v$ is equivalent to the SAP with respect to $G \ominus_\mu v$.

Without loss of generality, assume that $v = 1$, and since $\mathcal{S}(G)$ is closed under nonzero scalar multiplication, we may assume that

$$M = \begin{pmatrix} 1 & \beta & \mathbf{0} \\ \beta^T & X & A \\ \mathbf{0} & A^T & B \end{pmatrix}.$$

Let

$$S = \begin{pmatrix} X' & A \\ A^T & B \end{pmatrix},$$

where $X' = X - \beta^T \beta$ so that S is the Schur complement of the (1,1)-entry of M in M .

Consider the symmetric matrices

$$Z = \begin{pmatrix} \mathbf{0} & C \\ C^T & D \end{pmatrix} \quad \text{and} \quad Z' = \begin{pmatrix} 0 & \mathbf{0} & \gamma \\ \mathbf{0} & \mathbf{0} & C \\ \gamma^T & C^T & D \end{pmatrix}.$$

If $MZ' = 0$, then

$$\begin{aligned} 0 &= \gamma + \beta C \\ 0 &= A\gamma^T \\ 0 &= AC^T \\ 0 &= \beta^T \gamma + XC + AD \\ 0 &= B\gamma^T \end{aligned}$$

$$\begin{aligned} 0 &= BC^T \\ 0 &= A^T C + BD. \end{aligned}$$

Given fixed γ and nonzero β satisfying the first equation, these equations are satisfied if and only if $SZ = 0$. As a result, if S does not have the SAP with respect to $G \ominus v$, there exists such a Z with $SZ = 0$, and then $MZ' = 0$ and hence M does not have the SAP with respect to G . Under the additional assumption that v is simplicial, $A(G) \circ Z' = I \circ Z' = 0$ if and only if $A(G \ominus v) \circ Z = I \circ Z = 0$ if and only if $A(G \ominus_{\mu} v) \circ Z = I \circ Z = 0$. Finally, since β is nonzero, the fourth equation above shows that Z is nonzero if and only if Z' is nonzero. Thus, M satisfies the SAP with respect to G if and only if S satisfies the SAP with respect to $G \ominus v$. \square

We can also improve somewhat upon the simplicial condition:

COROLLARY 3.4. *Let v be a vertex of a multigraph G with the property that the complement of the graph $G[N(v)]$ is a forest. Suppose that $A \in \mathcal{S}(G \ominus v)$ has the SAP for $G \ominus v$, and x is a $|G|$ -by-1 column vector whose support corresponds to $N[v]$. If $xx^T + (0 \oplus A) \in \mathcal{S}(G)$, then $xx^T + (0 \oplus A)$ has the SAP for G .*

Proof. Suppose the result is false. By the support requirement of x and by scaling if needed, we may assume that we have an equation of the form

$$\begin{pmatrix} 1 & \beta & \mathbf{0} \\ \beta^T & X & A \\ \mathbf{0} & A^T & B \end{pmatrix} \begin{pmatrix} 0 & \mathbf{0} & \gamma \\ \mathbf{0} & Y & C \\ \gamma^T & C^T & D \end{pmatrix} = 0.$$

Also by the requirement on the support of x , β must have no zero entries. The equation implies $\beta Y = 0$, so that Y must have at least two nonzero entries in every column. Since the graph of Y is a forest, this is only possible if $Y = 0$. The result now follows from the arguments in the proof of Lemma 3.3. \square

The proofs of Lemma 3.3 and Corollary 3.4 follow closely the analysis of the so-called ΔY and $Y\Delta$ transforms in the survey [17]. Indeed, the $Y\Delta$ transform is the special case of Lemma 3.3 for a vertex of degree three whose neighbors are pairwise nonadjacent, and the small degree enables the proof of Corollary 3.4 to work for the reverse transform, ΔY , without any additional assumptions.

4. Chordal graphs and ν . Throughout this section, for a given graph G , we call a matrix $A \in \mathcal{P}(G)$ a ν -optimal matrix for G if A satisfies the SAP and the nullity of A is equal to $\nu(G)$.

THEOREM 4.1. *If v is a vertex of a multigraph G and there exists a ν -optimal matrix for G whose diagonal entry corresponding to v is nonzero, then $\nu(G) \leq \nu(G \ominus v)$. If v is also simplicial, then $\nu(G) = \nu(G \ominus v)$.*

Proof. If $A = [a_{ij}]$ is a ν -optimal matrix for G with $a_{vv} > 0$ and S is the Schur complement of a_{vv} in A , then $S \in \mathcal{P}(G \ominus v)$ by Lemma 3.2, the nullity of S is equal to $\nu(G)$ by Lemma 3.1 and the assumption that A is ν -optimal, and S satisfies the SAP for $G \ominus v$ by Lemma 3.3.

If v is simplicial, then given a ν -optimal matrix S for $G \ominus v$, there exists a vector x whose support corresponds to $N[v]$ such that $A = xx^T + (0 \oplus S) \in \mathcal{P}(G)$ [5]. Then $a_{vv} > 0$ and S is the Schur complement of a_{vv} in A . By Lemma 3.1 and Lemma 3.3, A satisfies the SAP for G and has nullity $\nu(G \ominus v)$. \square

REMARK 4.2. The existence of a vector x such as in the previous proof has been shown under other conditions as well, for example when v is adjacent to each of its neighbors by a single edge and the complement of $G[N(v)]$ is a star forest [10]. Combined with Corollary 3.4, we have $\nu(G) = \nu(G \ominus v)$ in this case as well.

THEOREM 4.3. *If G is a chordal multigraph that is not complete, then there exists a simplicial vertex v of G and a ν -optimal matrix for G whose diagonal entry corresponding to v is nonzero.*

Proof. Every chordal graph that is not complete has two nonadjacent simplicial vertices, u and v . If either u or v is not singly-isolated, then we are done. Assume both u and v are singly isolated, and M is a ν -optimal matrix for G whose diagonal entries corresponding to both u and v are zero. We claim that M does not have the SAP with respect to G .

Since M is positive semidefinite, the rows and columns of M corresponding to u and v are all zero. Since u and v are not adjacent in G , the symmetric matrix S with 1 in the (u, v) and (v, u) positions and zeros elsewhere satisfies $MS = A(G) \circ S = I \circ S = 0$. \square

While the next three results are known [12, 13, 15], our analysis thus far allows for shorter alternate proofs.

LEMMA 4.4. *If K is a complete graph on two or more vertices, then $\nu(K) = |K|$ if K is multicomplete and $\nu(K) = |K| - 1$ otherwise.*

LEMMA 4.5. *If G is a chordal multigraph, then $\omega(G) - 1 \leq \nu(G) \leq \omega(G)$.*

Proof. The lower bound follows from minor monotonicity and Lemma 4.4. We will prove the upper bound by induction. The statement is true for a graph with one vertex. Assume the upper bound holds for all graphs with at most k vertices, and

let G be a graph on $k + 1$ vertices. If G is complete, we are done. If G is chordal and not complete, then Theorem 4.3 guarantees that we can orthogonally remove a simplicial vertex (which cannot increase ω), and apply Theorem 4.1 and the induction hypothesis. \square

COROLLARY 4.6. *For any simple graph G , $\nu(G) \leq \text{tw}(G) + 1$.*

Proof. The graph G is a subgraph of a chordal $\text{tw}(G)$ -tree H [4], and $\omega(H) = \text{tw}(H) + 1 = \text{tw}(G) + 1$. Using Lemma 4.5 and the minor-monotone property of ν , $\nu(G) \leq \nu(H) \leq \omega(H) = \text{tw}(G) + 1$. \square

THEOREM 4.7. *Suppose G is a chordal multigraph. Then there exists a sequence of vertices v_1, \dots, v_m in G such that if we define the graphs $G_0 = G$, $G_1 = G_0 \ominus v_1$, $G_i = G_{i-1} \ominus v_i$, for $i = 1, 2, \dots, m$, then v_i is simplicial in G_{i-1} and G_m is isomorphic to $K_{\bar{k}}$ where $\omega(G) - 1 \leq k \leq \omega(G)$ and $\nu(G) = k$.*

Proof. If G is not complete, there exists a simplicial vertex that can be orthogonally removed with the equality of Theorem 4.1 by Theorem 4.3. If G is complete and has a non-singly isolated vertex, then that vertex may be orthogonally removed with the equality of Theorem 4.1. If G is isomorphic to $K_{\bar{k}}$, then $\omega(G) - 1 \leq k \leq \omega(G)$ by Lemma 4.4 and Lemma 4.5. \square

We state and prove our main observations on ν for chordal graphs.

THEOREM 4.8. *Let G be a chordal multigraph. If every maximal clique of G has a single edge that is not contained in any other maximal clique, then $\nu(G) = \text{tw}(G)$.*

Proof. If every maximal clique of G has a single edge that is not contained in any other clique, then the removal process of Theorem 4.7 cannot terminate in $K_{\bar{\omega(G)}}$. \square

THEOREM 4.9. *Let G be a chordal multigraph. Then $\nu(G) = \text{tw}(G) + 1$ if and only if there exists a maximal clique of G with every single edge contained in at least two maximal cliques of G .*

Proof. Theorem 4.8 gives one direction. For the other, we use a proof by induction: Note that the result holds for a graph with one vertex by Lemma 4.4. Assume that the result is true for all graphs on at most k vertices, and let G be a chordal multigraph on $k + 1$ vertices. Suppose that there exists a maximal clique C of G with every single edge of C contained in at least two cliques of G . If G is complete, then this can only happen if G is isomorphic to $K_{\bar{\omega(G)}}$, and the result follows from Lemma 4.4.

If G is not complete, choose a simplicial vertex v that does not belong to C . If v is singly-isolated, replace the multiedges adjacent to v with single edges to obtain a new graph H . Otherwise, set $H = G$. In either case, H is a subgraph of G , so that $\nu(H) \leq \nu(G)$, C is a maximal clique of H with every single edge contained in at

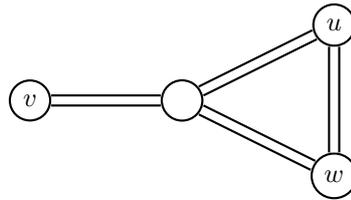


FIG. 4.1. The graph G in Example 4.11.

least two cliques of H , and v is not singly-isolated in H , so that $\nu(H \ominus v) = \nu(H)$ by Theorem 4.1. Let C' be the subgraph induced by the vertices of C in $H \ominus v$. Then C is a subgraph of C' , and any single edge of C that was part of $N(v)$ in H is a multiedge in $H \ominus v$, so that every single edge of C' is contained in at least two cliques of $H \ominus v$. As a result, $\omega(H \ominus v) = \omega(H) = \omega(G)$. By the induction hypothesis,

$$\text{tw}(G) + 1 = \text{tw}(H) + 1 = \nu(H \ominus v) = \nu(H) \leq \nu(G).$$

The result follows from Lemma 4.5. \square

EXAMPLE 4.10. To demonstrate the utility of Theorems 4.8, and 4.9, consider the following examples. Suppose G is a tree (on at least 3 vertices), then G satisfies the hypotheses of Theorem 4.8, and hence, $\nu(G) = \text{tw}(G) = 1$. A *linear k -tree* is a k -tree in which no more than two maximal cliques share an edge and every maximal clique shares an edge with at most two maximal cliques. Another application of Theorem 4.8 gives $\nu(G) = \text{tw}(G) = k - 1$ for any linear k -tree

On the other hand, if G is the graph G_3 from [7] (sometimes referred to as the supertriangle), then there is a maximal clique in G_3 in which each edge is contained in at least two maximal cliques, and hence, it follows from Theorem 4.9 that $\nu(G_3) = \text{tw}(G_3) + 1 = 3$. For the graph G known as the *pinwheel* (see [2, Fig. 2.1]), it follows that $\nu(G) = 3$ (the maximum clique size) by a simple application of Theorem 4.9.

EXAMPLE 4.11. For completeness we offer the following example to help shed some light on the removal process guaranteed by Theorem 4.7. For the chordal multigraph G in Figure 4.1, $\nu(G \ominus v) = \nu(G) = 3$ but $\nu(G \ominus u) = 2$. In particular, if there exists at least one non-singly-isolated simplicial vertex, then any such vertex may be selected next. If all simplicial vertices are singly-isolated, then a simplicial vertex not in a maximal clique must be selected next if one is available.

5. Chordal graphs and μ . For a given graph G , we call a matrix $A \in \mathcal{S}(G)$ an μ -optimal matrix for G if A satisfies M1–M3 and the nullity of A is equal to $\mu(G)$.

A simple graph is called a *split graph* if it can be partitioned into a clique and an independent set. Split graphs are exactly those simple chordal graphs whose complement is chordal. In [8], μ was determined for the case of split graphs: A split graph with maximum clique C and independent set S is *Type II* if there exist vertices $v \in C$ and $u_1, u_2 \in S$ with $N(u_1) = N(u_2) = C - v$, otherwise it is called *Type I*. If G is a split graph, then $\mu(G) = \text{tw}(G) + 1$ if and only if G is Type II. They also proved

THEOREM 5.1 ([8]). *For any simple graph G , $\mu(G) \leq \text{tw}(G) + 1$.*

We now determine $\mu(G)$ for all simple chordal graphs, thus extending the result in [8] for split graphs, by first establishing a relationship between $\mu(G)$ and $\mu(G \ominus_\mu v)$.

THEOREM 5.2. *If v is a vertex of a multigraph G and there exists a μ -optimal matrix for G whose diagonal entry corresponding to v is positive, then $\mu(G) \leq \mu(G \ominus_\mu v)$. If G is simple and v is simplicial, then $\mu(G) = \mu(G - v)$.*

Proof. If $A = [a_{ij}]$ is a μ -optimal matrix for G with $a_{vv} > 0$ and S is the Schur complement of a_{vv} in A , then S satisfies M1 for G by Lemma 3.2, the nullity of S is equal to $\nu(G)$ by Lemma 3.1 and the assumption that A is μ -optimal, and S satisfies the SAP for $G \ominus_\mu v$ by Lemma 3.3. If G is simple and v is simplicial, then $G \ominus_\mu v = G - v$ and the result follows from minor monotonicity. \square

In [1], the inequality of Theorem 5.2 was already known in special cases, and equality was shown to hold for edge subdivision and the ΔY transform if μ is assumed to be sufficiently large. A related result holds for \ominus_μ :

THEOREM 5.3. *Let v be a vertex of a simple graph G . If $\mu(G \ominus_\mu v)$ is greater than the degree of v , then $\mu(G) \leq \mu(G \ominus_\mu v)$.*

Proof. Let H be the graph obtained from G by adding an edge between any nonadjacent neighbors of v . Then H is a clique sum of the complete graph K_{d+1} , where d is the degree of v , and $G \ominus_\mu v$. Since $\mu(G \ominus_\mu v) > d = \mu(K_{d+1})$, $\mu(H) = \mu(G \ominus_\mu v)$ [17, Corollary 2.10], and $\mu(G) \leq \mu(H)$ by minor monotonicity. \square

REMARK 5.4. Theorem 4.3 can fail for μ . For example, the reader may verify that the graph G in Figure 5.1 has $\mu(G) = 2$ and the rank two matrix

$$M = \begin{pmatrix} 0 & -1 & -1 & 0 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & -2 & -1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$$

is μ -optimal for G . If there existed a μ -optimal matrix for G with a positive diagonal entry corresponding to a simplicial vertex, Theorem 5.2 would imply that $\mu(K_3) = 1$, a contradiction.

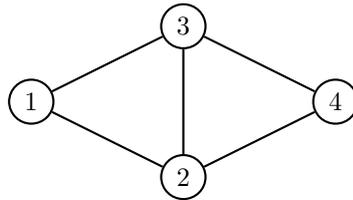


FIG. 5.1. The graph G for Remark 5.4.

THEOREM 5.5. *Let G be a chordal simple graph that is not complete and let u be a simplicial vertex of G . Then either there exists a μ -optimal matrix for G whose diagonal entry corresponding to u is strictly positive or there exists a simplicial vertex w of G that is not adjacent to u and $N(u) = N(w)$.*

Proof. Since G is not complete, it has at least two non-adjacent simplicial vertices u and w . Let M be a μ -optimal matrix for G , and let $A = [a_{ij}]$ be the corresponding matrix obtained from Lemma 2.1 that satisfies A1–A3. By construction, $A - J$ satisfies M1–M3 for G , $\text{rank}(A - J) = \text{rank}(M)$, and we may consider the diagonal entry of $A - J$ corresponding to u : Since A is positive semidefinite, it can be viewed as the Gram matrix of a vector representation of G , with \vec{u} and \vec{w} representing vertices u and w . The diagonal entries of A corresponding to u and v are $\|\vec{u}\|^2$ and $\|\vec{w}\|^2$ respectively, and are both 1 more than the corresponding entries of $A - J$. Since $\langle \vec{u}, \vec{w} \rangle = 1$, if $\|\vec{u}\| \leq 1$ and $\|\vec{w}\| \leq 1$, then both are unit vectors and $\vec{u} = \vec{w}$. To satisfy condition A1, $\vec{u} = \vec{w}$ if and only if $N(u) = N(w)$. \square

THEOREM 5.6. *If G is a chordal simple graph with exactly two non-adjacent simplicial vertices v and u , then $\mu(G - v) = \mu(G)$.*

Proof. If $N(v) \neq N(u)$, then Theorem 5.5 applies. If $N(v) = N(u)$, then G is a Type I split graph. \square

We now prove our main result about $\mu(G)$ for chordal simple graphs G , which extends some of the recent work in [8] for the case of computing μ for split graphs.

THEOREM 5.7. *If G is a chordal simple graph that contains a Type II split graph H as an induced subgraph where $\omega(H) = \omega(G)$, then $\mu(G) = \text{tw}(G) + 1$. Otherwise, $\mu(G) = \text{tw}(G)$.*

Proof. The first statement holds by minor monotonicity and $\text{tw}(G) + 1$ is an upper bound by Theorem 5.1. We prove the second statement by induction, noting first that the result holds for complete graphs. Suppose that G is a connected chordal simple graph that is not complete. By assumption, if G has three or more simplicial

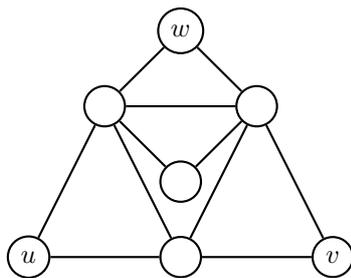


FIG. 5.2. The graph G in Example 5.8.

vertices, then there must be at least two that have non-equal neighborhoods, so by Theorem 5.2, Theorem 5.5, and Theorem 5.6, there exists a simplicial vertex v such that $\mu(G) = \mu(G - v)$. Since G does not contain a Type II split graph H where $\omega(H) = \omega(G)$ as an induced subgraph, neither does $G - v$, and the result holds by induction. \square

EXAMPLE 5.8. In Section 4, we have already demonstrated examples of chordal graphs for which ν is equal to both the tree-width and to the tree-width plus one. We now include examples of chordal simple graphs for which all four possible combinations of μ and ν occur. Observe that for K_3 , we have $\mu(K_3) = \nu(K_3) = \text{tw}(K_3) = 2$. For the remainder of this discussion, we will refer to the graph G in Figure 5.2. Observe that the tree-width of the chordal graph G is two. Furthermore, by Theorem 4.9, we know that $\nu(G) = 3$, and since G is not outerplanar, $\mu(G) \geq 3$, and hence, must then be equal to three. Consider the graph $G - w$ (which is the graph G_3 from [7]). The tree-width of $G - w$ is still two, but now it follows that $\mu(G - w) = 2$, by Theorem 5.7 (also note $G - w$ is outerplanar), but $\nu(G - w) = 3$, by Theorem 4.9. Now, consider the graph $G - u - v$. Again, the tree-width of $G - u - v$ is two, but since $G - u - v$ contains a Type II split subgraph, it follows that $\mu(G) = 3$ and $\nu(G) = 2$ by applying Theorem 5.7 and using Theorem 4.8. Finally, observe that $H = G - u, v, w$ is a Type II split graph, from which we conclude $\mu(H) = \text{tw}(H) + 1 = 3$, by an application of Theorem 5.7 again.

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